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Abstract

This paper explores a two-sector model of endogenous growth with AK technologies and production externalities. Unlike the standard one-sector AK model, the two-sector model may involve the relative price dynamics, so that there may exist transitional processes. It is shown that under certain conditions for the relative magnitudes of factor intensities, the balanced growth equilibrium exhibits local indeterminacy.

1 Introduction

The AK model presents the simplest framework in endogenous growth theory. Although it is extremely simple, the AK formulation captures the key factors that determine the long-term growth rate. From the empirical point of view, the AK model, however, has a very unattractive feature: the model does not involve transition dynamics, unless we introduce other complexities such as habit formation, adjustment costs of investment, etc. The standard AK economy always stays in the balanced-growth equilibrium and it instantaneously jumps to a new balanced-growth path when a shock hits the

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fundamentals of the economy. Therefore, the literature has treated the AK model as a convenient shortcut for analyzing the long-run equilibrium rather than as a useful framework to discuss growth dynamics.[1]

The purpose of this paper is to examine a generalized AK model in which consumption good and investment good are produced by using different technologies. Rebelo (1991) also examined a two-sector version of the AK model, but his specification assumed that both consumption and investment good sectors employ capital alone.[2] Due to this specification, the dynamic behavior of Rebelo's two-sector model is the same as that of the one-sector AK model. We generalize the Rebelo model by assuming that both sectors employ labor as well as capital. In the presence of externality generated by the aggregate capital, each sector has an AK technology. We show that, given our generalization, the dynamic behavior of the model depends on the relative magnitudes of factor intensities used in both production sectors. If the consumption good sector uses more capital intensive technology than the investment good sector, then we obtain the usual result, i.e. the economy has no transition process. However, under the more plausible assumption in which the investment good sector uses more capital intensive technology than the consumption good sector, there may exists a continuum of equilibria converging to the balanced-growth equilibrium. Therefore, in this case the economy may involve transition dynamics if the initial position of the economy is set out of the balanced-growth equilibrium.

2 A Two-Sector *AK* Model

Suppose that the production side of the economy consists of two sectors, investment good sector (sector 1) and the consumption good sector (sector 2). Production function of each sector is

$$Y_i = F_i\left(K_i, A_i\left(K\right)L_i\right), \quad i = 1, 2,$$

where Y_i is output, K_i is capital stock, L_i is labor input and K denotes the aggregate stock of capital. $A_i\left(K\right)$ represents positive externalities generated by capital stock of the economy at large and it is assumed to be an increasing

[1] See Barro and Sala-i-Martin (1995, Chapter 4) for a detailed exposition of the one-sector AK model.

[2] Rebelo (1991) presents a variety of convex models of endogenous growth. The first model discussed in his paper is a two-sector AK model.

function of K. Implication of this formulation is the same as that of Romer (1986). It is assumed that function $F_i(.)$ is linearly homogenous, increasing and strictly quasi concave with respect to capital K_i and effective labor $A_i(K)L_i$.[3] We also assume that labor and capital are perfectly mobile across sectors and that capital does not depreciate. Due to the assumptions made above, the production function is expressed as

$$Y_i = A_i(K)L_i f_i(x_i), \quad i = 1, 2, \tag{1}$$

where $x_i \equiv K_i/A_i(K)L_i$, $f_i(x_i) \equiv F_i(K_i/A_i(K)L_i, 1)$, $f_i'(x_i) > 0$ and $f_i''(x_i) < 0$. Furthermore, $f_i(x_i)$ satisfies the Inada conditions: $\lim_{x_i \to 0} f_i'(x_i) = \infty$ and $\lim_{x_i \to \infty} f_i'(x_i) = 0$.

