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## Abstract

We present a quasi-analytical method for pricing multi-dimensional American options based on interpolating two arbitrage bounds, along the lines of Johnson (1983). Our method allows for the close examination of the interpolation parameter on a rigorous theoretical footing instead of empirical regression. The method can be adapted to general diffusion processes as long as quick and accurate pricing methods exist for the corresponding European and perpetual American options. The American option price is shown to be approximately equal to an interpolation of two European option prices with the interpolation weight proportional to a perpetual American option. In the Black-Scholes model, our method achieves the same efficiency as Barone-Adesi and Whaley's (1987) quadratic approximation with our method being generally more accurate for out-of-the-money and long-maturity options. When applied to Heston's stochastic volatility model, our method is shown to be extremely efficient and fairly accurate.

**JEL Classification:** C02; C63; G13

**Keywords:** American option; Interpolation method; Quasi-analytical approximation; Critical boundary; Heston's Stochastic volatility model

## I. Introduction

The pricing of American options is one of the oldest problems in finance and its study has attracted experts from various other fields, including mathematics, statistics, economics and operational research. Earliest attempts include McKean (1965), Samuelson (1967), and Merton (1973). By now, hundreds of papers have been devoted to the study of various aspects of American options. They can be roughly divided into two groups, one focusing more on the theoretical issues and the other on the computational issues.

Papers focusing on theoretical aspects include: the property of the price function such as differentiability, convexity, and put-call symmetry (Bergman, Grundy and Wiener (1996), McDonald and Schroder (1998), Shroder (1999)); the property of the critical stock price boundary (van Moerbeke (1976), Barles, Burdeau, Romano and Samsoen (1995), Peskir (2005), Zhu (2006)); different formulations such as partial differential equation with free boundary (McKean (1965)), optimal stopping problem and its dual formulation (Bensoussan (1984), Karatzas (1988), Rogers (2002)), variational inequality (Jaillet, Lamberton and Lapeyre (1990)), and early exercise premium representation (Kim (1990), Jacka (1991), Carr, Jarrow and Myneni (1992));<sup>1</sup> convergence and other properties of numerical methods (Lamberton (1993, 1998), Amin and Khanna (1994), Clément, Lamberton, and Protter (2002), Stentoft (2004)); and finally various generalizations such as general diffusion processes (Detemple and Tian (2002)), multiple underlying assets (Detemple, Feng and Tian (2003)), and more exotic American-type derivatives (see Detemple (2006)).

This paper belongs to the second group that looks at the issue of computing American option prices. By now, more than a dozen well-established pricing methods exist, as well as many others that have not been widely adopted. For a relatively recent but brief survey, see Barone-Adesi (2005). These methods differ in dimensions such as efficiency, accuracy, generality, robustness, portability to other derivatives or other processes, ease of implementation, etc. In addition to offering a broader menu of pricing methods for practitioners to choose from, the development of new methods often offers considerable additional insights on various aspects of American options. Because of the huge variety of proposed methods, a clear and unambiguous classification of all the different methods is difficult. Nonetheless, they can be roughly divided into several somewhat distinctive groups. The first group is the numerical methods based on solving the valuation partial differential equation. Those include the binomial tree method (Cox, Ross and Rubinstein (1979), Rendleman and Barrter (1979), Curran (1995)), trinomial tree methods and other variations (Parkinson (1977), Breen (1991), Kamrad and Ritchken (1991),

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<sup>1</sup>For a survey of different mathematical formulations of the American option pricing problem, see Myneni (1992).

Figlewski and Gao (1999)), finite difference methods (Brennan and Schwartz (1977, 1978), Hull and White (1990)), method of lines (Carr and Faguet (1996)), and penalty method (Khaliqa, Vossb, and Kazmic (2005)). The second group of methods focuses more on the exercise boundary and usually makes use of the early exercise premium representation. Papers include: Geske and Johnson (1984), Omberg (1987), Bunch and Johnson (1992), Huang, Subramanyam and Yu (1996), Carr (1998), Ju (1998), Bunch and Johnson (2000), Sullivan (2000), and Kim and Jang (2008). The third group of methods looks at simulation techniques to value American options, an idea first proposed for European options by Boyle (1977). Examples include Broadie and Glasserman (1997), Longstaff and Schwartz (2001), Moreno and Navas (2003), and Stentoft (2004)). Numerical techniques such as Richardson extrapolation and variance reduction are often employed in these three groups of methods. Yet a fourth group of methods uses regression techniques to fit analytical approximations, often based on lower and upper bounds of the American option. These include Johnson (1983), Kim (1994), and Broadie and Detemple (1996). The final group looks at analytical or quasi-analytical approximations of the American option price, usually by making various approximations to the valuation partial differential equations. These include MacMillan (1986), Barone-Adesi and Whaley (1987), and Ju and Zhong (1999).

In this paper, we develop a new quasi-analytical method which approximates the American put option price as an interpolation of two bounds. Thus, this paper is very similar in spirit to Johnson (1983) and Broadie and Detemple (1996). In fact, the bounds we use are identical to the ones used in Johnson (1983), except that we formulate our approximation under the framework of general diffusion processes while Johnson (1983) assumes Black-Scholes model with zero dividend rate. Another important difference is that instead of using a regression procedure as in these two papers, we work out the interpolation parameter by looking at the partial differential equation the parameter has to satisfy, an idea fruitfully employed in MacMillan (1986), Barone-Adesi and Whaley (1987), and Ju and Zhong (1999). Our final solution expresses the American put option price as a linear interpolation of two European put option prices, with the interpolation weight proportional to the price of the perpetual American option. Thus, as long as efficient methods for pricing the European put option and perpetual American option exist, our method can be adapted to general diffusion processes. We first specialize our analysis to the important case of geometric Brownian motion. In this case, closed-form expressions exist for the European call and perpetual American options. We show that our interpolation method approaches the correct limits for many limiting cases, most interestingly, as the maturity increases to infinity. The method is then further adapted to one of the leading multi-dimensional models for pricing derivatives, namely, Heston's stochastic volatility model. Our method has significant value since the literature is still scarce in terms of pricing American options under this multi-dimensional model.

In our numerical study, we examine the performance of our method using both short-term, medium-term and long-term options under the geometric Brownian motion model. We select a broad set of parameter values for the interest rate, dividend rate, volatility, and current stock prices. Our method gives fairly accurate approximations for the critical stock price, the interpolation parameter, and the option value. In particular, our approximation allows us to have a quick look at the dependence of the interpolation parameter on various parameters, fulfilling a task not carried out completely in Johnson (1983). In general, the interpolation parameter increases in the interest rate, the volatility, and the time to maturity, but decreases in the dividend rate. By comparing our method with one of the most efficient methods, the quadratic approximation, we see that our method achieves roughly the same efficiency as the quadratic approximation and roughly the same order of accuracy for short-term options. However, in general our method performs better for out-of-the-money options and long-term options. Finally, numerical study shows that our method is also extremely efficient and accurate for Heston's stochastic volatility model. The mean percentage error is often well below 1% while the computational time is usually 0.02 seconds for each option. The properties of the critical stock price, the interpolation parameter, the American put option price, as well as the early exercise premium as functions of parameters in the stochastic volatility model are studied in detail. These properties are affected by the sign of the correlation between the stock price shock and instantaneous variance shock. The early exercise premium behaves similarly to the critical stock price for in-the-money put options but similarly to the American put option price for out-of-the-money put options. In particular, the early exercise premium is decreasing in the instantaneous volatility for in-the-money put options, but increasing for out-of-the-money options.

Although our new method does not necessarily dominate all the existing well-established methods in terms of both efficiency and accuracy, our method has several attractive features. First, our method has a good combination of efficiency and accuracy. For the Black-Scholes case, while achieving relatively similar efficiency and accuracy to the quadratic approximation, our method is generally more accurate than the quadratic approximation for out-of-the-money options and long-term options. For Heston's model, our method provides a unique new method which is both efficient and accurate. While slightly less accurate than Medvedev and Scaillet (2009), our method seems to be more than five times faster. Second, our method, which expresses the American put prices as an interpolation of two simple European put option prices, is conceptually and aesthetically appealing. While it is hard to quantify the approximation error in both the quadratic approximation and the interpolation method, our method cannot be too wrong because the final prices will always be within two arbitrage bounds. This adds considerable robustness to our method, especially when one encounters extreme parameter values for

which the performance of some other methods has not been fully tested.<sup>2</sup> The quasi-analytical nature of our solution also allows us to carry out a detailed and systematic study on the interpolation parameter. Third, our method is applicable for general diffusion processes, as long as closed-form solutions (or even just a quick algorithm) to pricing the corresponding European put option and perpetual American options exist. In particular, our method applies to the geometric Brownian motion model with nonzero dividend rate.<sup>3</sup> Thus, the approximation can be obviously applied to other American options such as options on futures, currency, etc. A unique and very attractive feature of our method is that it can be applied to multi-dimensional models such as stochastic volatility. Fourth, from a programmer's perspective, our method is very easy to implement. Indeed, the core algorithm for our interpolation method is less than 20 lines of MATLAB code. Finally, unlike many less efficient methods, our solution is in closed form. The current critical stock price and the value of the American put option come out simultaneously. This allows us to compute greeks such as Delta and Gamma without much additional cost.

The paper is organized as follows. In the next section, we derive the interpolation method from the valuation partial differential equation, first under general diffusion processes and then specialized to the geometric Brownian motion and stochastic volatility cases. Section III performs a numerical study of the interpolation method in the Black-Scholes and Heston's stochastic volatility models. It is shown that the interpolation method is both efficient and accurate. Section IV concludes.

## II. Derivation of the Interpolation Method

### A. Derivation for put options under general diffusion processes

We assume that an American put option with strike price  $K$  and maturity time  $T$  is written on some underlying continuous process  $S(t)$ , where  $S(t)$  can be the price of some traded underlying asset or quantities such as crude oil futures price, exchange rate, or even temperature. Assume that the process  $S(t)$  can be modeled as a diffusion process under the risk-neutral measure  $\mathbb{Q}$ :

$$dS(t) = \mu(S(t), Z(t), \theta)dt + \sigma(S(t), Z(t), \theta)dW(t), \quad (1)$$

where  $\theta$  is a parameter vector and  $W(t)$  is a standard Brownian motion under the measure  $\mathbb{Q}$ . We will always refer to  $S(t)$  as the stock price process. Here  $Z(t)$  is a vector of state variables

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<sup>2</sup>For example, when pricing moderately long-term American options, we have encountered situations in which equations (22) and (23) in Bunch and Johnson (2000) have no solution so the method breaks down. Zhu and He (2007) give an explanation for the failure of the Bunch and Johnson method to provide a good approximation in some cases. They also provide a solution for that problem.

<sup>3</sup>It is worth pointing out that some old methods need zero dividend rate for them to work, or need to be reworked out for the nonzero dividend case, examples including Johnson (1983), Geske and Johnson (1984) and Omberg (1987). Even more recent papers, for example, Bunch and Johnson (2000) and Zhu (2006), work with a zero dividend rate.

which follows an  $N$ -dimensional diffusion process:

$$dZ_i = \mu_i(S(t), Z(t), \theta)dt + \sigma_i(S(t), Z(t), \theta)dW_i(t), \quad (i = 1, \dots, N) \quad (2)$$

where the Brownian motions are correlated with  $\mathbb{E}[dW_i dW_j] = \rho_{ij} dt$  and  $\mathbb{E}[dW_i dW_j] = \rho_{0j} dt$  for  $i, j = 1, \dots, N$ . Notice that this specification incorporates two important cases: the Black-Scholes model and Heston's stochastic volatility model. In the Black-Scholes model,  $N = 0$  and there is no extra state variable, while in the stochastic volatility model,  $N = 1$  and  $Z(t) = v(t)$  is the instantaneous variance.

