Licensing Probabilistic Patents and Liability Rules: The Duopoly Case

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December 2008

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September 9, 2009

Abstract

In this paper a dynamic game is used to compare licensing of a cost reduction innovations under lost profits (LP) and unjust enrichment (UE), both damage rules used by courts in the calculation of damages when a patent has been infringed.

The innovation, whose property right belongs to a firm (patent holder) has a positive probability to be declared invalid in a court. The market is composed by two homogeneous firms that compete in quantities (Cournot).

Licensing by using royalty rates is preferred compared with fixed fees, it is observable little licensing (just big innovations).

LP is not always better or worst than UE, but the major of the cases consumers and society are better off under UE and in the major of cases LP benefits more to the patent holder.

1 Introduction

One of the most important mechanisms made for to compensate and to incentive innovation is the *Patent System*. In this system there is an authority (i.e. the EPO in Europe) that gives rights of property on pieces of knowledge to an agent, this rights are known as *patents*. Clearly not everything is patentable and not every patent is important and just a small part of them become important tools in market competence between firms.

Patents commonly are related with the exclusivity right to exploit the commercial potential of a innovation trough a monopoly, but also it is possible obtains rewards trough *licensing*. Licensing is a settlement that permits to a third part use the innovation. Commonly are used *fixed fees, royalty rates and auctions* as payment mechanism in a licensing settlement.

There is a huge literature that have analyzed licensing under indisputable property rights, known as *ironclad property rights* in the literature. But there are reasons to believe that the patent system is not perfect and patents are granted to inventions that do not meet the requirements of being a patentable subject matter (is a machine, process, etc) with some utility for the society being a novelty and a non obvious invention.

The mechanism of licensing has been widely studied. The common approach used was game theory. In this approach the patent holder and one or several players are involve in a dynamic game of three stages: At the first stage of the game, the patent holder decides how much ask for the licenses and how many licenses he will offer. At the second stage potential licensees decide to get the license of continues using the backstop technology [1]. Finally in the last stage, firms compete in the market[2].

However, there are reasons to think that many of the patents have disputable property rights, one reason is the big volume of applications received by the patent authorities and the competence of examiners and the complexity of the matter.

According to the EPO in 2003 were filed 123 700 applications[3], where on average a granted patent was published 47 months after the application was received (at 2008), and finally the quantity of patents granted in 2008 was 58,819 patents[4], being not so accurate it means than approximate 50% of applications ends as european patents. If we assume that 21.1 per thousand of patents are filed[5] is expected around of 1,000 suits per year.

The fact that many patents could be declared invalid within a court procedure creates uncertainty over the property rights, a factor that plays a important role in licensing , when property rights are probabilistic (meaning disputable)threat points of patentees and potential licensees suffered modifications, and is expected also modifications in the behavior of firms, one

[1] The best technology available without the use of the innovation

[2] see Kamien and Tauman [2002] and Sen and Tauman [2007] for a survey about licensing games under ironclad rights

[3] see Office [2004]

[4] see Office [2008]

[5] see Lanjouw and Schankerman [2004]

observable result was the explosion of patent litigation in the last years [6].

When potential users decide infringe a patent, the patent holder could enforce the property rights by using the legal system, in this arena the patent holder will try to prove *infringement* and the infringers will try to invalid the patent. If the patent holder is successful in to prove infringement, the court could authorize the payment of *damage payments* and order another actions in order to enforce the property rights, in such way that the patent holder will be compensated by the infringement, so then the legal system has a important role when licensing terms are defined. Commonly two liability rules are used for to calculate damage payments: *Lost Profits (LP)* and *Unjust Enrichment (UE)*.

The impacts of this damages have been studied in different contexts as vertical relationship and horizontal competition. In the case of vertical relationship Schankerman and Scotchmer [2001] have analyzed how liability rules protect patents, they conclude that UE protect better the patent holder than LP in the case of research tools, however in the case of cost reduction innovations these results are reverse.

My work is very related with the work of Anton and Yao [2007] and Choi [2009], In the case of Anton and Yao they explore the impacts of the LP rule on competence and innovation, assuming a linear demand scheme and a non drastic innovation, finishing their analysis concluding that infringement is a dominant situation even under the use of different liability rules. In the other fold Choi compares different liability rules assuming a drastic innovation and a more general demand function.

The starting point of my research is the contribution developed by Wang [1998], where he develops a duopoly model to study licensing under ironclad patents under a Cournot scenario. In this model royalty rate scheme is compared against fixed fee licensing for drastic and non-drastic innovations. Under this base model I added the development made it by Anton and Yao and Choi (AYC) to include probabilistic patents in a take or live it ex-ante licensing situation. In a difference of AYC I use a simple linear demand with homogeneous firms and homogeneous costs, this specification allow me to study drastic and non-drastic innovations, also I compare the royalty rate scheme against the fixed fee scheme assuming probabilistic patents.

My results show that surprisingly licensing it is not possible under UE and just big innovation are licensed under the LP rule, for another side it is

[6]see Lemley and Shapiro [2005]

showed that licensing using a royalty rate is better than a fixed fee scheme front the point of view of the patent holder. Comparison analysis show also that LP protect better the patentee for big innovations and small ones are better protected by UE rule.

The document is organized as follows. In the section 2 are established the assumptions and description of a licensing game. In the sections 3, 4 and 5 the game is solved. In section 6 a comparative analysis between LP and UE is executed. In section 7 the conclusions and important remarks of this work are analyzed. Proofs of the propositions are showed in the text and lengthy proofs are treated in an appendix.

2 The Game

The game is a non cooperative game that involves two players: patent holder (firm 1) and a competitor (firm 2), they produce the same good under fixed marginal costs c_i, with $i = 1, 2$.

Let $p = a - q_1 - q_2$ be the inverse linear demand function that both face, where q_i is the quantity offered by the firm i and $0 < c_i < a < \infty$. Let c be the fixed marginal cost of the backstop technology (old technology). The *firm* 1 has a patented a cost reduction innovation that reduces the marginal cost from c to $c - \epsilon$, where $0 < \epsilon < c$, then it produces under $c_1 = c - \epsilon$.

The another firm's marginal cost c_2 could be equal to c whether the *firm* 2 decides just use the old technology, or c_2 should be equal to $c - \epsilon$ when *firm* 2 uses the innovation. Where the last situation is achievable when the patent holder grants a license to the competitor or when the *Firm* 2 infringes the patent.

One useful expression is the *relative size of the innovation*,

$$\gamma = \frac{\epsilon}{a - c}$$

that is going to be used extensively along this document, without loss of generality it is assumed that $a - c = 1$.

Let $\pi_i^s(q_i, q_j) = (1 + \gamma - q_1 - q_2)q_i$ be the profit function associated with the use of the new cost reduction technology by the firm i and let $\pi_i^i(q_i, q_j) = (1 - q_1 - q_2)q_i$ be the profit function associated with the use of the old technology. Notice that the profit function for the patent holder is always π_1^s.

At the very beginning of the game the patent holder decides whether to license ($\mathcal{L}'$) or not ($\mathcal{N}'$), if decides licensing offers a fixed fee (F) or a royalty rate(r), the offer is a take it or leave one.