The commodity markets are competitive, so that profit maximization of the firms equates the marginal product of each factor input to its real price. The profit maximizaion conditions are thus given by

$$\begin{aligned} r = f_1'(x_1), \qquad & w = A_1(K)\left[f_1(x_1) - x_1 f_1'(x_1)\right], \\ r/p = f_2'(x_2), \qquad & w/p = A_2(K)\left[f_2(x_2) - x_2 f_2'(x_2)\right], \end{aligned} \tag{2}$$

where r, w and p respectively denote real rent, real wage and the price of consumption good. We take the investment good as the numèraire. Using (2), we obtain:

$$\frac{w}{A_i(K)r} + x_i = \frac{f_i(x_i)}{f_i'(x_i)}, \quad i = 1, 2. \tag{3}$$

In order to keep the AK structure of the model, we assume that the externality effects are expressed by linear functions of the aggregate capital:

$$A_i(K) = a_i K, \quad a_i > 0, \quad i = 1, 2. \tag{4}$$

Given this specificating, (3) gives the following relation:

$$x_i = x_i(\omega), \quad x_i'(\omega) = -a_i f_i'^2 / f_i'' f_i > 0, \quad i = 1, 2, \tag{5}$$

[3] The specification of technology used here differs from the two-sector models with physical and human capital studied by, for example, Mino (1996). While models with human capital may sustain endogenous growth even when the technology satisfies convexity, the possibility of continuing growth in our model comes from the assumption of non-convex technology. In the existing literature, the two-sector model with externalities examined by Benhabib and Farmer (1996) is most closely related to our formulation. Their modeling is more general than ours, because they consider both aggregate and sector specific externalities generated by labor as well as by capital. On the other hand, they does not consider the possibility of endogenous growth and focus on the dynamics under the condition that factor intensities are identical across the two sectors.

where $\omega = w/rK$. As a result, the relative price of consumption good is expressed as

$$p = \frac{f_1'(x_1(\omega))}{f_2'(x_2(\omega))} \equiv p(\omega). \tag{6}$$

It is easy to show that the relation between ω and p satisfies the following:

$$\text{sign } p'(\omega) = \text{sign } \left[\frac{a_2}{x_2(\omega)+\omega} - \frac{a_1}{x_1(\omega)+\omega}\right]. \tag{7}$$

Thus if the externality effects are symmetric (i.e. $a_1 = a_2$), the sign of $p'(\omega)$ is determined by the magnitudes of private factor intensities K_1/L_1 and K_2/L_2.

As for consumers' side of the economy, we use the standard representative family model with fixed labor supply. Each household provides one unit of labor in each moment and maximizes a discounted sum of utilities

$$U = \int_0^\infty \frac{C^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt, \quad \sigma > 0,\ \sigma \neq 1, \quad \rho > 0,$$

subject to the intertemporal budget constraint:

$$K_0 + \int_0^\infty w_s \exp\left(-\int_0^s r_s d\xi\right) ds = \int_0^\infty p_s C_s \exp\left(-\int_0^s r_s d\xi\right) ds,$$

where K_0 is the given initial holding of capital. Letting q be the implicit price of capital, the necessary conditions for optimization are given by:

$$C^{-\sigma} = pq, \tag{8}$$

$$\dot{q} = q(\rho - r). \tag{9}$$

It is assumed that the number of households is normalized to unity, so that C also denotes the aggregate consumption and the total labor supply is one.

Finally, the market clearing conditions for goods and factor inputs are given by the following:

$$\dot{K} = Y_1, \quad C = Y_2, \tag{10}$$

$$K_1 + K_2 = K, \quad L_1 + L_2 = 1. \tag{11}$$

3 The Dynamic System

Observe that the full employment conditions in (11) and the definition of x_i yield

$$a_1 L_1 x_1(\omega) + a_2(1 - L_1) x_2(\omega) = 1.$$

Thus the labor devoted to investment good production is written as

$$L_1 = \frac{1 - a_2 x_2(\omega)}{a_1 x_1(\omega) - a_2 x_2(\omega)} \equiv L_1(\omega),$$

and the production function of both sectors are expressed as

$$Y_i = y_i(\omega) K, \quad i = 1, 2, \tag{12}$$

where

$$\begin{aligned} y_1(\omega) &\equiv a_1 L_1(\omega) f_1(x_1(\omega)), \\ y_2(\omega) &\equiv a_2 [1 - L_1(\omega)] f_2(x_2(\omega)). \end{aligned}$$