Let the current time be 0. Let  $\tau = T - t$  denote the time to maturity. We assume that the risk-free interest rate  $r$  is a positive constant and interpret  $\delta(S) \equiv r - \mu(S, Z, \theta)/S$  as the dividend. Throughout the paper, we will write  $S_\tau$  for  $S(T - \tau)$  and similarly for other quantities. We assume that a smooth critical boundary  $S_\tau^*$  exists for  $\tau \in (0, T]$ . Detemple (2006) gives one sufficient condition on  $\mu(\cdot)$  and  $\sigma(\cdot)$  so that this assumption is true. In particular, the constant elasticity of variance (CEV) process in which the diffusion function is  $S(t)^{\theta/2}$  for some  $\theta > 0$  satisfies this assumption. See Detemple and Tian (2002). It is however, possible for the critical boundary to have a discontinuity at  $\tau = 0$ , as is the case for the American put option under the geometric Brownian motion model when  $r < \delta$ , where  $r$  and  $\delta$  are the constant risk-free interest rate and dividend rate, respectively.

In contrast to the fair amount of research done on pricing American option under the Black-Scholes setup, relatively little work has been done for other diffusion processes, notable exceptions being Kim and Yu (1996) and Detemple and Tian (2002). Considering general diffusion processes is useful for a few reasons. First, in practice American options are often written on financial quantities other than stock prices, for example, commodity prices which often exhibit mean-reverting behavior. Practitioners often use processes other than the geometric Brownian motion, such as the log-OU process or the CEV process, to model these quantities. Second, even if we assume the underlying follows a geometric Brownian motion, it is still useful for the purpose of pricing exotic American options to consider more general diffusion processes. For example, consider an American option with payoff function  $(K - f(S_\tau))^+$  if exercised at  $t = T - \tau$ , where  $f(\cdot)$  is an increasing smooth function such as the logarithm and  $S(t)$  is a geometric Brownian motion process. If we define a new process  $Y(t)$  by  $Y = f(S)$ , then we have a standard American put option written on the process  $Y$ , which is no longer necessarily a geometric Brownian motion. Finally, considering general processes can offer additional insights on the behavior of the American option price under the Black-Scholes setup. For example, by considering the general CEV process, we can gain understanding on how the free parameter  $\theta$  affects the option prices and the early exercise boundary.

We will often suppress the dependence of  $Z$  and simply let  $p(S, \tau, K, r, \theta)$  denote the value

of a European put option with time to maturity  $\tau$  and strike price  $K$  when the current stock price is  $S$ . Let  $P(S, \tau, K, r, \theta)$  denote the value of a corresponding American put option. We will often omit some of the arguments when there is no confusion. For example, we will often write  $P(S, \tau)$ ,  $P(K)$ , or  $P(S, \tau, K)$ . The fundamental valuation partial differential equation (PDE) for the American put option is

$$\mathcal{L}P(S, \tau) - rP = 0, \quad (3)$$

where

$$\mathcal{L} \equiv -\frac{\partial}{\partial \tau} + \mathcal{A}, \quad (4)$$

with the infinitesimal generator  $\mathcal{A}$  given by

$$\mathcal{A} = \mu(S, \theta) \frac{\partial}{\partial S} + \frac{1}{2} \sigma(S, \theta)^2 \frac{\partial^2}{\partial S^2} + \sum_{i=1}^N \left[ \mu_i \frac{\partial}{\partial Z_i} + \frac{1}{2} \rho_{ij} \sigma_i \sigma_j \sum_{j=1}^N \frac{\partial^2}{\partial Z_i \partial Z_j} + \rho_{0i} \sigma_i \frac{\partial^2}{\partial Z_i \partial S} \right]. \quad (5)$$

Here for simplicity, we have omitted the dependence of  $\sigma$ ,  $\mu_i$  and  $\sigma_i$  on  $S, Z$  and  $\theta$  in the above expression. The terminal boundary condition is  $P(S, 0) = (K - S)^+$ . There is a far-field boundary condition  $\lim_{S \rightarrow \infty} P(S, \tau) = 0$ . The boundary conditions on  $Z$  should depend on the problem. We will assume that the free boundary is nicely behaved. At the free boundary  $S_\tau^*$ , we have the value matching and high contact conditions, namely,

$$P(S_\tau^*, \tau) = K - S_\tau^*, \quad (6)$$

$$\frac{\partial P}{\partial S}(S_\tau^*, \tau) = -1. \quad (7)$$

For theoretical discussions on the free boundary, see Detemple (2006).

We assume that a fast algorithm to compute the European put price  $p(S, \tau, K)$  exists. The approach we take in this paper is to interpolate the American put price by two European puts along the lines of Johnson (1983). That is, we want to find out the function  $\alpha = \alpha(S, \tau, K, \theta)$ , such that

$$P(S, \tau, K) = \alpha p(S, \tau, Ke^{r\tau}) + (1 - \alpha)p(S, \tau, K). \quad (8)$$

Another way to write the same equation is

$$P(S, \tau, K) = p(S, \tau, K) + \alpha (p(S, \tau, Ke^{r\tau}) - p(S, \tau, K)) \equiv p(S, \tau, K) + \alpha D(S, \tau, K). \quad (9)$$

This equation has the interpretation that we are approximating the early exercise premium by the term  $\alpha D(S, \tau, K)$ , where  $D$  has been defined to be the price difference of the two European put options.

Johnson (1983) considers only the geometric Brownian motion case. Even more restrictively, he considers only a zero dividend rate. His method of finding  $\alpha$  is purely based on empirical fitting. In this paper, we want to explore Johnson's idea further and see if this method of interpolation can be carried out further and if there could be a more systematic method of finding the optimal  $\alpha$ . It is also worthwhile pointing out that there are other American pricing methods which are based on interpolation, for example, the LUBA approximation in Broadie and Detemple (1996).

For the geometric Brownian motion case  $\mu(S, Z, \theta) = (r - \delta)S$  with a constant dividend rate  $\delta > 0$ , by simple no-arbitrage arguments, we have  $p(K) \leq P(K) \leq p(Ke^{r\tau})$ , so  $0 \leq \alpha \leq 1$  and we have an interpolation as opposed to extrapolation. For general diffusion processes, if the process  $S(t)$  is not too pathologic so that it is always not worse off to pay one share of stock later than earlier, then  $P(K) \leq p(Ke^{r\tau})$ , and thus equation (8) is also an interpolation. The reason is simple. Whenever we early exercise the American put option, we receive  $K$  and deliver the stock. Delivering the stock earlier is not as good as delivering the stock at maturity because we lose the dividends. Receiving  $K$  earlier on the other hand is not as good as receiving  $Ke^{r\tau}$  at maturity. In particular, we have strict interpolation for the stochastic volatility model. Throughout the paper, we will assume this is the case. However, our method also applies if this is not the case. For that situation, we would need to interpret the term interpolation in the more general sense to include both "interpolation" and "extrapolation."

Notice that when  $\tau = 0$ , equation (8) is valid for any  $\alpha$ . To find  $\alpha$  for positive  $\tau$ , we start by plugging the interpolated price in equation (8) into the valuation PDE in equation (3). We have

$$\mathcal{L}p(S, \tau, K) - rp(S, \tau, K) = 0 \quad (10)$$

since  $p(S, \tau, K)$  is the price of a traded asset. However,  $p(S, \tau, Ke^{r\tau})$  with  $\tau$  changing is not the price of a traded asset (the quantity  $p(S, \tau, Ke^{rT})$  for fixed  $T$  is). Instead, we have

$$\mathcal{L}p(S, \tau, Ke^{r\tau}) - rp(S, \tau, Ke^{r\tau}) = -p'_3(S, \tau, Ke^{r\tau})rKe^{r\tau}, \quad (11)$$

where  $p'_3(S, \tau, Ke^{r\tau})$  denotes the partial derivative of  $p(S, \tau, K)$  with respect to the third argument, evaluated at  $Ke^{r\tau}$ . With the help of the above two equations, the fundamental valuation PDE becomes

$$D \mathcal{A}\alpha - \alpha p'_3(S, \tau, Ke^{r\tau})rKe^{r\tau} + \epsilon = 0, \quad (12)$$

where

$$\epsilon = -D \frac{\partial \alpha}{\partial \tau} + \sigma^2(S, \theta) \frac{\partial D}{\partial S} \frac{\partial \alpha}{\partial S} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \rho_{ij} \sigma_i \sigma_j \frac{\partial D}{\partial Z_i} \frac{\partial \alpha}{\partial Z_j} + \sum_{i=1}^N \sigma_i \rho_{0i} \frac{\partial D}{\partial S} \frac{\partial \alpha}{\partial Z_i}. \quad (13)$$

We now make the following approximation

$$D = p(S, \tau, Ke^{r\tau}) - p(S, \tau, K) \approx p'_3(S, \tau, Ke^{r\tau})Ke^{r\tau}\Phi(\tau, \theta), \quad (14)$$

for some function  $\Phi(\tau, \theta)$  to be determined which has no  $S$  dependence. We will often simply write  $\Phi$  or  $\Phi(\tau)$ . If one thinks of the above equation as a first-order Taylor expansion around  $Ke^{r\tau}$ , then  $\Phi \approx 1 - e^{-r\tau}$ . The nice feature of this crude approximation is that it works for general diffusion processes. Also, in our numerical study for geometric Brownian motion case, we find that it already gives quite accurate results. However, for any specific diffusion process, we recommend fitting the above equation using a regression procedure to find the best functional form for  $\Phi$ . This idea of regression was also very fruitfully utilized in other well-known research works such as Johnson (1983) and Broadie and Detemple (1996). For example, Broadie and Detemple (1996) use a regression approach to produce an improved approximation from their lower and upper price bounds.