In the second stage the competitor decides between three alternatives: 1) accept the offer of the patent holder when is offered ($\mathcal{L}$); 2) uses the backstop technology ($\mathcal{N}$) and 3) Infringe the patent ($\mathcal{I}$) (see Figure 1 abode).

In the last stage the firms decide the quantities offered in the market as solution of a Cournot game. Once the competitor infringes the patent the patent holder reacts by starting a process in a court, with the objective to enforce its property rights.

The result of the trial is unknown, but there is a common knowledge probability $\theta \in (0, 1)$ that the patent will be declared valid after the trial, this parameter also reflects the *strength of the patent*.

When the patent holder shows the existence of infringement, the court pass to calculate damage payments. In this work are considered two options: at the first option the court calculates damages using LP and at the second option the court calculates damages using the UE rule.

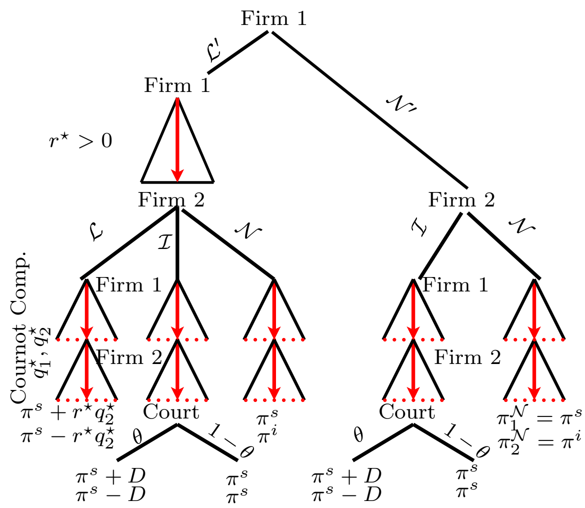

Figure 1: Game tree (the royalty rate case)

In this case the payoffs are characterized through the actions of the competitor, by example if the patent holder plays $\mathcal{N}'$ and the competitor plays

$\mathcal{N}$, the payoff obtained is the same that is obtained in the case when, the patent holder plays $\mathcal{L}'$ and asks a royalty rate r and the competitor plays $\mathcal{N}'$, where in both situation players choose the same quantities.

Then by using this consideration the payoffs are:

1. In the case that the competitor plays $\mathcal{N}$ the payoff for the players are $\pi_1^{\mathcal{N}} = \pi_1^s(q_1, q_2)$ and $\pi_2^{\mathcal{N}} = \pi_2^i(q_1, q_2)$, meaning that the patent holder uses the innovation and the other firm uses the *backstop technology* (old method of production).

2. When the competitor plays $\mathcal{L}$ the payoffs are $\pi_1^{\mathcal{L}} = \pi_1^s(q_1, q_2) + L(q_2)$ and $\pi_2^{\mathcal{L}} = \pi_2^s(q_1, q_2) - L(q_2)$, where $L(q_2) = F$ when a fixed fee F is offered labeled as $\mathcal{L}, \mathcal{F}$ and $L(q_2) = rq_2$ when a royalty rate r is offered in exchange of a license this case is labeled as $\mathcal{L}, \mathcal{R}$, because royalties and fixed fees are endogenously determinated our two cases become four for the inclusion of the liability rules, i.e. $\mathcal{L}, \mathcal{F}, \mathcal{LP}$ means that licensing has been played when a fixed fee has been offered and LP is used as liability rule.

3. In the case that the competitor plays $\mathcal{I}$ the payoffs are $\pi_1^{\mathcal{I}} = \pi_1^s(q_1, q_2) + \theta D(q_1, q_2)$ and $\pi_2^{\mathcal{I}} = \pi_2^s(q_1, q_2) - \theta D(q_1, q_2)$, where D is the damage payment calculated using the LP or the UE rule, meaning that $\mathcal{I},\mathcal{UE}$ represents played $\mathcal{I}$ when the UE rule is used.

The solution criterion for the game described above is the *Sub-Game Perfect Nash Equilibrium (SPNE)*,that is going to solve in the next tree sections.

3 Competition Stage

Given a defined rule for the calculations of damages (LP or UE), a level of technology chosen by the incumbent firm ($\mathcal{N}$, $\mathcal{I}$, $\mathcal{L}$) and a licensing policy defined by the patent holder (to offer or not a license to the competitor using a fixed fee or a royalty rate), both firms compete by choosing quantities. This section is devoted to calculate the payoffs under different scenarios as a solution of the Cournot problem.

At the case when the competitor decides to use the backstop technology ($\mathcal{N}$), the Nash Equilibrium (NE) is granted when

$$(q_1^{\mathcal{N}}, q_2^{\mathcal{N}}) = \begin{cases} \left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}, 0\right) & \text{if } 1 \leq \gamma \end{cases} \tag{1}$$

As was noted by Arrow [1962] big innovations could permit to the patent holder to reduce the price till levels below the competitive prices, meaning that just the patent holder can remain in the market, this kind of innovations are called *drastic*. In this particular setup an innovation is *non-drastic* if $0 \leq \gamma < 1$ and is defined *drastic* if $\gamma \geq 1$.

Payoffs are

$$\begin{aligned} \pi_1^{\mathcal{N}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \\ \pi_2^{\mathcal{N}} &= \begin{cases} \left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ 0 & \text{if } 1 \leq \gamma \end{cases} \end{aligned} \tag{2}$$

A more complex situation emerges when the competitor infringes the patent ($\mathcal{I}$), once infringement is played the patent holder will try to enforce the property rights by suing the incumbent firm. When the patent holder is successful in the court (gains the trial), it is assumed here that the court will calculate a damage payment based in the LP or UE rule.

$$\begin{aligned} \pi_1^{\mathcal{I}} &= (1 - q_1 - q_2 + \gamma)q_1 + \theta D(q_1, q_2) \\ \pi_2^{\mathcal{I}} &= (1 - q_1 - q_2 + \gamma)q_2 - \theta D(q_1, q_2) \end{aligned} \tag{3}$$

Then the payoffs are characterized by eq.(3), notice that the first is the part $(1 - q_i - q_j + \gamma)q_i$ is the profit gained by the sales and the second part based on the damage payments $\theta D(q_1, q_2)$.

Damages could be calculated in different ways, the most common way to do it is using the LP rule or the UE rule. Both rules are based in a profile scenario, this scenario is *"no infringement"*, the idea behind LP is to compensate the share of profit lost by the patent holder caused by the infringement. In the case of UE, the profit excess above the competitor's profit relative to the *"no infringement"* scenario is transfered to the patent holder, this rule is also called disgorgement.

Basically UE and LP both need a comparison scenario of *"no infringement"*, in our model $\pi_1^{\mathcal{N}}$ is used as the comparison payoff when LP is the liability rule used by the court. The damage payment in this case is

$$D^{LP} = \max\left\{\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1, 0\right\} \tag{4}$$

When the court uses UE as liability rule, the damage (D^{UE}) is calculated in base to excess of profit for the competitor respect to $\pi_2^{\mathcal{N}}$, then

$$D^{UE} = \max\left\{(1 - q_1 - q_2 + \gamma)q_2 - \pi_2^{\mathcal{N}}, 0\right\} \tag{5}$$

The NE when damages are calculated by using the LP rule, and when the incumbent firm decides to infringe the patent deserve a special treatment [7].