Keeping our assumptions in mind, it can be shown that

$$\begin{aligned} \text{sign } y_1'(\omega) &= \text{sign } [a_2 x_2(\omega) - a_1 x_1(\omega)], \\ \text{sign } y_2'(\omega) &= \text{sign } [a_1 x_1(\omega) - a_2 x_2(\omega)]. \end{aligned} \tag{13}$$

Remember that $a_i x_i = K_i/KL_i$, which represents the ratio of capital allocation rate, K_i/K, and labor allocation rate, L_i, to sector i. Again, if $a_1 = a_2$, then the sign of $y_i'(\omega)$ is determined by the relative magnitude of private input ratio, K_i/L_i.

The commodity market equilibrium conditions (10) present:

$$\dot{K}/K = y_1(\omega), \tag{14}$$

$$C/K = y_2(\omega). \tag{15}$$

On the other hand, the optimization conditions for the household's consumption plan (8) and (9) give

$$-\sigma \frac{\dot{C}}{C} = \rho - f_1'(x_1(\omega)) + \frac{\dot{p}}{p}. \tag{16}$$

Equations (6) and (15) respectively yield $\dot{p}/p = [p'(\omega)/p(\omega)]\dot{\omega}$ and $\dot{C}/C - \dot{K}/K = [y_2'(\omega)/y_2(\omega)]\dot{\omega}$. Hence, by use of (14) and (16), we obtain the following dynamic equation that summarizes the entire model:

$$\dot{\omega} = \Omega(\omega)\left\{\frac{1}{\sigma}[f_1'(x_1(\omega)) - \rho] - y_1(\omega)\right\}, \tag{17}$$

where

$$\Omega(\omega) \equiv \frac{\sigma y_2(\omega) p(\omega)}{\sigma y_2'(\omega) p(\omega) + y_2(\omega) p'(\omega)}.$$

In the balanced-growth equilibrium ω stays constant. Thus the steady-state value of ω^*, if it exists, should satisfy

$$f_1'(x_1(\omega^*)) = \sigma y_1(\omega^*) + \rho. \tag{18}$$

When $\omega = \omega^*$, from (14) and (15) $\dot{K}/K$ and C/K also stay constant. Consequently, letting g be the balanced-growth rate, the long-run equilibrium of our model is characterized by $\dot{C}/C = \dot{K}/K = \dot{Y}_i/Y_i = \dot{w}/w = g$. Furthermore, ω, r, p and x_i do not change in the long-run equilibrium.

4 Discussion

As well as the standard two-sector neoclassical growth models (e.g. Uzawa 1964), the dynamic behavior of our model depends on the relative magnitude of factor intensity used in both production sectors. For analytical simplicity, the following discussion assumes that the externality effects in both sectors are symmetric ($a_1 = a_2$). Given this assumption, (7) and (13) respectively become

$$\begin{aligned} \text{sign } p'(\omega) &= \text{sign } [x_1(\omega) - x_2(\omega)], \\ \text{sign } y_1'(\omega) &= \text{sign } [x_2(\omega) - x_1(\omega)], \\ \text{sign } y_2'(\omega) &= \text{sign } [x_1(\omega) - x_2(\omega)]. \end{aligned} \tag{19}$$

We also assume that no factor reversal condition globally holds so that $x_1(\omega) > x_2(\omega)$ or $x_2(\omega) < x_1(\omega)$ for all feasible values of ω. Note that by definition $0 \leq L_1(\omega) \leq 1$. Since L_1 is a monotonic function of ω under the no factor intensity reversal condition, there exist the minimum and the maximum levels of ω. Thus from (19) the relative price defined should also satisfy $p \in [p(\omega_{\min}), p(\omega_{\max})]$ for $x_1 > x_2$ and $p \in [p(\omega_{\max}), p(\omega_{\min})]$ for $x_1 < x_2$.