Consistent with the idea of treating the  $S$  dependence in  $D$  as small, we will omit the last term  $\epsilon$  in equation (12). When  $\tau$  is small,  $D$  is close to 0. When  $\tau$  is large, the dependence of  $\alpha$  on  $\tau$  is small. This is similar to the approximation used in the quadratic approximation in MacMillan (1986), Barone-Adesi and Whaley (1987), and Ju and Zhong (1999). Putting all three approximations together, we get the following differential equation for  $\alpha$ :

$$\mathcal{A}\alpha(S, \tau) - \frac{r}{\Phi(\tau)}\alpha(S, \tau) = 0. \quad (15)$$

This is exactly the differential equation for a perpetual American option under the general diffusion process with the only change being that  $r/\Phi(\tau)$  now plays the role of the interest rate. The  $\tau$  dependence of  $\Phi$  should not concern us, because the above equation only involves spatial variables in which  $\tau$  is a fixed free parameter. We will assume that a closed-form solution for the perpetual American when interest rate is  $r$  exists and denote it as  $P_\infty(S, K, r)$ . The solution to  $\alpha$  can then be written as

$$\alpha(S, \tau) = A \frac{P_\infty(S, K, r/\Phi(\tau))}{P_\infty(S_\tau^*, K, r/\Phi(\tau))} \quad (16)$$

for  $A$  not depending on  $S$ . Notice that  $A = \alpha(S_\tau^*, \tau)$  and is a function of  $r$ ,  $\tau$ ,  $K$  and the parameter vector  $\theta$ , but for a fixed option, it is a constant. Going back to our interpolation equation, we have that if  $S \geq S_\tau^*$ ,

$$P(S, \tau, K) = A \frac{P_\infty(S, K, r/\Phi(\tau))}{P_\infty(S_\tau^*, K, r/\Phi(\tau))} p(S, \tau, Ke^{r\tau}) + \left(1 - A \frac{P_\infty(S, K, r/\Phi(\tau))}{P_\infty(S_\tau^*, K, r/\Phi(\tau))}\right) p(S, \tau, K), \quad (17)$$

and

$$P(S, \tau, K) = K - S \quad (18)$$

if  $S < S_\tau^*$ . Equation (17) is the main result of this paper. It expresses the American option prices under general diffusion processes as an interpolation of two European option prices, where interestingly the weights depend on the price of a perpetual American option. Notice that this result holds for every model in our general diffusion framework, so no derivation is needed at all if we consider a new model. We could go directly from equation (17). Also, notice that we do not need closed form solutions to apply the method, but just a quick and accurate pricing of the European put option and the perpetual American option. Hence it covers pricing by transform analysis in affine or quadratic models, for example pricing by fast Fourier transform in the Heston model.

Another way of writing the interpolation equation is

$$P(S, \tau, K) = p(S, \tau, K) + A \frac{P_\infty(S, K, r/\Phi(\tau))}{P_\infty(S_\tau^*, K, r/\Phi(\tau))} D(S, \tau, K). \quad (19)$$

This equation allows one more interpretation of  $A$  and  $\alpha$  that can help us gain some insight on the interpolation method. The European put option with strike  $Ke^{r\tau}$  is more valuable than the American option because effectively one receives the cash amount  $K$  at time 0 and delivers the stock at maturity. For the American option, you have to receive the cash amount  $K$  and deliver the stock at the *same* time, although this exercise time is random. Thus, we can think of  $D(S, \tau, K)$  as the maximum amount of premium we can receive over the European put option with strike  $K$ , if we were able to choose different times for receiving the cash and for delivering the stock. The quantities  $\alpha$  and  $A$  are then accounting for the fact that we can only receive the cash and deliver the stock at the same time.

There are still two quantities to determine, namely, the critical stock price  $S_\tau^*$ , below which the American put should be exercised immediately, and the value of  $A$ . Before we close the system with the two smooth pasting conditions, we give the expression for the greek Delta  $\Delta_P$  for the American put option first. The Delta is a very important quantity for risk management as it is frequently used to design hedging strategies and to compute Value-at-Risk. The expression is:

$$\Delta_P(S, \tau, K) = \Delta_p(S, \tau, K) + A \left( \frac{D(S, \tau, K)}{P_\infty(S_\tau^*, K, r/\Phi(\tau))} \frac{\partial P_\infty}{\partial S} + \frac{P_\infty(S, K, r/\Phi(\tau))}{P_\infty(S_\tau^*, K, r/\Phi(\tau))} \frac{\partial D}{\partial S} \right), \quad (20)$$

where  $\Delta_p(S, \tau, K) = \frac{\partial p}{\partial S}(S, \tau, K)$  is the Delta for the corresponding European option. This equation is obtained by differentiating the interpolation equation (17). Thus, as soon as we compute the value of  $A$ , we can simultaneously get the price of the American option and the option delta. This is a big advantage over some of the other non-analytical methods. For example, in the binomial tree methods or Monte Carlo simulation methods, one would often have to compute option prices twice in order to compute the Delta by finite difference approximation.

We are now ready to close the system. Evaluating the Delta's at the free boundary, we get

$$\Delta_P(S_\tau^*, \tau, K) = \Delta_p(S_\tau^*, \tau, K) + A \left( D \frac{\partial \log P_\infty}{\partial S}(S_\tau^*, K, r/\Phi(\tau)) + \frac{\partial D}{\partial S}(S_\tau^*, \tau, K) \right), \quad (21)$$

The high contact condition  $\Delta_P(S_\tau^*, \tau, K) = -1$  gives

$$A = \frac{1 + \frac{\partial p}{\partial S}(S_\tau^*, \tau, K)}{-D \frac{\partial \log P_\infty}{\partial S}(S_\tau^*, K, r/\Phi(\tau)) - \frac{\partial D}{\partial S}(S_\tau^*, \tau, K)}. \quad (22)$$

The value matching condition gives one more equation

$$P(S_\tau^*, \tau, K) = K - S_\tau^* = Ap(S_\tau^*, \tau, Ke^{r\tau}) + (1 - A)p(S_\tau^*, \tau, K). \quad (23)$$

The above two equations together give an equation for  $S_\tau^*$  which can be solved by suitable numerical methods. After solving  $S_\tau^*$ , equation (22) gives the value of  $A$  and equation (17) gives the value of the American put option. This methodology is very similar to the quadratic approximation used in Barone-Adesi and Whaley (1987), in which the critical stock price is also solved with a similar equation (equation (24) in their paper).

That our method solves the critical boundary and the American option price at the same time is an advantage of our method over some other methods. This is because the critical stock price is almost as important as the price itself. It allows us to determine whether it is optimal to exercise the option immediately. Also, many numerical approximation methods such as those utilizing the early exercise premium representation rely on the accurate computation of the critical boundary.

Another hidden attractive feature of our method is that after  $S_\tau^*$  is found, equation (17) allows us to compute American options with different initial stock price  $S$  without much additional computational cost. This is in stark contrast to the Binomial tree method, which requires a completely different tree for each value of  $S$ . One might ask that since in practice we always have a single observed current stock price, whether a closed-form expression in  $S$  such as equation (17) is useful other than helping us understand the  $S$  dependence of the American put option price. We can quickly think of two other instances in which such an explicit expression is useful. One is to value compound options on an American put option (Geske (1979) considers compound options written on European options). In such a case, a closed-form expression for the American put option price would allow one to compute the compound option price by numerical integration. Another instance is to value American call options with different strike prices  $K_i$  and current stock price  $S$  under the geometric Brownian motion model. By the put-call symmetry, we can transform these call options into put options with different current stock prices  $K_i$  and common strike price  $S$ .

Finally, while it is hard to quantify and control the errors in both the quadratic approximation and our method, our put option price always lies between two arbitrage bounds. This is one advantage of the interpolation method over the quadratic approximation because the put option price in the latter method can potentially violate the upper bound  $p(S, \tau, K e^{r\tau})$  we have used.

A straight-forward extension of our method is to allow for explicit and deterministic time dependence on the drift and diffusion functions of  $S$ . Our method still works as long as the corresponding European option and perpetual American option have closed-form solutions.<sup>4</sup> For example, if  $r_u$  is a deterministic process, we could approximate the American option price by interpolating the prices  $p(S, \tau, K)$  and  $p(S, \tau, K \exp(\int_0^T r_u du))$ . However, the accuracy of the approximation might be affected because we have omitted the time dependence of  $\alpha$  on  $\tau$  in our derivation.

## B. Specializing to geometric Brownian motion

We now specialize the above general method to the case of geometric Brownian motion. The process for the underlying asset  $S(t)$  is modeled as

$$dS(t) = bS(t)dt + \sigma S(t)dW_t, \quad (24)$$

where  $b$  is a constant cost of carry and  $\sigma$  is a constant. The cost of carry is assumed to be smaller than or equal to the risk-free interest rate  $r$ . The European put option price  $p(S, \tau, K)$  under this model is given by the famous Black-Scholes formula (Black and Scholes, 1973)

$$p(S, \tau, K) = Ke^{-r\tau}N(-d_2(S, K)) - Se^{(b-r)\tau}N(-d_1(S, K)), \quad (25)$$

where

$$d_1(S, K) = \frac{\log(Se^{b\tau}/K)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}, \quad d_2 = d_1 - \sigma\sqrt{\tau}, \quad (26)$$

and  $N(\cdot)$  is the cumulative normal distribution function. The Delta of the European put option is  $-e^{(b-r)\tau}N(-d_1(S, K))$ . The value of the perpetual American put was first worked out by Merton (1973) as follows:

$$P_\infty(S, K, r) = CS^{q(b,r)}, \quad (27)$$

if  $S$  is above a constant critical stock price, and  $P_\infty(S, K, r) = K - S$  otherwise. Here,  $C$  is a constant not very interesting to us, and  $q(b, r)$  is a negative number given by

$$q(b, r) \equiv \frac{\sigma^2 - 2b}{2\sigma^2} - \frac{1}{2\sigma^2}\sqrt{(\sigma^2 - 2b)^2 + 8r\sigma^2}. \quad (28)$$

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<sup>4</sup>The perpetual American option price can be computed very fast using numerical methods since it satisfies an ODE instead of a PDE. So the only really necessary requirement of our method is the existence of a closed-form solution for the European put price.

Notice that if  $r > 0$ , then  $q$  is always negative. One can easily check that this solution satisfies the following ordinary differential equation

$$bS \frac{\partial P_\infty}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_\infty}{\partial S^2} - rP_\infty = 0. \quad (29)$$

In the classical Black-Scholes setup,  $b = r - \delta$ , where  $\delta$  is a constant dividend rate. By the above result for the perpetual American put option and equation (16), we see that the weight  $\alpha$  in our interpolation method is given by

$$\alpha = A \left( \frac{S}{S_\tau^*} \right)^{q(r-\delta,r/\Phi)}. \quad (30)$$

Our interpolation equation for  $S > S_\tau^*$  now becomes

$$P(S, \tau, K) = A \left( \frac{S}{S_\tau^*} \right)^{q(r-\delta,r/\Phi)} p(S, \tau, Ke^{r\tau}) + \left( 1 - A \left( \frac{S}{S_\tau^*} \right)^{q(r-\delta,r/\Phi)} \right) p(S, \tau, K), \quad (31)$$

and the equation for  $A$  becomes

$$A = \frac{1 - N(-d_1(S_\tau^*, K))e^{-\delta\tau}}{[N(-d_1(S_\tau^*, Ke^{r\tau})) - N(-d_1(S_\tau^*, K))]e^{-\delta\tau} - q(r - \delta, r/\Phi)D(S, \tau, K)/S_\tau^*}. \quad (32)$$

The above equation when substituted into the value matching condition in equation (23) gives an equation for  $S_\tau^*$  which needs to be solved numerically. One way to do this is to use the method suggested in Barone-Adesi and Whaley (1987), in which they provide an initial estimate of  $S_\tau^*$  and then use Newton-Raphson method to find the root of equation (23). Although our formula looks a little bit more complicated than Barone-Adesi and Whaley (1987), the amount of computation needed in these two methods is almost identical. To implement the Newton-Raphson algorithm, we need the derivative of the right-hand side of (23) with respect to  $S_\tau^*$ . Since  $A$  has explicit  $S_\tau^*$  dependence, we recommend a variant of the Newton-Raphson algorithm in which we approximate this derivative by using the value of  $A$  from the previous iteration and treating it as a constant in (23). Since  $A$  converges extremely fast, this method is very efficient. In our actual implementation, we usually need very few iterations for  $S_\tau^*$  to converge.