Lemma 1. *The Cournot solution when competitor infringes and court uses LP rule for calculate damages is,*

$$(q_1^{\mathcal{I},\mathcal{LP}}, q_2^{\mathcal{I},\mathcal{LP}}) = \begin{cases} \left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right) & \text{if } \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right) & \text{if } \gamma \geq \frac{\theta}{3-2\theta} \end{cases} \tag{6}$$

, it produces

$$\begin{aligned} \pi_1^{\mathcal{I},\mathcal{LP}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma < \frac{\theta}{3-2\theta} \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \\ \pi_2^{\mathcal{I},\mathcal{LP}} &= \begin{cases} \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \end{aligned} \tag{7}$$

When $q_2^{\mathcal{I},\mathcal{LP}} = q_2^{\mathcal{N}}$ eq. (6), the patent holder gets the same profit that in the situation of no infringement but the competitor stays in a better situations because enjoys a lower cost and produce the same quantity that should be produced under no infringement, Anton and Yao [2007] calls this equilibrium *Passive Infringement*, because the damage payment does not reflect the effects of the infringement.

However, when $\gamma > \frac{\theta}{3-2\theta}$, damage payments calculated with the lost profit rule are positive in equilibrium, then in equilibrium a *Active Infringement* is present.

[7] Interested readers could see Anton and Yao [2007] for a more detailed analysis for non drastic innovation

Lemma 2. *The Cournot solution when competitor infringes and court uses UE as liability rule is,*

$$(q_1^{\mathcal{I},\mathcal{UE}}, q_2^{\mathcal{I},\mathcal{UE}}) = \left((1-\theta)\frac{1+\gamma}{3-\theta}, \frac{1+\gamma}{3-\theta}\right) \tag{8}$$

Results in the lemmas 1 and 2 cannot be considered trivial, because the best replies that produces the NEs are non-smooth in both cases. Proofs of this lemmas are considered in the appendix[8].

using the lemma 1,

$$\begin{aligned} \pi_1^{\mathcal{I},\mathcal{UE}} &= \begin{cases} \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0<\gamma<1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \\ \pi_1^{\mathcal{I},\mathcal{UE}} &= \begin{cases} (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0<\gamma<1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } \gamma \geq 1 \end{cases} \end{aligned} \tag{9}$$

When the incumbent firm decides to accept the offer of the patent holder against a given fixed fee (F) or a given royalty rate (r), the following NEs are obtained. In the fixed fee case

$$\left(q_1^{\mathcal{L},\mathcal{F}}, q_2^{\mathcal{L},\mathcal{F}}\right) = \left(\frac{1+\gamma}{3}, \frac{1+\gamma}{3}\right) \tag{10}$$

and finally for a given royalty rate (r)

$$\left(q_1^{\mathcal{L},\mathcal{R}}, q_2^{\mathcal{L},\mathcal{R}}\right) = \left(\frac{1+\gamma+r}{3}, \frac{1+\gamma-2r}{3}\right) \tag{11}$$

these results produce the following payoffs for the fixed fee case

$$\begin{aligned} \pi_1^{\mathcal{L},\mathcal{F}} &= \left(\tfrac{1+\gamma}{3}\right)^2 + F \\ \pi_2^{\mathcal{L},\mathcal{F}} &= \left(\tfrac{1+\gamma}{3}\right)^2 - F \end{aligned} \tag{12}$$

and

$$\begin{aligned} \pi_1^{\mathcal{L},\mathcal{R}} &= \left(\tfrac{1+\gamma+r}{3}\right)^2 + r\tfrac{1+\gamma-2r}{3} \\ \pi_2^{\mathcal{L},\mathcal{R}} &= \left(\tfrac{1+\gamma-2r}{3}\right)^2 \end{aligned} \tag{13}$$

for the royalty rate case.

[8] Anton and Yao [2007] have been proved the lemma 1 and have claim that the lemma 2 is truth, in the appendix I offer the proof for the lemma 2 and an alternative proof for the lemma 1.

4 Competitor's Technology Stage

By assuming that the policy of the patent holder es know means a license offer against a license fee (royalty rate or fixed fee), the next step for to solve the game is to analyze the behavior of the competitor respect to the technology choice,where the alternatives are. not infringe the patent $\mathcal{N}$ (use the backstop technology). Infringe the patent $\mathcal{I}$ (use the new technology without a permission of the patent holder). And accept to pay for the use the new technology if a license is offered $\mathcal{L}$

Because

Lemma 3. *If the courts calculates damages using the* LP *rule or the* UE $\pi_2^{\mathcal{I}} \geq \pi_2^{\mathcal{N}}$.

the competitor always prefer to infringe instead to use the backstop technology independently of the liability rule, even more this result could be proved for more general specifications as different rates of efficiency in the use of the innovation (see AY&Ch) .

The game at this point is not completely solve, the values of F and r are unknown, then it is not possible to compare $\pi_2^{\mathcal{I}}$ against $\pi_2^{\mathcal{F}}$ or $\pi_2^{\mathcal{R}}$, but instead it is possible to know for which values of F and r, $\pi_2^{\mathcal{F}} \geq \pi_2^{\mathcal{I}}$ ($\pi_2^{\mathcal{R}} \geq \pi_2^{\mathcal{I}}$) holds.

Let $\underline{F}$ be a fixed fee $F > 0$ such that $\pi_2^{\mathcal{L},\mathcal{F}} - \pi_2^{\mathcal{I}} = 0$, then

$$\underline{F} = \left(\frac{1+\gamma}{3}\right)^2 - \pi_2^{\mathcal{I}} \tag{14}$$

, notice that if $\underline{F}$ is negative there is no positive fixed fee that makes the license option as good as infringe for the competitor, in the appendix is prove that

Lemma 4. $\underline{F}^{\mathcal{LP}} \geq 0$ *but* $\underline{F}^{\mathcal{UE}} \geq 0$ *just if* $\gamma \geq \delta_1$, *where*

$$\delta_1 = \frac{12 - 5\theta + \theta^2 - 2\sqrt{(3-\theta)^2(3+\theta)}}{6 - 7\theta + \theta^2}$$

.

In the case of the royalty rate, let $\underline{r}$ be a royalty rate r that makes $\pi_2^{\mathcal{L},\mathcal{R}} - \pi_2^{\mathcal{I}} = 0$, then if exists

$$\underline{r} = \frac{1 + \gamma - 3\sqrt{\pi_2^{\mathcal{I}}}}{2} \tag{15}$$

From eq. 2.12 and 2.13 $\partial(\pi_2^{\mathcal{L},\mathcal{F}} - \pi_2^{\mathcal{I}})/\partial F < 0$, $\partial(\pi_2^{\mathcal{L},\mathcal{R}} - \pi_2^{\mathcal{I}})/\partial r < 0$ and $\pi^{\mathcal{L},\mathcal{F}} = \pi^{\mathcal{L},\mathcal{R}}$ for $F = r = 0$. It is possible to create a one to one function between $\underline{r}$ and $\underline{F}$, then by using the lemma 4

Lemma 5. *In the LP case always exist a positive fixed fee $\underline{F}$ (or royalty rate $\underline{r}$) such that $\pi_2^{\mathcal{L},\mathcal{F}} \geq \pi_2^{\mathcal{I},\mathcal{LP}}$ (or $\pi_2^{\mathcal{L},\mathcal{R}} \geq \pi_2^{\mathcal{I},\mathcal{LP}}$).*

However in the UE case the last statement is true just for $\gamma > \delta_1$.