Case (i) $x_1(\omega) < x_2(\omega)$: First, assume that the consumption good sector employs more capital intensive technology than the investment good sector. By (19), it is seen that under this condition $p'(\omega) < 0$, $y_1'(\omega) > 0$ and $y_2'(\omega) < 0$, so that $\Omega(\omega) < 0$. Furthermore, since in this case the left hand side of (18) decreases with ω, while its right hand side increases with ω, there exists a unique ω^*. Hence, we find that (17) yields:

$$\left.\frac{d\dot{\omega}}{d\omega}\right|_{\omega=\omega^*} = \Omega\left(\omega^*\right)\left[\left(1/\sigma\right) f_1'' x_1'\left(\omega^*\right) - y_1'\left(\omega^*\right)\right] > 0.$$

Because the initial value of ω $(= W/KR)$ is not predetermined under perfect foresight, the above means that a competitive equilibrium is uniquely determined around the steady state value of ω^*. Furthermore, since ω^* is unique, the above inequality is satisfied for all $\omega \in [\omega_{\min}, \omega_{\max}]$. Thus the globally determinacy holds and the economy should always stay in the balanced growth equilibrium. Accordingly, we obtain the same conclusion as in the standard AK model.

Case (ii) $x_1(\omega) > x_2(\omega)$: In contrast, if we assume a more plausible condition under which the investment good sector uses a more capital intensive technology than the consumption good sector, we may have various possibilities. In this case, (19) means that $p'(\omega) > 0$. $y_1'(\omega) < 0$, $y_2'(\omega) > 0$, and $\Omega(\omega) > 0$. As a result, if $f_1'' x_1'(\omega^*) < \sigma y_1'(\omega^*)$, we obtain

$$\left.\frac{d\dot{\omega}}{d\omega}\right|_{\omega=\omega^*} = \Omega\left(\omega^*\right)\left[\left(1/\sigma\right) f_1'' x_1'\left(\omega^*\right) - y_1'\left(\omega^*\right)\right] < 0,$$

which means that there locally exists a continuum of conversing equilibria around the balanced-growth equilibrium. Since in case (ii) both right and left hand sides of (18) are decreasing functions, there may exist multiple steady states. Figures 1 depicts two examples of the global dynamics of the model. Figure 1-a shows the case of a unique steady state, while the Figure 1-b for the case of multiple steady states. As figures suggest, for both cases, the initial value of ω can be set anywhere on its feasible domain. For example, suppose the economy initially stays at point A in Figure 1-a. Then ω starts to increase towards its steady state level ω^*. Equations (13), (14) and (15) show that during the transition $\dot{K}/K$ monotonically decreases, while C/K monotonically increases, and hence $\dot{K}/K < \dot{C}/C < 0$ for all $t \geq 0$.[4]

The familiar policy effects should also be reexamined for case (ii). As an example, consider capital income taxation. Letting τ be the rate of income

[4] The case (ii) discussed above is probably one of the simplest examples of endogenous growth model that may exhibit indeterminacy of equilibria. A through survey on the indeterminacy issues in growth models is presented by Benhabib and Gali (1995).

tax, the balanced growth condition is modified as

$$(1-\tau) f_1'(x_1(\omega^*)) = \sigma y_1(\omega^*) + \rho.$$

Using this and (14), the effect of a change in the rate of tax on the long-term growth rate is given by

$$\frac{dg}{d\tau} = \frac{y_1'(\omega^*) f_1'(x_1(\omega^*))}{(1-\tau) f_1'' x_1'(\omega^*) - \sigma y_1'(\omega^*)}.$$

Thus for the case of $x_1(\omega) < x_2(\omega)$, we obtain the usual result: $dg/d\tau < 0$. However, in the case of $x_1(\omega) > x_2(\omega)$, it holds that $dg/d\tau > 0$ in the balanced-growth equilibrium with continuum converging paths where $(1-\tau) f_1''(x_1(\omega^*)) > \sigma y_1(\omega^*)$. Namely, if the investment good sector's technology is more capital intensive that of the consumption good sector, a higher capital income tax could accelerate long-run growth.

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