From the value matching equation (23), we have  $0 \leq A \leq 1$  since for any  $S$ ,  $\tau$  and  $K$ , we have

$$p(S, \tau, K) \leq P(S, \tau, K) \leq p(S, \tau, Ke^{r\tau}). \quad (33)$$

Equation (31) is then indeed an interpolation since  $q < 0$  and  $S \geq S_\tau^*$ . This is one attractive feature of our method since the American option price is by construction confined within the two no-arbitrage bounds. Thus, we expect that our method becomes more accurate when the

two bounds are close to each other, as is the case for small interest rate, large dividend rate, or out-of-the-money put options.

We still have one stone unturned and that is  $\Phi = \Phi(\tau, \theta)$ . Determining the best  $\Phi$  is a slightly tricky part of our approximation. A simple and aesthetically appealing choice would just be the result from Taylor series approximation:

$$\Phi(\tau, \theta) \approx 1 - \exp(-r\tau). \quad (34)$$

Interestingly, our numerical study shows this crude approximation already gives fairly accurate results. However, to further improve the accuracy, we use the idea of fitting which was also employed in Johnson (1983) and Broadie and Detemple (1996). The starting point is equation (14) which, when specialized to the Black-Scholes model, states that

$$\Phi(\tau, \theta) \approx \frac{D(S, \tau, K)}{N(-d_2(S, Ke^{r\tau})K)}. \quad (35)$$

However, in the actual fitting, we also take into consideration the last term in equation (12) that we have omitted in our approximation, because now we have an explicit solution for  $\alpha$ . We postulate the following simple parametric form of  $\Phi$ :

$$\Phi(\tau, \theta) \approx 1 - \exp(-|k_1 r\tau + k_2 \delta\tau + k_3 \sigma \sqrt{\tau}|). \quad (36)$$

The absolute value sign is to guarantee that  $\Phi$  is positive. The left-hand side of equation (14) is then simulated for different values of  $S$ ,  $r$ ,  $\delta$  and  $\sigma$  while fixing  $K = 100$  since it is homogeneous of degree 1 in  $K$ . The optimal values of  $k_1$ ,  $k_2$  and  $k_3$  are obtained by minimizing the maximum error of the fit. In the end, we recommend the following values:

$$k_1 = 1.239; \quad k_2 = -0.264; \quad k_3 = 0.0215. \quad (37)$$

One can improve the accuracy of the interpolation method by proposing a more complicated parametric form for  $\Phi$  in equation (34), for example, by including terms such as  $\tau$ ,  $(r\tau)^2$ , and  $S_\tau^*$ , or even putting back some  $S$  dependence into  $\Phi$ . One particular parametric form that was adopted in Li (2009) is the following:

$$\Phi(\tau, \theta) \approx 1 - \exp(-|\lambda_1 r\tau + \lambda_2 \delta\tau + \lambda_3 \sigma \sqrt{\tau} + \lambda_4 \min(r, \delta)\tau|), \quad (38)$$

with

$$\lambda_1 = 1.386; \quad \lambda_2 = -0.125; \quad \lambda_3 = 0.00542; \quad \lambda_4 = -0.255. \quad (39)$$

This choice tries to capture more dependence of the American option price on the interplay between the interest rate and the dividend rate.

Alternatively, one could fit  $\Phi$  separately for different regions of the parameter values. For example, one could fit short-term and long-term options separately. However, all the above possible approaches require a large number of true American put option prices and a large scale calibration exercise. In this paper, we decide not to do this and be content with the simple parametric form in equation (34).

Equation (31) together with our choice of parametric form of  $\Phi$  in equation (34) has some nice asymptotic properties. First, if we let  $r$  goes to 0 while fixing other parameters,  $p(S, \tau, Ke^{r\tau})$  and  $p(S, \tau, K)$  approach the same limiting European put option price. In turn, the American put option value in equation (31) approaches that of the European option. This agrees with early findings and our intuition that if there is no benefit of receiving cash earlier, then we should never exercise the American put option early. Second, as  $K$  goes to 0, or  $S$  goes to infinity, our interpolation method gives the correct limiting value for the American put option, which is zero. Third, when  $\tau$  goes to 0, the interpolated price goes to the correct limit  $(K - S_0)^+$ . The final and probably the most attractive asymptotic property of our interpolation method is the behavior when  $\tau$  goes to infinity. In this case, we have  $p(S, \tau, K) \rightarrow 0$ ,  $p(S, \tau, Ke^{r\tau}) \rightarrow K$ , and  $\Phi(\tau) \rightarrow 1$ . These together show that  $P(S, \tau, K)$  approaches the perpetual American put option price.

In the next section, we take a close look at the numerical performance of the interpolation method under the geometric Brownian motion model. We only need to consider put options because of the extremely simple put-call symmetry in this model:

$$C(S, \tau, K, r, \delta, \sigma) = P(K, \tau, S, \delta, r, \sigma). \quad (40)$$

That is, to find the price of an American call option, we can just switch the roles of  $S$  and  $K$  and the roles of  $r$  and  $\delta$ , and then pretend that we have an American put option. The critical stock price functions  $S_{\tau, \text{Call}}^*(K, r, \delta, \sigma)$  and  $S_{\tau, \text{Put}}^*(K, r, \delta, \sigma)$  are also nicely related by

$$S_{\tau, \text{Call}}^*(K, r, \delta, \sigma) = \frac{K^2}{S_{\tau, \text{Put}}^*(K, \delta, r, \sigma)}. \quad (41)$$

Because of the above two equations, American call options in the geometric Brownian motion model can be reduced completely to put options.

### C. Specializing to Heston's stochastic volatility model

There now exist quite a number of different methods for pricing American options under the Black-Scholes model. However, such pricing methods for general diffusion processes are still fairly scarce, with the vast majority of them completely numerical in nature. A few examples of multi-dimensional models include Finucane and Tomas (1996), Chung (2000), Zhylyevskyy

(2009), and Medvedev and Scaillet (2009). One of the most attractive features of our quasi-analytical method is that it can be adapted to multi-dimensional models since the interpolation approximate in equation (17) is derived under a very general setup. The numerical study for the stochastic volatility model in the next section shows that the method can be both very efficient and accurate even for general diffusion processes.

To introduce the notation, Heston's stochastic volatility model is specified as follows (see Heston (1993))

$$dS(t) = (r - \delta)S(t)dt + \sqrt{v(t)}S(t)dW_S(t), \quad (42)$$

$$dv(t) = \kappa_v(v(t) - \theta_v)dt + \sigma_v\sqrt{v(t)}dW_v(t). \quad (43)$$

Here  $v(t)$  is the instantaneous variance process,  $\kappa_v$  the mean-reverting strength,  $\theta_v$  the long-run mean instantaneous variance, and  $\sigma_v$  the volatility of variance coefficient. The two Brownian motions  $W_S(t)$  and  $W_v(t)$  have a constant correlation  $\rho$ .

To use our interpolation method, we just need to use the interpolation approximation in equation (17). We need a few things. First, we need the European option price. Second, we need the perpetual American option price. Finally, in the high contact condition we need several partial derivatives with respect to  $S$ .

The European option prices can be done easily through an inverse Fourier transformation using the characteristic function of  $\log S(T)/S$ . Specifically, a European call option with strike price  $K$  maturing at time  $T = t + \tau$  is computed by numerical integration as (see Li and Pearson (2009))

$$c(S, \tau, K) = \frac{Se^{-r\tau}}{\pi} \operatorname{Re} \int_{0+i\operatorname{Im} u}^{+\infty+i\operatorname{Im} u} e^{-iu \log \frac{K}{S}} \cdot \frac{\phi_{s_\tau}(u-i)}{iu(iu+1)} du \quad (\operatorname{Im} u < 0), \quad (44)$$

where  $\operatorname{Re}$  and  $\operatorname{Im}$  denote the real and imaginary parts of a complex quantity, respectively, and the integration path is a half-line parallel to the  $x$ -axis in the complex plane. The price of the European put option can be obtained through the put-call parity

$$c(S, \tau, K) - p(S, \tau, K) = Se^{-\delta\tau} - Ke^{-r\tau}. \quad (45)$$

Let  $v_t$  denote the current instantaneous variance. The characteristic function for  $s_\tau \equiv \log S(T)/S$  is given by (see Li and Pearson (2009))

$$\phi_{s_\tau}(u) = e^{iu(r-\delta)\tau} \exp\left(-a(\tau)v_t - b(\tau)\right), \quad (46)$$

where

$$a(\tau) = \frac{2\lambda(1 - e^{-\xi\tau})}{2\xi - (\xi - \kappa^*)(1 - e^{-\xi\tau})}; \quad (47)$$

$$b(\tau) = \frac{\kappa_v\theta_v}{\sigma_v^2} \left( 2 \log \left( 1 - \frac{\xi - k^*}{2\xi} (1 - e^{-\xi\tau}) \right) + (\xi - \kappa^*)\tau \right), \quad (48)$$

with  $\lambda = (iu + u^2)/2$ ,  $\kappa^* = \kappa_v - iu\sigma_v\rho$ , and  $\xi = \sqrt{\kappa^{*2} + 2\sigma_v^2\lambda}$ .

The partial derivatives needed can be computed numerically or analytically once we have the European and perpetual American option prices. No analytical approximations for the perpetual American option exists in the literature as far as we know. Therefore, it need to be solved numerically or by using a volatility of variance  $\sigma_v$ -expansion similar to the one in Lewis (2000). If one is willing to sacrifice some accuracy, we can perform a zeroth order  $\sigma_v$ -expansion for the perpetual American option price in the following sense. This amounts to set  $\sigma_v = 0$  in the  $v(t)$  process. In this case, the variance process  $v(t)$  is still time-dependent, but no longer stochastic. Indeed, we can solve the process  $dv(t) = \kappa_v(v(t) - \theta_v)dt$  to give

$$v(t) = \theta_v + (v_t - \theta_v)e^{-\kappa_v t}. \quad (49)$$

If  $t = 0$ , then  $v(t) = v_t$ . If  $t$  goes to infinity,  $v(t)$  goes to the long-run mean  $\theta_v$ . Solving the perpetual American option price with time-varying but deterministic variance is surprisingly difficult. Therefore, we make the following approximation. We assume that the variance process is a constant  $\tilde{\sigma}^2$  for some value of  $\tilde{\sigma}$ . Under this approximation, the  $v$ -dependence then all drops out in the partial differential equation for the perpetual American option to give the following Merton-type solution

$$P_\infty(S, K, r) = CS^{\tilde{q}(b,r)}. \quad (50)$$

However, now we have

$$q(b, r) \equiv \frac{\tilde{\sigma}^2 - 2b}{2\tilde{\sigma}^2} - \frac{1}{2\tilde{\sigma}^2} \sqrt{(\tilde{\sigma}^2 - 2b)^2 + 8r\tilde{\sigma}^2}, \quad (51)$$

with  $b = r - \delta$ . It remains a question what value of  $\tilde{\sigma}$  we should use. Two simple but extreme choices would be to use  $\tilde{\sigma} = \sqrt{v_t}$  or  $\tilde{\sigma} = \sqrt{\theta_v}$ . More reasonable choice would be to use something in between these two. One choice would be to use the Black-Scholes implied volatilities from the European put prices  $p(K)$  or  $p(Ke^{r\tau})$ .<sup>5</sup> Another choice is to use the time-averaged risk-neutral expected variance given by

$$\tilde{\sigma}^2 = \frac{1}{\tau} \int_0^\tau \mathbb{E}v(u)du = \theta_v + (v_t - \theta_v) \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau}. \quad (52)$$

The above equation is true regardless of whether we have  $\sigma_v = 0$  or not. By the way, the  $\tilde{\sigma}$  above is identical to the mechanism behind the construction of the CBOE VIX index. Using this last approximation tends to underestimate the true value of the perpetual American option. See Theorem 2.5 in Lewis (2000) for a corresponding statement for European put options.