5 Licensing Stage

In a take it or leave it bargaining the patent holder will ask for the fixed fee that makes the competitor indifferent between take the license or to infringe.

From eq 2.12 it is observable that the patent holder will choose the greater F that makes the competitor as good as in infringement then , $F^\star = \underline{F}$.

In the case of the royalty rate from the eq 2.13 it is known that the profit of the patent holder reach maximum at $r = \frac{1+\gamma}{2}$ and $\pi_2^{\mathcal{L},\mathcal{R}} = 0$, then $\underline{r} \leq \frac{1+\gamma}{2}$, then the patent holder will ask $r^\star = \underline{r}$ as a royalty rate in exchange of a license, summarizing

Lemma 6. *The patent holder will ask for $F^\star = \underline{F}$ as a fixed fee and $r^\star = \underline{r}$ as a royalty rate.*

By using the definition of $\underline{r}$ (eq 2.15) in the payoff function $\pi_1^{\mathcal{L},\mathcal{R}}$ (eq 2.13), the patent holder's payoff is

$$\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\pi_2^{\mathcal{I}} \tag{16}$$

, from eq 2.14 it is known that $\pi_2^{\mathcal{I}} = \left(\frac{1+\gamma}{3}\right)^2 - F^\star$, then using this result in the last equation produces

$$\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left[\left(\frac{1+\gamma}{3}\right)^2 - F^\star\right] \tag{17}$$

by using eq 2.12

$$\pi_1^{\mathcal{L},\mathcal{R}} - \pi_1^{\mathcal{L},\mathcal{F}} = F^\star/4$$

, summarizing.

Proposition 1. *The patent holder will prefer to license using a royalty rate scheme instead or a fixed fee scheme.*

The final step for solve the game is to compare the situation of licensing against the situation of no licensing. When patent holder does not offer a license, the competitor begins to infringe the patent, so then the patent holder has to compare $\pi_1^{\mathcal{L}}$ against $\pi_1^{\mathcal{I}}$ in order to offer or not a license. Then by comparing this profits is observable that

Proposition 2. *The patent holder will never license under UE. However under LP a royalty rate's license is offered if* $\gamma > \delta_2$*, where*

$$\delta_2 = \frac{\theta(3-2\theta) + 3\sqrt{(3-\theta)^2(2-\theta)}}{18 - 15\theta + 4\theta^2}$$

.

6 LP vs UE

Now at this point is possible to compare the results obtained under UE and the results obtaining under the LP rule.

When the LP rule is used there are at least three situations: 1)Passive infringement $\gamma \leq \theta/(3-2\theta)$, 2)Active Infringement $\gamma > \theta/(3-2\theta)$ and 3) Licensing by a royalty rate $\gamma > \delta_2$, where $0 \leq \theta/(3-2\theta) \leq \delta_2 \leq 1$, then

Lemma 7. *When LP is used as a liability rule, there is licensing if* $\gamma > \delta_2$*, otherwise the patent holder does not offer a license and the competitor infringes the patent. In the case of UE there is no licensing and the competitor infringes the patent in equilibrium.*

the equilibrium quantities under both regimes are

$$\left(\pi_i^{\mathcal{LP}}, \pi_i^{\mathcal{UE}}\right) = \begin{cases} \left(\pi_i^{\mathcal{I},\mathcal{LP}}, \pi_i^{\mathcal{I},\mathcal{UE}}\right) & 0 \leq \gamma < \delta_2 \\ \left(\pi_i^{\mathcal{L},\mathcal{R},\mathcal{LP}}, \pi_i^{\mathcal{I},\mathcal{UE}}\right) & \gamma \geq \delta_2 \end{cases}$$

where $i = 1, 2$. In consequence

$$\pi_1^{\mathcal{LP}} = \begin{cases} \left(\frac{1+2\gamma}{3}\right)^2 & \text{if } 0 < \gamma \leq \delta_1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \delta_1 \leq \gamma < \delta_2 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2\right) & \text{if } \delta_2 < \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2\right) & \text{if } 1 \leq \gamma \end{cases} \tag{18}$$

$$\pi_2^{\mathcal{LP}} = \begin{cases} \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma \leq \delta_1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \delta_1 < \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases}$$

In the case of UE infringement is always present, then the payoffs under this situation are

$$\pi_1^{\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases}$$

$$\pi_2^{\mathcal{UE}} = \begin{cases} (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{19}$$

and by comparing payoffs under LP against UE, is established that

Proposition 3. *The patent holder, the competitor and the industry are better off under LP for drastic innovations. however under non drastic the situation depends on values of θ and γ.*

Because the demand is linear the consumer surplus $csp = (q_1 + q_2)^2/2 = Q^2/2$, when LP rule is used

$$Q^{\mathcal{LP}} = \begin{cases} \frac{2+\gamma}{3} & \text{if } 0 < \gamma < \delta_1 \\ (2-\theta)\frac{1+\gamma}{3-\theta} & \text{if } \delta_1 \leq \gamma < \delta_2 \\ \frac{2(1+\gamma)-r}{3} & \text{if } \gamma \geq \delta_2 \end{cases}$$

then

$$Q^{\mathcal{LP}} = \begin{cases} \frac{2+\gamma}{3} & \text{if } 0 < \gamma < \delta_1 \\ (2-\theta)\frac{1+\gamma}{3-\theta} & \text{if } \delta_1 < \gamma < \delta_2 \\ \frac{(1+\gamma)}{3} + \sqrt{\pi_2^I} & \text{if } \gamma \geq \delta_2 \end{cases}$$