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<sup>5</sup>On a practical note, the implied volatility function in MATLAB uses a Dekker-Brent algorithm and is not very efficient. A rational approximation in Li (2008) or a successive over-relaxation algorithm in Li and Lee (2009) would be useful.

Since our method approaches the perpetual American option price when  $\tau$  goes to infinity, this approximation tends to underestimate the American option prices. However, our numerical approximation in the next section shows that this downward bias is quite small. Small advantages of using this time-averaged volatility instead of the Black-Scholes implied volatilities is that it has lower computational cost and we know that the computed American option prices are almost always downward biased.

From the programmer's perspective, the above further approximation is very attractive. It means that we can recycle the code for the Black-Scholes case. The only two changes we need to make is to replace  $\sigma$  with some  $\tilde{\sigma}$ , and to replace the Black-Scholes European put prices with the ones computed from the inverse Fourier transformation. This amounts to modifying a couple lines of code only.

## D. Approximation for American call option prices

So far we have only considered American put options. The literature on American options has largely concentrated on American put options for possibly a few reasons. First, for the Black-Scholes model and other well-behaved constant dividend rate model, a put-call symmetry for American options (and also for the corresponding critical boundaries) similar to equations (40) and (41) might be established, which reduces the problem of computing American call option prices completely to the problem of American put option prices. Second, for general diffusion processes especially with a complicated dividend process, the pricing of American call option is actually sometimes more difficult than for American put options for various reasons. For example, the domain for an put option is a compact region while the domain for a call option is unbounded. Therefore, it is worthwhile developing an interpolation method for American call options.

Let  $C(S, \tau, K, r, \theta)$  and  $c(S, \tau, K, r, \theta)$  denote the values of an American option and a European option, respectively. Same as before, we will often omit variables for the sake of simplicity. Similar to put options, we seek the following interpolation scheme:

$$C(S, \tau, K) = \tilde{\alpha}c(S, \tau, Ke^{-r\tau}) + (1 - \tilde{\alpha})c(S, \tau, K) \equiv c(S, \tau, K) + \tilde{\alpha}\tilde{D}(S, \tau, K), \quad (53)$$

where

$$\tilde{D}(S, \tau, K) \equiv c(S, \tau, Ke^{-r\tau}) - c(S, \tau, K) \quad (54)$$

is the price difference of the two European call options. Now unlike for put options, although  $\tilde{\alpha}$  is still strictly positive, it is no longer guaranteed to be smaller than 1. That is, we no longer necessarily have  $C(S, \tau, K) < c(S, \tau, Ke^{-r\tau})$ . The reason is that it might be beneficial to receive the stock earlier to capture the dividends even though we have to pay the cash  $K$  earlier.

For notational simplicity, below we will still use  $S_\tau^*$  to denote the critical stock price for the American call option. Readers are cautioned not to confuse this with the critical stock price for the American put option, which will have a different numerical value. With derivation extremely similar to the put option case and thus omitted, we have the following result

$$C(S, \tau, K) = \tilde{A} \frac{C_\infty(S, K, r/\Phi(\tau))}{C_\infty(S_\tau^*, K, r/\Phi(\tau))} c(S, \tau, K e^{-r\tau}) + \left(1 - \tilde{A} \frac{C_\infty(S, K, r/\Phi(\tau))}{C_\infty(S_\tau^*, K, r/\Phi(\tau))}\right) c(S, \tau, K), \quad (55)$$

where we use  $C_\infty(S_\tau^*, K, r/\Phi(\tau))$  to denote the price of a perpetual American call option with interest rate  $r/\Phi(\tau)$ . The high contact condition  $\Delta_C(S_\tau^*, \tau, K) = 1$  gives

$$\tilde{A} = \frac{-1 + \frac{\partial c}{\partial S}(S_\tau^*, \tau, K)}{-\tilde{D} \frac{\partial \log C_\infty}{\partial S}(S_\tau^*, K, r/\Phi(\tau)) - \frac{\partial \tilde{D}}{\partial S}(S_\tau^*, \tau, K)}. \quad (56)$$

The above two equations together with the value matching condition  $C(S_\tau^*, \tau, K) = S_\tau^* - K$  allow us to solve for  $\tilde{A}$  and  $S_\tau^*$ .

Notice that the above method breaks down if  $r = 0$ . If  $r$  is close to 0, we might have to extrapolate a lot so that the method might not be very accurate. However, it is possible to use a strict interpolation if both the dividend rate  $\delta > 0$  and interest rate  $r$  are constants. In this case, we will have a genuine interpolation if we use  $c(S e^{\delta\tau}, \tau, K)$  as the upper bound. The reason is that whenever we early exercise the American call option, we receive the stock and pay  $K$  early. Receiving the stock earlier is not as good as receiving  $S e^{\delta\tau}$  at maturity. Furthermore, having to pay  $K$  early is not as good as paying  $K$  at maturity. Therefore, no arbitrage argument gives us  $C(S, \tau, K) < c(S e^{\delta\tau}, \tau, K)$ . Similar derivation as before will give us the strict interpolation as below:

$$C(S, \tau, K) = \hat{A} \frac{C_\infty(S, K, \delta/\Phi(\tau))}{C_\infty(S_\tau^*, K, \delta/\Phi(\tau))} c(S e^{\delta\tau}, \tau, K) + \left(1 - \hat{A} \frac{C_\infty(S, K, \delta/\Phi(\tau))}{C_\infty(S_\tau^*, K, \delta/\Phi(\tau))}\right) c(S, \tau, K), \quad (57)$$

The high contact condition  $\Delta_C(S_\tau^*, \tau, K) = 1$  gives

$$\hat{A} = \frac{-1 + \frac{\partial c}{\partial S}(S_\tau^*, \tau, K)}{-\hat{D} \frac{\partial \log C_\infty}{\partial S}(S_\tau^*, K, \delta/\Phi(\tau)) - \frac{\partial \hat{D}}{\partial S}(S_\tau^*, \tau, K)}, \quad (58)$$

with  $\hat{D} \equiv c(S e^{\delta\tau}, \tau, K) - c(S, \tau, K)$ . If we trace the derivation, we will see that now the  $\Phi$  function should be chosen to be something close to  $1 - e^{-\delta\tau}$  instead of something close to  $1 - e^{-r\tau}$  as is the case for American put options. Also notice that  $\delta/\Phi$  is now playing the role of the interest rate in the perpetual American call option price.

Same as we did before, we could specialize the above method to the important cases of Black-Scholes model and Heston's stochastic volatility model. For the sake of conserving space, we omit the details here.

It is also possible to do the analysis more measure-theoretically instead of looking at the PDE. The idea is to employ the put-call symmetry (see McDonald and Schroder (1998)). For this, the interest rate is now no longer required to be a constant, but we require the dividend  $\delta \equiv r(S) - \mu(S)/S$  to be a positive constant. We can then define a measure change using the dividend-adjusted stock price as the numeraire. That is, define a new measure  $\mathbb{Q}^*$  by using the Radon-Nikodym derivative  $\Lambda_t = e^{-\int_0^t r_u du} S(t) e^{\delta t} / S(0)$ . This would transform the call option into a put option in the new measure. Thus the same idea above can be applied if the European put price and the perpetual American put price are in closed form for the process  $S(t)$  under  $\mathbb{Q}^*$ . This is the case for the geometric Brownian motion model and Heston's stochastic volatility model.

### III. Numerical Study

#### A. American option pricing in the Black-Scholes model

Fairly large number of methods for pricing American options have been proposed. They differ in accuracy and efficiency, as well as some other important dimensions. In this paper, we are more concerned with speed than accuracy. There are a few justifications for this. First, a quick look at the quoted prices of equity derivatives tells us that their bid-ask spreads are still quite large compared to stocks, which are now often penny priced. Thus, when one calibrates his model to market data, he might not need excessive accuracy because he only has rough ranges for the true prices. Second, when one is pricing an American derivative, he is often faced with issues such as parameter uncertainty, market imperfection such as commissions, transactions costs, existence of irrational investors, tax and other regulatory issues, etc. Thus he would not be interested in whether a given method can produce an accuracy up to four digits or not. Third, for institutional investors or hedge funds with a large number of different positions in American options, a quick and robust estimation of the value and risk of the portfolio is much appreciated, even though one might have to sacrifice some accuracy. The focus of this paper is to provide a method that is able to value hundreds or even thousands of American options within one second. Indeed, we do not claim that our method dominates all the other methods in terms of both accuracy and efficiency. In fact, under the Black-Scholes model, there are a few other methods that are clearly more accurate than ours, notably binomial tree methods with large number of time steps and Broadie and Detemple (1996). Both of these two methods are not as efficient as ours, although Broadie and Detemple (1996) is much more efficient than the binomial

tree method. The method that is closest to our interpolation method both in terms of technique of approximation and level of sophistication is the quadratic approximation of Barone-Adesi and Whaley (1987). Thus, we will mostly compare our method with the quadratic approximation of Barone-Adesi and Whaley (1987).<sup>6</sup> In order to have a benchmark for the true option prices and critical boundary, we also implement a binomial tree model as in Cox, Ross and Rubinstein (1979). We use 10000 time steps most of the time. We check the accuracy of the Binomial tree model against the put-call symmetry in equations (40) and (41). Whenever the two sides of either equations differ by more than 0.02%, we switch to a binomial tree with 15000 time steps (which is extremely time-consuming, especially for the purpose of searching for the critical stock price).

We first take a look at the critical stock prices below which the put option should be exercised immediately. Figure 1 plots the critical price  $S_\tau^*$  as a function of time to maturity  $\tau$  for two fixed parameter vectors  $(r, \delta, \sigma)$ . In both subplots, we set  $K = 100$ , and  $\sigma = 0.4$ . The top subplot is for  $r = 0.04$ , and  $\delta = 0.08$ , while the bottom subplot uses  $r = 0.08$  and  $\delta = 0.04$ . A very nice common feature of the quadratic approximation and the interpolation method is that they both have the correct asymptotic critical stock price as  $\tau \rightarrow 0^+$ , which is given by

$$S_{0^+}^* = \min(K, rK/\delta). \quad (59)$$

The choices of  $r$  and  $\delta$  are made such that the theoretical asymptotic critical stock prices at  $\tau = 0^+$  are different for the two subplots. For the top subplot, the theoretical value is 50, while it is 100 for the bottom subplot. The general conclusion we can draw from this figure is that at least for the particular parameters used in the figure, both the quadratic approximation and interpolation method produce relatively accurate critical prices.