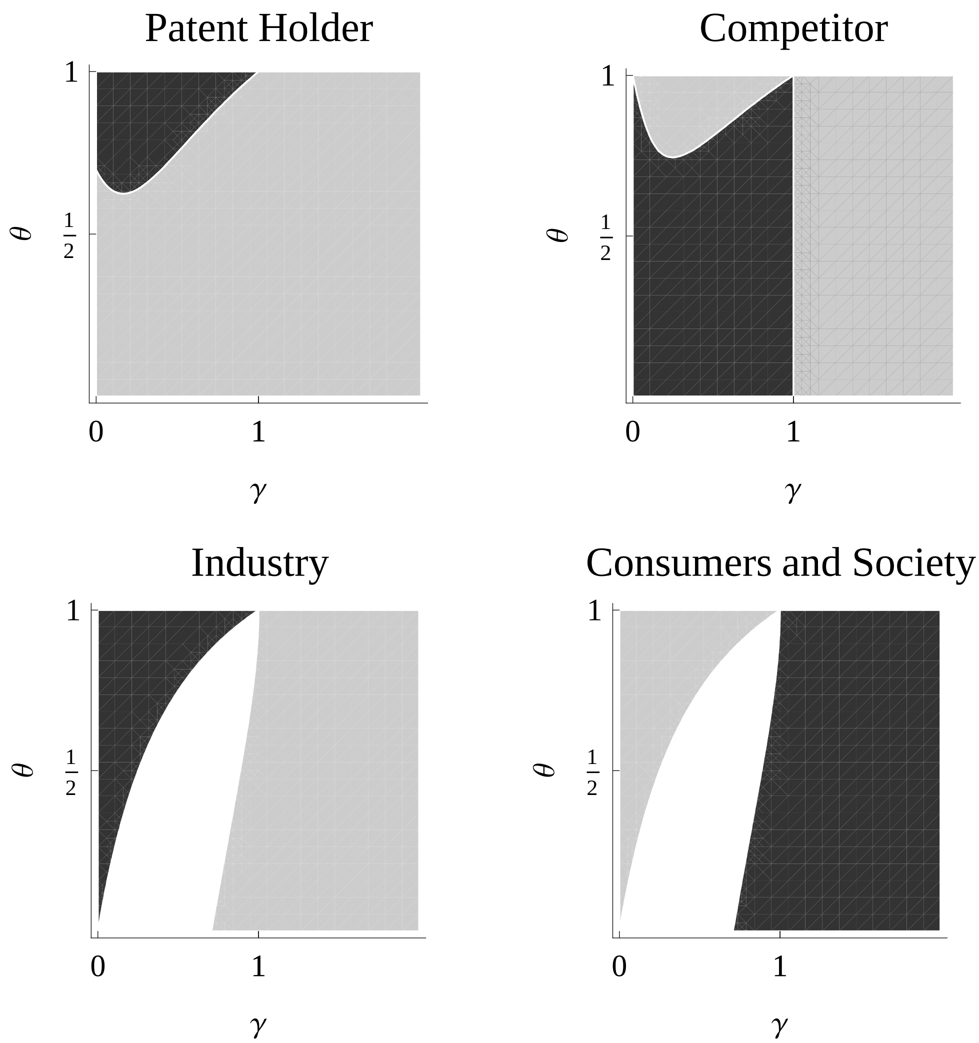

Figure 2: Equilibrium results under LP and UE

then

$$
Q^{\mathcal{LP}} = \begin{cases} \frac{2+\gamma}{3} & \text{if } 0 < \gamma < \delta_1 \\ (2-\theta)\frac{1+\gamma}{3-\theta} & \text{if } \delta_1 \leq \gamma < \delta_2 \\ \frac{(1+\gamma)}{3} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2} & \text{if } \delta_2 \leq \gamma < 1 \\ \frac{(1+\gamma)}{3} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2} & \text{if } \gamma \geq 1 \end{cases} \tag{20}
$$

For UE we get that

$$
Q^{\mathcal{UE}} = (2-\theta)\frac{1+\gamma}{3-\theta} \tag{21}
$$

7 Conclusions

Appendix

Proof Lemma 1. It is important to notice is that the best response function $\phi_1(q_2)$ is the same whether $D^{LP} > 0$ or $D^{LP} = 0$. The best response when $q_2 \in [0, a-c+\epsilon]$ is

$$
\phi_1(q_2) = \frac{1 - q_2 + \gamma}{2}
$$

The best response of the competitor deserves a special treatment. Let,

$$
\begin{aligned}
x(q_1, q_2) &= (1 - q_1 - q_2 + \gamma)q_2 - \theta \max\left\{\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1, 0\right\} \\
&= x_1(q_1, q_2) - \theta \max\{x_2(q_1, q_2), 0\}
\end{aligned}
$$

be the payoff of the competitor.

When $q_1 > 1 + \gamma$ the price becomes negative for any $q_2 \geq 0$, then in this case

$$
\phi_2(q_1) = 0 \quad \text{if } q_1 > 1 + \gamma
$$

.

If the innovation is drastic $\pi_1^{\mathcal{N}}$ is the monopoly profit in consequence $\pi_1^{\mathcal{N}} - (1 - q_1 - q_2 + \gamma)q_1 \geq 0$ for any $q_1, q_2 \geq 0$, then $D^{LP} > 0$ and, in consequence

$$
\phi_2(q_1) = \frac{1 + \gamma - (1+\theta)q_1}{2} \quad \text{if } \gamma/(1) \geq 1 \text{ and } q_1 \in [0, 1+\gamma)
$$

When the innovation is non drastic, for a given $q_1 \in [0, 1+\gamma)$, $x(q_1, q_2)$ reach maximum at $\tilde{q_2}$, where $0 < \tilde{q_2} < \hat{q_2} = (1+\gamma-q_1)/2$ and $\hat{q_2}$ is the maximum of $x_1(q_1, q_2)$. Then $\partial x_1(q_1, q_2)/\partial q_2 > 0$ for $q_2 \in [0, (1+\gamma-q_1)/2)$. And $\partial x_2(q_1, 0)/\partial q_2 = q_1$ for any q_2. Then the best response depends on the sign of $x_2(q_1, q_2)$, this sign could be positive, negative or zero. There are two values of q_1 that make $x_2(q_1, 0) = 0$,

$$\begin{aligned} q_1^{a,b} &= \frac{(1+\gamma) \pm \sqrt{(1+\gamma)^2 - 4\pi_1^{\mathcal{N}}}}{2} \\ &= \frac{(1+\gamma)}{2} \pm \sqrt{\left(\frac{1+\gamma}{2}\right)^2 - \left(\frac{1+2\gamma}{3}\right)^2} \end{aligned}$$

, where a refers to the inferior value and b to the superior one. For a given q_1 $x_2(q_1, 0)$ reach minimum at $q_1^c = \frac{(1+\gamma)}{2}$, this results plus the fact that $\gamma/(1) < 1$ allow to see that

$$0 < q_1^a < q_1^{\mathcal{N}} < q_1^c < q_1^b < 1+\gamma$$

.

In consequence $x_2(q_1, 0) > 0$ for $q_1 \in (0, q_1^a) \cup (q_1^b, 1+\gamma)$ and $x_2(0) \leq 0$ when $q_1 \in [q_1^a, q_1^b]$, then

$$\phi_2(q_1) = \frac{1+\gamma-(1+\theta)q_1}{2} \quad \text{if } q_1 \in (0, q_1^a) \cup (q_1^b, 1+\gamma) \text{ and } \gamma/(1) < 1$$

.