Table 1 presents a more thorough study by considering different parameter combinations of  $r, \delta, \sigma$  and  $\tau$ . Three maturities are chosen, namely, 0.5, 1, and 3 years, representing short-term, medium-term, and long-term options, respectively. The interest rates and dividend rates are set to be either 0.02, 0.04, or 0.08. Thus the ratio  $r/\delta$  ranges from small (0.25) to fairly large (4.00). The volatilities are set to be either 0.2, 0.4 or 0.6 to represent low, medium and high volatilities. For all the options, the strike price  $K$  is fixed at 100. Results from three methods are presented: the Cox, Ross and Rubinstein's binomial tree method with 10000 time steps (treated as the true values), the quadratic approximation in Barone-Adesi and Whaley (1987), and the interpolation method in this paper. The average absolute error (AAE) and average percentage error (APE) for the relative columns are also presented. The average percentage error is defined as the ratio of the absolute approximation error to the binomial tree value. A quick look tells us

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<sup>6</sup>Broadie and Detemple (1996), Ju and Zhong (1999), and many others, already contain detailed analysis on the tradeoff of efficiency and accuracy of fairly large set of established methods for pricing American options.

that the two methods give very similar results on average. For all three maturities, the average percentage error is about 2% for both methods. The larger errors of the interpolation method seem to concentrate in the parameter region where  $\delta$  is small.

One of the motivations of this paper, besides searching for an efficient and accurate analytical approximation, is to take a more systematic look at the interpolation method first proposed in Johnson (1983) and extend it beyond the simple case of zero dividend rate. The quantity of a lot of interest in this interpolation method is the weight  $\alpha$ . In Johnson's paper,  $\alpha$  is fitted by regression using a limited number of American options from Parkinson (1977) and Rubinstein and Cox (1982). Our closed-form expression in equation (30) allows us to take a detailed look at  $\alpha$  without too much computational burden. In our approximation, the  $S$  dependence of  $\alpha$  is captured completely in the factor  $S^{q(r-\delta,r/\Phi)}$ , and the constant  $A$  has no  $S$  dependence. By the value matching condition in equation (23), the constant  $A$  has an interpretation of being the weight of interpolation at the critical stock price. Table 2 presents the weight  $A$  in our interpolation method as a function of volatility  $\sigma$ , interest rate  $r$ , and dividend rate  $\delta$  for the same parameter combinations we used in Table 1. As the value matching condition is satisfied in both the binomial tree model and the interpolation method, the differences in  $A$  in these two methods are purely due to the differences in the values for  $S_\tau^*$ . As we see, the values of  $A$  have a wide range as the parameters change. The smallest value of  $A$  from the binomial tree model in the table is 0.087 while the largest is 0.686. The interpolation method gives fairly accurate estimates of  $A$ . The average absolute errors are about 0.2 for all three maturities, while the average percentage errors are 7% or below.

By looking more carefully the numbers in Table 2, we see that at least for all the parameter combinations presented in the table,  $A$  is an increasing function of  $\sigma$  if we keep  $r$ ,  $\delta$  and  $\tau$  constant. However, the dependence of  $A$  on  $r$ ,  $\delta$ , and  $\tau$  are not as simple. In general,  $A$  is an increasing function of  $r$ , but there are a couple of exceptions, for example, when  $\sigma = 0.2$ ,  $\delta = 0.08$  and  $\tau = 0.5$ . Also,  $A$  is generally decreasing in  $\delta$ , except for the first three rows. Finally,  $A$  is usually an increasing function of  $\tau$ , the parameter combination  $\sigma = 0.2$ ,  $r = 0.08$  and  $\delta = 0.02$  being one exception. Figure 2 gives us a more continuous look at the dependence of  $A$  on the parameters  $(\sigma, r, \delta, \tau)$ . The subplots in the figure reinforce the rough observation we made from Table 2 that  $A$  generally increases with  $r$ ,  $\sigma$  and  $\tau$ , but decreases with  $\delta$ . That  $A$  is increasing in  $r$  and decreasing in  $\delta$  agrees with our intuition. Recall that  $A$  can be considered a penalization for having to deliver the stock and receiving the cash at the same time. If  $r$  is much larger than  $\delta$ , then the difference of costs between delivering the stock at some random time and delivering it at maturity decreases. On the other hand, we are more likely to early exercise in order to receive the cash earlier. These two push the American option price closer to  $p(S, \tau, Ke^{r\tau})$  and therefore  $A$  becomes larger. However, the behavior of  $A$  with respect to  $\sigma$

and  $\tau$  is not intuitive to us. Thus, the numerical study of the interpolation parameter  $A$  adds our understanding on the interpolation method, initially studied in Johnson (1983).

We are now ready to take a look at the pricing performance for the American put options. Figure 3 plots the American put option prices as a function of the current stock price  $S$  (left subplots), as well as the approximation errors for the quadratic and interpolation methods (right subplots). In all the subplots,  $K = 100$ ,  $\sigma = 0.4$ , and  $\tau = 1$ . The pairs of interest rate  $r$  and dividend rate  $\delta$  used in the top, middle and bottom panels are  $(0.04, 0.04)$ ,  $(0.04, 0.08)$ , and  $(0.08, 0.04)$ , respectively. As we see, both the quadratic and interpolation methods are quite accurate. All three pricing curves seem to be on top of each other in the three left subplots. The right subplots give us a finer look at the pricing performance. The two methods give similar errors, with the interpolation method seems to be more accurate for out-of-the-money options. This is to be expected, since for out-of-the-money options, the two bounds in the interpolation get closer to each other.

Figure 4 allows us to take a quick look at the performance of the methods along a different dimension, namely, the time to maturity  $\tau$ . The left two subplots plot the option prices from the three methods, while the right subplots plot the pricing errors of the interpolation and quadratic approximation methods. In all the subplots,  $K = 100$ ,  $\sigma = 0.4$ , and  $r = \delta = 0.04$ . The current stock prices used in the top and bottom subplots are  $S = 85$  and  $S = 115$ , respectively. These two approximately represent out-of-the-money and in-the-money options. As we see, both the quadratic and interpolation methods are still quite accurate for  $\tau$  ranging from 0.5 to 3 years. However, at least for these parameters, the interpolation method performs better for longer-maturity options than the quadratic approximation. This is consistent with the findings in Ju and Zhong (1999) that the performance of the quadratic approximation deteriorates when  $\tau$  increases.

Tables 3, 4 and 5 take a more detailed look at the performance of the quadratic approximation and interpolation methods in pricing American put options by looking at short-term, medium-term and long-term options separately. More specifically, we fix the maturity  $\tau$  in Tables 3, 4 and 5 to be 0.5, 1 and 3 years, respectively. The interest and dividend rates are set to be either 0.02, 0.04, or 0.08. The volatilities are set to be either 0.2, 0.4 or 0.6. The strike  $K$  is fixed at 100. The current stock prices are chosen to be 85, 100, 115, representing roughly in-the-money, near-the-money and out-of-the-money put options. Table 3 shows that the performance of the two methods is extremely similar for short-term options, with the average percentage error for both methods within 0.5%. Tables 4 and 5 show that the accuracy of both methods decreases somewhat as the maturity  $\tau$  increases, especially for the quadratic approximation. In all three tables, the relatively larger errors for the interpolation method are usually associated with small  $\delta$ . This is to be expected, as the approximation in equation (14) gets better as  $\delta$  increases.

In general, the interpolation method performs better than the quadratic approximation for long-term options and out-of-the-money options. Although not reported in the tables, we also compute option prices for  $S = 130$  and  $145$ . For these stock prices, the interpolation method clearly outperforms the quadratic approximation.

## B. American option pricing in Heston's stochastic volatility model

We now take a look at the performance of our interpolation method for Heston's stochastic volatility model. For simplicity, we use the zeroth-order  $\sigma_v$ -expansion which amounts to replace  $\sigma$  with  $\tilde{\sigma}$  in equation (52). In the first part of our numerical study, we use the same parameters as in Ito and Toivanen (2006) and Medvedev and Scaillet (2009) in order to compare the accuracy. That is,  $r = 0.1$ ,  $\delta = 0$ ,  $\kappa_v = 5$  or  $2.5$ ,  $\theta_v = 0.16$ ,  $\sigma_v = 0.9$  or  $0.45$ ,  $\rho = 0.1$ . Time-to-maturity is 3 months (0.25), and strike price is set at 10. Two different instantaneous volatilities are considered:  $v_t = 0.0625$  and  $v_t = 0.25$ .

Before we look at the accuracy, it is worthwhile pointing out that our method takes about 0.02 seconds to price one option under the stochastic volatility model in MATLAB. The bulk of the computing time is spent computing the inverse Fourier transformations as each iteration would require the computation of two European option prices. Since the approximation in Medvedev and Scaillet (2009) takes about 0.1 seconds to compute one option, it seems that our method is about 5 times faster (some caution has to be taken with this claim as the comparison might not be fair due to possible different computer powers). Also, our method is not as “bulky”. The core algorithm of our method takes less than 20 lines of MATLAB code apart from the inverse Fourier transformation for computing the European option prices.

Table 6 reports the results for the performance of our interpolation method (coupled with a zeroth-order  $\sigma_v$ -expansion using the time-averaged expected variance) together with other existing methods under Heston's stochastic volatility model. The other models include: the numerical method in Ito and Toivanen (2006), the two 5-th order approximations in Medvedev and Scaillet (2009), and Monte-Carlo method (taken from Medvedev and Scaillet (2009)) in Longstaff and Schwartz (2000). These are abbreviated as Interp, IT, MS1, MS2, and MC, respectively. Also reported are the critical stock price, two European put option prices  $p(K)$  and  $p(Ke^{r\tau})$  used as lower and upper bounds, and the interpolation weight  $\alpha$  (which is not the interpolation parameter  $A$ ). The average absolute error (AAE) and average percentage error (APE) are also presented assuming that the prices from MC are the true prices. As we see from the table, our method is fairly accurate with an average percentage error of only 0.6%, and slightly more accurate than the approximation 1 in Medvedev and Scaillet (2009). It is also evident that for all the options considered, our method gives prices that are all downward biased, a result of the zeroth-order  $\sigma_v$ -expansion. Not reported in detail here, other choices of  $\tilde{\sigma}$  do not

change the result much. For example, using the instantaneous variance  $v_t$  itself or the Black-Scholes implied volatilities for  $p(K)$  or  $p(Ke^{r\tau})$  in place of  $\sigma$  results in average percentage errors of 0.8%, 0.7% and 0.7%, respectively. The approximation 2 in Medvedev and Scaillet (2009) seems to be more accurate than our interpolation method. The same is true for Ito and Toivanen (2006), although our method is faster than both of these two other methods. Also, one has to take caution that the parameters in this table are not very typical. For example, the magnitude of  $\rho$  and  $\tau$  are fairly small. Since Medvedev and Scaillet (2009) relies on a small- $\tau$  expansion while our method approaches the true perpetual American option price as maturity goes to infinity, it is possible that our method will have an advantage for long-maturity options. Also, for computational efficiency, we have coupled the interpolation approximation with a zeroth-order  $\sigma_v$ -expansion which understates the option prices. The accuracy of the interpolation method would be improved if we use numerical methods to compute the perpetual American option price and its partial derivative, although at the cost of lower computational efficiency.

The availability of an efficient and accurate quasi-analytical American option pricing algorithm under stochastic volatility model allows us to take a detailed look at the dependence of various quantities such as  $S_\tau^*$ ,  $A$ , and  $C(S, \tau, K)$  on parameters of interest such as  $v_t$ ,  $\sigma_v$ ,  $\kappa_v$ ,  $\rho$ ,  $\theta_v$ , etc. This has considerable value as a detailed understanding of these dependence properties is still lacking in the literature.