The next case appears when $q_1 \in [q_1^a, q_1^b]$, in consequence $x_2(q_1, 0) \leq 0$, then by looking for some q_2^a that makes $x_2(q_1, q_2^a) = 0$

$$\begin{aligned} q_2^a &= (1+\gamma) - q_1 - \pi_1^{\mathcal{N}}/q_1 \\ &\geq (1+\gamma) - q_1^b - \pi_1^{\mathcal{N}}/q_1^a = 0 \end{aligned}$$

, in consequence $0 \leq q_2^a < 1+\gamma$. Now by evaluating the derivative on the right of x at (q_1, q_2^a) (or directional derivative in the direction $(0, 1)$),
$\partial^+ x/\partial q_2(q_1, q_2^a) = 1+\gamma-2q_2^a - q_1 - \theta q_1$, then

$$\phi_2(q_1) = q_2^a \quad \text{if } q_1 \in [q_1^a, q_1^b] \wedge \gamma/(1) < 1 \wedge 1+\gamma-2q_2^a - q_1 - \theta q_1 \leq 0$$

or

$$\phi_2(q_1) = \frac{1+\gamma-(1+\theta)q_1}{2}$$
$$\text{if } q_1 \in [q_1^a, q_1^b] \wedge \gamma/(1) < 1 \wedge 1+\gamma-2q_2^a - q_1 - \theta q_1 > 0$$

When is assumed that $x_2 > 0$ the Nash equilibrium is
$\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$
, now if $x_2 \leq 0$, $q_2^a(\phi_1) = q_2^{\mathcal{N}}$, so then $\phi_1(q_2^{\mathcal{N}}) = q_1^{\mathcal{N}}$ and the condition $\gamma/(1) < 1 \wedge 1+\gamma-2q_2^a - q_1 - \theta q_1 > 0$ becomes in $\gamma/(1) < \theta/(3-2\theta)$, this condition implies that $\frac{1+\gamma}{3-\theta} \in [q_1^a, q_1^b]$, then if $\gamma/(1) < \theta/(3-2\theta)$ holds $x_2 < 0$ in equilibrium and the Nash equilibrium is $\left(\frac{1+2\gamma}{3}, \frac{1-\gamma}{3}\right)$.

When $\gamma/(1) < \theta/(3-2\theta)$ does not hold $x_2 > 0$, the Nash equilibrium is $\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$ □

Proof Lemma 2. There is a symmetry respect to last proof, this time $\phi_2(q_1)$ is the same whether $D^{UE} > 0$ or $D^{UE} = 0$, when $q_1 \in [0, 1+\gamma]$ is

$$\phi_2(q_1) = \frac{1-q_1+\gamma}{2}$$

and 0 if $q_1 > 1+\gamma$.

Let,

$$\begin{aligned} y(q_1,q_2) &= (1-q_1-q_2+\gamma)q_1 + \theta \max\left\{(1-q_1-q_2+\gamma)q_2 - \pi_2^{\mathcal{N}}, 0\right\} \\ &= y_1(q_1,q_2) - \theta \max\{y_2(q_1,q_2), 0\} \end{aligned}$$

be the payoff of the patent holder.

When $q_2 > 1+\gamma$ the price becomes negative for any $q_1 \geq 0$, then in this case

$$\phi_1(q_2) = 0 \quad \text{if } q_2 > 1+\gamma$$

.

If the innovation is drastic $\pi_2^{\mathcal{N}} = 0$, then

$$\phi_1(q_2) = \frac{1+\gamma-(1+\theta)q_2}{2} \quad \text{if } \gamma/(1) \geq 1 \text{ and } q_2 \in [0, 1+\gamma)$$

When the innovation is non drastic, for a given $q_2 \in [0, 1+\gamma)$, $y(q_1,q_2)$ reach maximum at $\tilde{q}_1$, where $0 < \tilde{q}_1 < \hat{q}_1 = (1+\gamma-q_2)/2$ and $\hat{q}_1$ is the maximum of $y_1(q_1,q_2)$. Then $\partial y_1(q_1,q_2)/\partial q_1 > 0$ for $q_1 \in [0, (1+\gamma-q_2)/2)$ and $\partial y_2(q_1,0)/\partial q_1 = -q_2$ for any q_1. Then the best response depends on the sign of $y_2(q_1,q_2)$, this sign could be positive, negative or zero. There are two values of q_2 that make $y_2(0,q_2)=0$,

$$\begin{aligned} q_2^{a,b} &= \frac{(1+\gamma) \pm \sqrt{(1+\gamma)^2 - 4\pi_2^{\mathcal{N}}}}{2} \\ &= \frac{(1+\gamma)}{2} \pm \sqrt{\left(\frac{1+\gamma}{2}\right)^2 - \left(\frac{1-\gamma}{3}\right)^2} \end{aligned}$$

where a refers to the inferior value and b to the superior one. $y_2(q_1,0)$ reach maximum at $q_2^c = \frac{(1+\gamma)}{2}$, in a consequence

$$0 < q_2^a < q_2^c < q_2^b < 1+\gamma$$

.

Also $y_2(0,q_2) < 0$ for $q_1 \in (0,q_2^a) \cup (q_2^b, 1+\gamma)$ and $y_2(0,q_2) \geq 0$ when $q_2 \in [q_2^a, q_2^b]$, then

$$\phi_1(q_2) = \frac{1+\gamma-q_2}{2} \quad \text{if } q_2 \in [q_1^a, q_1^b]$$

.

if $\phi_1(q_2) = \frac{1+\gamma}{3}$ is played the best response of the other player is $\frac{1+\gamma}{3} \in [q_2^a, q_2^b]$, in a consequence it is not a NE.

Because $q_2^b > \hat{q}_2$, the best response belong to the interval $[0, q_2^a)$ when $y(0,q_2) > 0$. There is

$$\begin{aligned} q_1^a &= (1+\gamma) - q_2 - \pi_2^{\mathcal{N}}/q_2 \\ &\geq (1+\gamma) - q_2^b - \pi_1^{\mathcal{N}}/q_2^a = 0 \end{aligned}$$

that makes $y_2(q_1^a, q_2) = 0$, where $0 \leq q_2^a < 1+\gamma$.

then the derivative on the left (or in direction (-1,0)),

$\partial^- y/\partial q_1(q_1^a, q_2) = -(1+\gamma-2q_1^a - q_2 - \theta q_2)$, then

$$\phi_2(q_1) = q_1^a \quad \text{if } q_2 \in [0,q_1^a) \wedge \gamma/(1) < 1 \wedge -(1+\gamma-2q_1^a-q_2-\theta q_2) \leq 0$$

$$\begin{aligned} \phi_2(q_1) = &\frac{1+\gamma-(1+\theta)q_1}{2} \\ &\text{if } q_2 \in [0,q_1^a) \wedge \gamma/(1) < 1 \wedge -(1+\gamma-2q_1^a-q_2-\theta q_2) > 0 \end{aligned}$$

If is assumed that $y_2 > 0$ the Nash equilibrium is
$\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$
, now if $y_2 \leq 0$ in equilibrium, $q_2^a(\phi_1) = (1+5\gamma)/3$, so then $\phi_2((1+5\gamma)/3) = q_2^{\mathcal{N}}$ and the condition $-(\gamma/(1) < 1 \wedge 1+\gamma-2q_2^a - q_1 - \theta q_1) \leq 0$ becomes in $-(1-2\gamma)/3 + (1+\theta)(1-\gamma)/3 \leq 0$, but the first term is always positive then the unique Nash equilibrium is $\left(\frac{1+\gamma}{3-\theta}, (1-\theta)\frac{1+\gamma}{3-\theta}\right)$ □

Proof Lemma 3. When $\gamma \leq \theta/(3-2\theta)$, $\pi_2^{\mathcal{I},\mathcal{LP}} = \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) > \left(\frac{1-\gamma}{3}\right)^2 = \pi_2^{\mathcal{N}}$. When $\theta/(3-2\theta) \leq \gamma < 1$,

$$G(\gamma,\theta) = \pi_2^{\mathcal{I},\mathcal{LP}} - \pi_2^{\mathcal{N}} = \left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2 - \left(\frac{1-\gamma}{3}\right)^2$$

, now notice that $G_{11} = \left(\frac{1}{3-\theta}\right)^2 - \frac{4\theta+1}{9}$, because at $\theta = 0$ $G_{11} = 0$ and because $dG_{11}/d\theta = 2(3-\theta)^{-3} - 4/9 < (2)^{-2} - 4/9 < 0$, $G_{11} < 0$ for $\theta \in (0,1)$, then G is concave in γ for $\theta \in (0,1)$.