Figure 5 plots the critical stock prices for American put options under Heston's stochastic volatility model as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . Three different values of shock correlation  $\rho$  are plotted on each of the four subplots:  $-0.6$  (the curves marked with  $*$ ),  $0$  (the curves with no markers), and  $0.6$  (the curves marked with  $+$ ). The base parameter values are:  $K = 10$ ,  $r = 0.1$ ,  $\delta = 0$ ,  $v_t = 0.25$ ,  $\sigma_v = 0.7$ ,  $\kappa_v = 3$ , and  $\theta_v = 0.10$ . The subplots have no dependence on the current stock price  $S$  since the critical stock price is not a function of the current stock price. On each of the four subplots, one parameter (the  $x$ -axis) is relaxed to vary in a fixed region as shown in the subplots. For example, the instantaneous variance is allowed to vary from  $0.01$  to  $0.49$  (corresponding to instantaneous volatility going from  $10\%$  to  $70\%$ ). Notice that the regions are chosen to be quite wide in order to have a complete view. As we see in the figure, regardless of the sign of the correlation coefficient  $\rho$ , the critical stock price  $S_\tau^*$  is always decreasing in the instantaneous variance  $v_t$  and the long-run mean  $\theta_v$ . In particular,  $S_\tau^*$  is quite sensitive in  $v_t$ . This agrees with our intuition that when the volatility gets higher, holders of American put options become more reluctant to exercise the option and the stock price has to drop much lower in order to induce him to exercise the option early. Consistent with established results such as those in Lewis (2000), when  $\rho = 0$ , the volatility of variance parameter  $\sigma_v$  and mean-reverting strength  $\kappa_v$  have little effect on  $S_\tau^*$ . However,

when  $\rho$  is large in magnitude,  $S_\tau^*$  is increasing in both  $\sigma_v$  and  $\kappa_v$  if  $\rho < 0$ , and decreasing in both  $\sigma_v$  and  $\kappa_v$  if  $\rho > 0$ . This runs somewhat against our intuition that the effect of  $\sigma_v$  and  $\kappa_v$  should be opposite to each other, as one parameter tends to amplify the effect of stochastic volatility while the other tends to diminish it. Nonetheless in extensive numerical analysis we find that  $S_\tau^*$  behaves similarly as  $\sigma_v$  and  $\kappa_v$  change is a very robust property. For example, changing the relative magnitude of  $\theta_v$  and  $v_t$  does not affect this property at all. Finally, in all four subplots, the critical stock price  $S_\tau^*$  is smaller when  $\rho > 0$  than when  $\rho < 0$ . This agrees with our intuition. The reasoning goes as follows. Suppose  $\rho > 0$ . Then when the stock price  $S$  drops, the option tends to be more valuable. However, because of the positive correlation, there is a larger possibility that the instantaneous variance  $v_t$  would also drop, which reduces the option value. The situation is different when  $\rho < 0$ , where the effect of a drop in  $S$  is reinforced by an increase in  $v_t$ . This is the origin of the volatility skew in Heston's stochastic volatility model. Therefore, when  $\rho < 0$ , an American put option holder would tend to exercise more than if  $\rho > 0$ , resulting in a larger  $S_\tau^*$  for  $\rho < 0$ .

Figure 6 plots the interpolation parameter  $A$  for American put options under Heston's stochastic volatility model as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . The parameter values are the same as those in Figure 5. Recall that  $A$  is the interpolation weight  $\alpha$  if  $S = S_\tau^*$  and is not a function of the current stock price  $S$ . As we see, the interpolation parameter  $A$  is not very sensitive to all the parameters, especially when  $\rho = 0$ . This probably can explain partially why our interpolation method is so accurate given its simplicity. Again, the interpolation parameter is larger when  $\rho < 0$  than when  $\rho > 0$  on all four subplots. The interpolation parameter  $A$  is always increasing in  $v_t$  in order to produce higher early exercise premium when volatility gets larger. The dependence of  $A$  on  $\theta_v$  is not very strong. Also, if  $\rho < 0$ ,  $A$  is increasing in  $\sigma_v$  and  $\kappa_v$ . The situation is opposite when  $\rho > 0$ .

Figures 7 and 8 further plot the interpolation parameter  $A$  for American put options under Heston's stochastic volatility model as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . The difference is that in Figure 7 we set  $S = 9$  so that the options are “in-the-money”, while in Figure 8 we set  $S = 11$  so that the options are “out-of-the-money”. Comparing these two graphs, we see that the effect of  $\rho$  is completely opposite for in-the-money and out-of-the-money options. A negative  $\rho$  tends to increase the price of an out-of-the-money American put option while decrease the price of an in-the-money put option, a result well-known for the European put option case. See the original paper of Heston (1993) or page 70 of Lewis (2000). The American put option price on both graphs is increasing in  $v_t$  and  $\theta_v$ , which agrees well with our intuition. A larger  $\kappa$  always result in smaller American put option value regardless of the signs

of  $\rho$  and moneyness. On the other hand, for in-the-money options, the American put option price is increasing in  $\sigma_v$  if for larger values of positive  $\rho$  and decreasing in  $\sigma_v$  if  $\rho < 0$ . The effect is opposite for out-of-the-money options. Numerical study not shown here reveals that these dependence properties of American put option prices on the stochastic volatility parameters are exactly the same as those of the corresponding European put options. Thus, none of these properties constitute a surprise.

It is very useful to look at the dependence of the early exercise premium (*EEP*) on the parameters of the stochastic volatility model. Here we define the *EEP* to be the difference between the American put option price and its corresponding European option price. Figure 9 plots the early exercise premium as a function of various parameters for the in-the-money case with  $S = 9$ . A short way to summarize this graph is that the monotonicity dependence of the early exercise premium for in-the-money put option on various parameters is exactly the same as the dependence of the critical stock price on these parameters (compare Figure 9 with Figure 5). This rule of thumb should be very useful in practice. Another very notable property is that *EEP* decreases with the instantaneous variance  $v_t$  if the option is in the money. With lower current volatility, the American feature of being able to exercise the option early becomes more valuable if the American option is in the money. However, the behavior of the *EEP* for out-of-the-money American puts is quite different, as shown in Figure 10. In this case, the dependence of the *EEP* on various parameters is exactly the same as the dependence of  $P$  on these parameters. In particular, the early exercise premium is now an increasing function of  $v_t$ . That the *EEP* is increasing in  $v_t$  if the put option is out-of-the-money and decreasing in  $v_t$  if in-the-money agrees with our intuition.<sup>7</sup>

## IV. Conclusion

The pricing of American option is one of the oldest problems faced by financial economists and practitioners. In this paper, we present a new quasi-analytical method for pricing American options based on interpolating two European option prices that under usual situations serve as the lower and upper bounds for the American option. The weight of the interpolation is derived analytically using a method similar to the quadratic approximation in Barone-Adesi and Whaley (1987) and is found to be approximately proportional to the perpetual American option price.

The interpolation method achieves similar efficiency as Barone-Adesi and Whaley's quadratic approximation, and is thus much faster than most other methods. In our numerical study on the Black-Scholes setup, we demonstrate that the above two methods also have about the same

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<sup>7</sup>Some caution also has to be taken since the early exercise premiums for out-of-the-money options are fairly small and sometimes comparable to the accuracy of our interpolation method. However, the monotonicity properties should still be quite reliable.

accuracy in many cases. However, our extensive numerical study shows that our method is generally more accurate for out-of-the-money and long-maturity options.

We also take a look at the dependence of the interpolation weight on various parameters, thus fulfilling a task not carried out completely in Johnson (1983). We found that the interpolation generally puts more weight on the upper bound as the interest rate, or the volatility, or time to maturity increases, but less weight as the dividend rate increases.

Finally, we carry out a numerical study for the stochastic volatility model. Our interpolation method coupled with a zeroth-order volatility of variance expansion gives very accuracy results and is faster than other methods such as Ito and Toivanen (2006) and Medvedev and Scaillet (2009).

Our method at its current form has some shortcomings, notably the inability to handle stochastic interest rates because in this case it is not clear how to choose a good upper bound. However, our method has quite a few distinctive and attractive features other than a good combination of efficiency and accuracy. The method is not as “bulky” as some other methods and the core algorithm requires less than 20 lines of code. There is really not much room for even a mediocre programmer to make mistakes with so few lines. Unlike many other methods, our method can be adapted to general diffusion processes as long as efficient pricing algorithms exist for the corresponding European and perpetual American options. The final approximated American price always lies within two arbitrage bounds and has some nice asymptotic properties which adds considerable robustness to our method. The method also allows us to compute the greeks and critical stock prices at the same time.

There are a few directions one can take to further the study we have done. Since we have examined only geometric Brownian motion and Heston’s stochastic volatility models in our numerical study, it would be useful to look at other diffusion processes in future research, for example, the CEV process. Also, the accuracy of our method might be improved. One possibility is to search for a better parametric form for  $\Phi$ . However, this runs the risk of overfitting and might not perform well out of the sample. The second approach is to reexamine our approximation of the valuation PDE and make it more accurate. Finally, it might be useful if one can come out with bounds which are tighter than the ones we used. If the forms are still simple and analytically tractable, then it might be possible to design a new approximation which is more accurate by mimicking the approach we have taken.

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**Table 1**  
**Critical Stock Prices of American Puts in the Black-Scholes Model**

This table presents the critical stock prices as a function of volatility  $\sigma$ , interest rate  $r$ , dividend rate  $\delta$ , and time to maturity  $\tau$  in the Black-Scholes model. Three maturities are chosen, namely, 0.5, 1, and 3 years. The interest and dividend rates are set to be either 0.02, 0.04, or 0.08. The volatilities are set to be either 0.2, 0.4 or 0.6. The strike  $K$  is fixed at 100. Results from three methods are presented: the binomial tree method with 10000 time steps (treated as the true values), the quadratic approximation in Barone-Adesi and Whaley (1987), and the interpolation method in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented.