$G(1,\theta) = \left(\frac{2}{3-\theta}\right)^2 - \theta$, moreover $G_2(1,\theta) = 8(3-\theta)^{-3} - 1 < 0$ for $\theta \in (0,1)$, $G(1,0) = \left(\frac{2}{3}\right)^2$ and $G(1,1) = 0$ then by continuity $G(1,\theta) > 0$ for $\theta \in (0,1)$.

$$G(\theta/(3-2\theta),\theta) = \left(\frac{1}{3-2\theta}\right)^2 - \theta\left(\frac{1}{3-2\theta}\right)^2 - \left(\frac{1-\theta}{3-2\theta}\right)^2 = \frac{\theta(1-\theta)}{(3-2\theta)^2} > 0$$

because G is concave in γ and $G(\theta/(3-2\theta),\theta), G(1,\theta) > 0$ $G > 0$ for $\gamma > \theta/(3-2\theta)$ and $\theta \in (0,1)$.

When $\gamma > 1$, $\pi_2^{\mathcal{I},\mathcal{LP}} \geq \pi_2^{\mathcal{N}} = 0$.

For the UE case, if $\gamma < 1$
$\pi_2^{\mathcal{I},\mathcal{UE}} = (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1-\gamma}{3}\right)^2 > \left(\frac{1-\gamma}{3}\right)^2 = \pi_2^{\mathcal{N}}$
and in the case $\gamma > 1$, $\pi_2^{\mathcal{I},\mathcal{UE}} \geq \pi_2^{\mathcal{N}} = 0$

□

Proof Lemma 4. By using eq 2.14 this definition

$$\underline{F}^{\mathcal{LP}} = \begin{cases} \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+2\gamma}{3}\right)\left(\frac{1-\gamma}{3}\right) & \text{if } 0 < \gamma \leq \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+2\gamma}{3}\right)^2 & \text{if } \frac{\theta}{3-2\theta} < \gamma < 1 \\ \left(\frac{1+\gamma}{3}\right)^2 - \left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta\left(\frac{1+\gamma}{2}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{22}$$

and after some algebra

$$\underline{F}^{\mathcal{LP}} = \begin{cases} \frac{1}{9}\gamma(1+3\gamma) & \text{if } 0 < \gamma \leq \frac{\theta}{3-2\theta} \\ \frac{\theta\left(3-5\theta+\theta^2+\gamma^2\left(30-23\theta+4\theta^2\right)+\gamma\left(24-22\theta+4\theta^2\right)\right)}{9(3-\theta)^2} & \text{if } \frac{\theta}{3-2\theta} < \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(57-50\theta+9\theta^2\right)}{36(3-\theta)} & \text{if } 1 \leq \gamma \end{cases}$$

it is straightforward to see that the first and third term are positive, in the case of the second term, this term is not always positive, but if $\frac{\theta}{3-2\theta} < \gamma < 1$, the term is also positive, then $\underline{F}^{\mathcal{LP}} \geq 0$.

Now in the case of UE,

$$\underline{F}^{\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{3}\right)^2 - (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1-\gamma}{3}\right)^2 & \text{if } 0 < \gamma < 1 \\ \left(\frac{1+\gamma}{3}\right)^2 - (1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 & \text{if } 1 \leq \gamma \end{cases} \tag{23}$$

and after some algebra

$$\underline{F}^{\mathcal{UE}} = \begin{cases} -\frac{\theta\left(6-7\theta+\theta^2+\gamma^2\left(6-7\theta+\theta^2\right)-2\gamma\left(12-5\theta+\theta^2\right)\right)}{9(-3+\theta)^2} & \text{if } 0 < \gamma < 1 \\ \frac{(1+\gamma)^2\theta(3+\theta)}{9(3-\theta)^2} & \text{if } 1 \leq \gamma \end{cases} \tag{24}$$

it is easy to see that the first term in this case is not always positive, however after find the roots of the polynomial it is easy to see that the expression is equal or greater than zero when $\gamma > \frac{12-5\theta+\theta^2-2\sqrt{27-9\theta-3\theta^2+\theta^3}}{6-7\theta+\theta^2}$. In the case of the second term is easy to see that is positive. Then $\underline{F}^{\mathcal{UE}} \geq 0$ if $\gamma > \frac{12-5\theta+\theta^2-2\sqrt{27-9\theta-3\theta^2+\theta^3}}{6-7\theta+\theta^2}$.
□

Proof Proposition 2. from 2.14 $\pi_1^{\mathcal{L},\mathcal{R}} = \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\pi_2^{\mathcal{I}}$

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} = \begin{cases} \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\frac{1+2\gamma}{3-\theta}\right)\left(\frac{1-\gamma}{3}\right) & 0 \leq \gamma < \frac{\theta}{3-2\theta} \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+2\gamma}{3}\right)^2\right) & \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4}\left(\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta\left(\frac{1+\gamma}{2}\right)^2\right) & \gamma \geq 1 \end{cases}$$

then after some algebra

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} - \pi_1^{\mathcal{I},\mathcal{LP}} = \begin{cases} -\frac{(1-\gamma)(5\theta+\gamma(9+7\theta))}{36(3-\theta)} & 0 \leq \gamma < \frac{\theta}{3-2\theta} \\ \frac{\theta\left(-9+3\theta+\theta^2+2\gamma\theta(-3+2\theta)+\gamma^2\left(18-15\theta+4\theta^2\right)\right)}{36(-3+\theta)^2} & \frac{\theta}{3-2\theta} \leq \gamma < 1 \\ \frac{(1+\gamma)^2(1-\theta)^2\theta}{16(3-\theta)^2} & \gamma \geq 1 \end{cases}$$

it is easy to see that the first term is negative, the third one is positive and the second one could be positive or negative this case is not always positive, however after find the roots of the polynomial this expression is equal or greater than zero when $\gamma >$. In the case of the second term is easy to see that is positive. Then $\pi_1^{\mathcal{L},\mathcal{R},\mathcal{LP}} - \pi_1^{\mathcal{I},\mathcal{LP}} \geq 0$ if $\gamma > \frac{\theta(3-2\theta)+3\sqrt{(3-\theta)^2(2-\theta)}}{18-15\theta+4\theta^2}$.

For the case of UE by preceding as in the LP case,

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{UE}} = \begin{cases} \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4} * \left((1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2 + \theta * \left(\frac{1-\gamma}{3}\right)^2\right) & 0 \leq \gamma < 1 \\ \left(\frac{1+\gamma}{2}\right)^2 - \frac{5}{4} * \left((1-\theta)\left(\frac{1+\gamma}{3-\theta}\right)^2\right) & \gamma \geq 1 \end{cases}$$

then after some algebra and using the definition of $\pi_1^{\mathcal{I},\mathcal{UE}}$

$$\pi_1^{\mathcal{L},\mathcal{R},\mathcal{UE}} - \pi_1^{\mathcal{I},\mathcal{UE}} = \begin{cases} -\frac{\theta\left(18-15\theta+\theta^2-2\gamma\theta(3+\theta)+\gamma^2\left(18-15\theta+\theta^2\right)\right)}{36(-3+\theta)^2} & 0 \leq \gamma < 1 \\ \frac{(1+\gamma)^2(-1+\theta)\theta}{4(-3+\theta)^2} & \gamma \geq 1 \end{cases}$$

The second term is clearly negative, in the case of the roots of the polynomial are imaginary then the term is positive or negative, because at $\theta = \gamma = 1/2$ the value is $-187/7200$, then $\pi_1^{\mathcal{L},\mathcal{R},\mathcal{UE}} - \pi_1^{\mathcal{I},\mathcal{UE}} < 0$.

$\square$

Proof Proposition 3. After some algebra,

$$\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} = \begin{cases} \frac{\theta\left(3-5\theta+\theta^2\right)-2\gamma\left(-9+21\theta-8\theta^2+\theta^3\right)+\gamma^2\left(27-15\theta-2\theta^2+\theta^3\right)}{9(-3+\theta)^2} & 0 \leq \gamma < \frac{\theta}{3-2*\theta} \\ \frac{\theta\left(9-12\theta+2\gamma(-6+\theta)\theta+2\theta^2+\gamma^2\left(36-30\theta+5\theta^2\right)\right)}{9(-3+\theta)^2} & \frac{\theta}{3-2*\theta} \leq \gamma < \delta_2 \\ \frac{\theta\left(2\gamma\theta(-9+2\theta)+3\left(3-5\theta+\theta^2\right)+\gamma^2\left(54-45\theta+8\theta^2\right)\right)}{12(-3+\theta)^2} & \delta_2 \leq \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(21-26\theta+5\theta^2\right)}{16(-3+\theta)^2} & \gamma \geq 1 \end{cases}$$

It easy to see that the third and fourth cases are positive, the second case has two roots under $\theta/(2-theta)$ and because at $\theta = \gamma = 1/2$ is positive the term is also positive in the region under study, the last case has both roots inside the study region, then after a straightforward analysis it is concluded that. $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} > 0$ if $\theta < \frac{-9+21\theta-8\theta^2+\theta^3 \pm 3\sqrt{9-51\theta+85\theta^2-50\theta^3+12\theta^4-\theta^5}}{27-15\theta-2\theta^2+\theta^3}$.

In the case of the competitor, after some algebra

$$\pi_2^{\mathcal{LP}} - \pi_2^{\mathcal{UE}} = \begin{cases} -\frac{\theta\left(6-7\theta+\theta^2\right)+\gamma\left(9-30\theta+11\theta^2-2\theta^3\right)+\gamma^2\left(27-12\theta-4\theta^2+\theta^3\right)}{9(-3+\theta)^2} & 0 \leq \gamma < \frac{\theta}{3-2*\theta} \\ -\frac{\theta\left(9-12\theta+2\gamma(-6+\theta)\theta+2\theta^2+\gamma^2\left(36-30\theta+5\theta^2\right)\right)}{9(-3+\theta)^2} & \frac{\theta}{3-2*\theta} \leq \gamma < 1 \\ -\frac{(1+\gamma)^2(-6+\theta)\theta^2}{9(-3+\theta)^2} & \gamma \geq 1 \end{cases}$$

in the last case is easy to see that the expression is positive, in the second case both roots are under $\theta/(3-2\theta)$ and the expression is negative at $\theta = \gamma = 1/2$, finally in the first case both roots are inside the region of interest then after some analysis is straightforward to see that $\pi_1^{\mathcal{LP}} - \pi_1^{\mathcal{UE}} > 0$ if $\theta > \frac{-9+30\theta-11\theta^2+2\theta^3-3\sqrt{9-132\theta+238\theta^2-116\theta^3+17\theta^4}}{2(27-12\theta-4\theta^2+\theta^3)}$.

In the case of the industry

$$\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} = \begin{cases} \frac{\theta(-3+2\theta)+\gamma^2\theta(-3+2\theta)+\gamma\left(9-12\theta+5\theta^2\right)}{9(-3+\theta)^2} & 0 \leq \gamma < \frac{\theta}{3-2*\theta} \\ 0 & \frac{\theta}{3-2*\theta} \leq \gamma < \delta_2 \\ \frac{\theta\left(-9+3\theta+\theta^2+2\gamma\theta(-3+2\theta)+\gamma^2\left(18-15\theta+4\theta^2\right)\right)}{36(-3+\theta)^2} & \delta_2 \leq \gamma < 1 \\ \frac{(1+\gamma)^2\theta\left(189-138\theta+29\theta^2\right)}{144(-3+\theta)^2} & \gamma \geq 1 \end{cases}$$

In the first case both roots are outside the region of study and at $\theta = 1/2, \gamma = 1/10$ the expression is negative, then $\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} < 0$ if $0 \leq \gamma < \frac{\theta}{3-2*\theta}$, the third case follows by notice that one of the roots is δ_2 (the other one is negative) and at $\theta = 1/10, \gamma = 9/10$ the expression is positive, then $\sum \pi_i^{\mathcal{LP}} - \sum \pi_i^{\mathcal{UE}} > 0$ if $\frac{\theta}{3-2*\theta} \leq \gamma < \delta_2$, and the last case follows directly. $\square$

Proof Proposition 4. after some algebra

$$Q^{\mathcal{LP}} - Q^{\mathcal{UE}} = \begin{cases} \frac{\theta-\gamma(3-2\theta)}{3(3-\theta)} & 0 \leq \gamma < \frac{\theta}{3-2*\theta} \\ 0 & \frac{\theta}{3-2*\theta} \leq \gamma < \delta_2 \\ -\frac{(1+\gamma)(-3+2\theta)}{3(-3+\theta)} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta * \left(\frac{1+2*\gamma}{3}\right)^2} & \delta_2 \leq \gamma < 1 \\ -\frac{(1+\gamma)(-3+2\theta)}{3(-3+\theta)} + \sqrt{\left(\frac{1+\gamma}{3-\theta}\right)^2 - \theta * \left(\frac{1+\gamma}{2}\right)^2} & \gamma \geq 1 \end{cases}$$

in the case when $0 \leq \gamma < \frac{\theta}{3-2*\theta}$, it is observable that $\partial(Q^{\mathcal{LP}} - Q^{\mathcal{UE}})/\partial\gamma < 0$ and at $\gamma = \frac{\theta}{3-2*\theta}$ $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} = 0$, then $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} > 0$ for $0 \leq \gamma < \frac{\theta}{3-2*\theta}$. In the third case, because both roots of the polynomial are below δ_2 then the term is positive or negative, because at $\theta = 1/10, \gamma = 9/10$ the expression is negative, by noticing that if the third term is negative this implies that the fourth it is also negative, then $Q^{\mathcal{LP}} - Q^{\mathcal{UE}} > 0$ for $\gamma \geq \delta_2$. $\square$

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