$(\sigma, r, \delta)$	$\tau = 0.5$			$\tau = 1$			$\tau = 3$		
	Binom	Quad	Interp	Binom	Quad	Interp	Binom	Quad	Interp
0.2, 0.02, 0.02	72.35	73.34	72.97	66.31	67.33	67.05	56.21	56.96	57.00
0.2, 0.02, 0.04	45.86	45.26	45.23	44.32	43.46	43.72	40.39	39.10	40.03
0.2, 0.02, 0.08	23.06	22.62	23.50	22.42	21.72	23.28	21.16	19.71	23.21
0.2, 0.04, 0.02	79.79	80.75	80.67	75.48	76.44	76.53	68.51	69.16	69.78
0.2, 0.04, 0.04	74.78	75.59	75.48	69.60	70.35	70.38	61.34	61.57	62.03
0.2, 0.04, 0.08	45.99	45.27	45.65	44.60	43.50	44.22	41.53	39.53	41.06
0.2, 0.08, 0.02	85.24	85.86	86.16	82.72	83.28	83.91	79.25	79.56	81.03
0.2, 0.08, 0.04	83.40	84.09	84.31	80.36	80.97	81.46	75.96	76.19	77.40
0.2, 0.08, 0.08	77.38	77.98	78.06	73.16	73.57	73.84	66.94	66.58	67.32
0.4, 0.02, 0.02	52.42	53.84	52.68	44.11	45.44	44.45	31.98	32.73	32.18
0.4, 0.02, 0.04	41.07	40.50	39.39	35.80	35.60	35.09	26.36	26.13	26.49
0.4, 0.02, 0.08	21.10	20.51	21.02	19.78	18.95	20.32	16.87	15.65	18.92
0.4, 0.04, 0.02	60.08	61.62	60.95	52.79	54.20	53.68	42.07	42.75	42.66
0.4, 0.04, 0.04	55.98	57.19	56.62	48.57	49.59	49.23	37.98	38.18	38.34
0.4, 0.04, 0.08	41.63	40.79	40.73	37.42	36.70	37.11	29.73	28.70	29.99
0.4, 0.08, 0.02	67.28	68.54	68.39	61.72	62.78	62.94	53.93	54.19	55.15
0.4, 0.08, 0.04	65.20	66.43	66.28	59.31	60.30	60.42	51.10	51.17	52.01
0.4, 0.08, 0.08	59.94	60.85	60.73	53.64	54.22	54.38	45.12	44.58	45.41
0.6, 0.02, 0.02	38.07	39.60	37.73	29.52	30.80	29.24	18.61	19.13	18.17
0.6, 0.02, 0.04	32.53	32.86	31.08	25.49	25.91	24.83	16.21	16.25	16.09
0.6, 0.02, 0.08	19.28	18.58	18.31	17.07	16.32	16.97	12.14	11.49	13.65
0.6, 0.04, 0.02	44.94	46.64	45.41	36.76	38.15	37.10	26.06	26.47	25.81
0.6, 0.04, 0.04	42.00	43.33	42.26	34.07	35.09	34.28	23.93	24.02	23.74
0.6, 0.04, 0.08	34.46	34.38	33.79	28.24	28.20	28.17	20.00	19.42	20.26
0.6, 0.08, 0.02	52.29	53.82	53.18	45.26	46.42	46.00	36.13	36.15	36.26
0.6, 0.08, 0.04	50.52	51.94	51.35	43.41	44.42	44.07	34.32	34.15	34.33
0.6, 0.08, 0.08	46.51	47.54	47.12	39.50	40.08	39.96	30.81	30.19	30.67
AAE		0.95	0.74		0.84	0.65		0.57	0.68
APE		0.02	0.02		0.02	0.01		0.02	0.02

**Table 2**  
**The Interpolation Parameter  $A$  in the Black-Scholes Model**

This table presents the weight  $A$  in our interpolation method as a function of volatility  $\sigma$ , interest rate  $r$ , and dividend rate  $\delta$  in the Black-Scholes model. The interest rates and dividend rates are set to be either 0.02, 0.04, or 0.08. The volatilities are set to be either 0.2, 0.4 or 0.6. Three times to maturities, 0.5, 1 and 3 years, are presented. For all the options, the strike price  $K$  is fixed at 100. Results from the binomial tree method with 10000 time steps are treated as the true values. The average absolute error (AAE) and average percentage error (APE) are also presented.

$(\sigma, r, \delta)$	$\tau = 0.5$		$\tau = 1$		$\tau = 3$	
	Binom	Interp	Binom	Interp	Binom	Interp
0.2, 0.02, 0.02	0.233	0.218	0.282	0.265	0.365	0.348
0.2, 0.02, 0.04	0.087	0.100	0.122	0.134	0.213	0.220
0.2, 0.02, 0.08	0.091	0.074	0.129	0.096	0.225	0.150
0.2, 0.04, 0.02	0.447	0.410	0.457	0.421	0.473	0.443
0.2, 0.04, 0.04	0.211	0.195	0.253	0.237	0.324	0.312
0.2, 0.04, 0.08	0.089	0.096	0.125	0.133	0.216	0.225
0.2, 0.08, 0.02	0.538	0.489	0.528	0.480	0.505	0.457
0.2, 0.08, 0.04	0.414	0.377	0.416	0.381	0.416	0.384
0.2, 0.08, 0.08	0.188	0.175	0.224	0.211	0.281	0.275
0.4, 0.02, 0.02	0.401	0.394	0.467	0.459	0.566	0.561
0.4, 0.02, 0.04	0.179	0.214	0.276	0.292	0.448	0.444
0.4, 0.02, 0.08	0.169	0.172	0.232	0.211	0.380	0.302
0.4, 0.04, 0.02	0.544	0.517	0.568	0.543	0.606	0.593
0.4, 0.04, 0.04	0.368	0.353	0.428	0.413	0.519	0.512
0.4, 0.04, 0.08	0.174	0.192	0.257	0.264	0.415	0.409
0.4, 0.08, 0.02	0.605	0.568	0.607	0.574	0.611	0.587
0.4, 0.08, 0.04	0.508	0.476	0.526	0.499	0.555	0.538
0.4, 0.08, 0.08	0.334	0.316	0.387	0.372	0.469	0.464
0.6, 0.02, 0.02	0.522	0.532	0.590	0.598	0.678	0.691
0.6, 0.02, 0.04	0.326	0.363	0.446	0.466	0.608	0.612
0.6, 0.02, 0.08	0.240	0.278	0.334	0.338	0.533	0.468
0.6, 0.04, 0.02	0.615	0.602	0.644	0.635	0.686	0.692
0.6, 0.04, 0.04	0.485	0.478	0.548	0.543	0.637	0.641
0.6, 0.04, 0.08	0.298	0.315	0.410	0.412	0.568	0.561
0.6, 0.08, 0.02	0.653	0.627	0.660	0.642	0.675	0.673
0.6, 0.08, 0.04	0.578	0.556	0.603	0.587	0.642	0.642
0.6, 0.08, 0.08	0.445	0.431	0.504	0.494	0.590	0.592
AAE		0.021		0.018		0.017
APE		0.07		0.05		0.05

**Table 3**  
**Short-term American Put Option Prices in the Black-Scholes Model ( $\tau = 0.5$ )**

This table presents the American put option prices as a function of volatility  $\sigma$ , interest rate  $r$ , dividend rate  $\delta$ , and current stock price  $S$  in the Black-Scholes model. The interest rates and dividend rates are set to be either 0.02, 0.04, or 0.08. The volatilities are set to be either 0.2, 0.4 or 0.6. For all the options, the strike price  $K$  is fixed at 100 and time to maturity 0.5 year. Results from three methods are presented: the binomial tree method with 10000 time steps (treated as the true values), the quadratic approximation in Barone-Adesi and Whaley (1987), and the interpolation method in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented.

$(\sigma, r, \delta)$	$S = 85$			$S = 100$			$S = 115$		
	Binom	Quad	Interp	Binom	Quad	Interp	Binom	Quad	Interp
0.2, 0.02, 0.02	15.71	15.70	15.71	5.59	5.59	5.59	1.28	1.28	1.28
0.2, 0.02, 0.04	16.38	16.38	16.38	6.06	6.06	6.06	1.45	1.45	1.45
0.2, 0.02, 0.08	17.84	17.84	17.84	7.08	7.08	7.08	1.85	1.85	1.85
0.2, 0.04, 0.02	15.29	15.24	15.23	5.17	5.16	5.14	1.12	1.13	1.12
0.2, 0.04, 0.04	15.63	15.62	15.62	5.55	5.55	5.54	1.27	1.27	1.27
0.2, 0.04, 0.08	16.94	16.94	16.94	6.50	6.50	6.50	1.63	1.63	1.63
0.2, 0.08, 0.02	15.00	15.00	15.00	4.47	4.47	4.37	0.87	0.89	0.84
0.2, 0.08, 0.04	15.05	15.02	15.01	4.77	4.77	4.70	0.98	1.00	0.96
0.2, 0.08, 0.08	15.50	15.48	15.47	5.46	5.48	5.46	1.24	1.26	1.24
0.4, 0.02, 0.02	19.41	19.41	19.42	11.15	11.16	11.16	5.97	5.98	5.97
0.4, 0.02, 0.04	19.93	19.94	19.94	11.58	11.58	11.58	6.27	6.27	6.27
0.4, 0.02, 0.08	21.07	21.07	21.07	12.49	12.49	12.49	6.91	6.91	6.91
0.4, 0.04, 0.02	18.86	18.83	18.83	10.69	10.69	10.68	5.66	5.67	5.65
0.4, 0.04, 0.04	19.28	19.28	19.28	11.06	11.08	11.07	5.92	5.93	5.92
0.4, 0.04, 0.08	20.30	20.31	20.31	11.91	11.92	11.92	6.52	6.52	6.52
0.4, 0.08, 0.02	17.95	17.89	17.84	9.88	9.89	9.81	5.09	5.12	5.05
0.4, 0.08, 0.04	18.29	18.25	18.20	10.20	10.22	10.15	5.32	5.35	5.29
0.4, 0.08, 0.08	19.04	19.05	19.02	10.90	10.93	10.89	5.82	5.85	5.82
0.6, 0.02, 0.02	23.96	23.96	23.98	16.66	16.66	16.68	11.42	11.42	11.43
0.6, 0.02, 0.04	24.39	24.41	24.42	17.05	17.06	17.06	11.74	11.75	11.75
0.6, 0.02, 0.08	25.35	25.36	25.36	17.89	17.89	17.89	12.43	12.43	12.43
0.6, 0.04, 0.02	23.40	23.38	23.40	16.18	16.18	16.18	11.02	11.03	11.02
0.6, 0.04, 0.04	23.79	23.79	23.80	16.53	16.55	16.54	11.32	11.34	11.33
0.6, 0.04, 0.08	24.63	24.67	24.66	17.30	17.32	17.32	11.96	11.98	11.97
0.6, 0.08, 0.02	22.42	22.38	22.36	15.29	15.31	15.25	10.30	10.34	10.27
0.6, 0.08, 0.04	22.76	22.74	22.71	15.61	15.64	15.58	10.57	10.61	10.55
0.6, 0.08, 0.08	23.47	23.50	23.47	16.28	16.33	16.28	11.14	11.19	11.14
AAE	0.02	0.02		0.01	0.02		0.01	0.01	
APE	0.001	0.001		0.001	0.003		0.004	0.003	

**Table 4**  
**Medium-term American Put Option Prices in the Black-Scholes Model ( $\tau = 1$ )**

This table presents the American put option prices as a function of volatility  $\sigma$ , interest rate  $r$ , dividend rate  $\delta$ , and current stock price  $S$  in the Black-Scholes model. The interest rates and dividend rates are set to be either 0.02, 0.04, or 0.08. The volatilities are set to be either 0.2, 0.4 or 0.6. For all the options, the strike price  $K$  is fixed at 100 and time to maturity 1 year. Results from three methods are presented: the binomial tree method with 10000 time steps (treated as the true values), the quadratic approximation in Barone-Adesi and Whaley (1987), and the interpolation method in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented.

$(\sigma, r, \delta)$	$S = 85$			$S = 100$			$S = 115$		
	Binom	Quad	Interp	Binom	Quad	Interp	Binom	Quad	Interp
0.2, 0.02, 0.02	16.94	16.93	16.93	7.84	7.84	7.83	3.01	3.02	3.01
0.2, 0.02, 0.04	18.10	18.11	18.11	8.74	8.74	8.74	3.51	3.51	3.51
0.2, 0.02, 0.08	20.73	20.73	20.73	10.77	10.77	10.77	4.70	4.70	4.70
0.2, 0.04, 0.02	16.04	15.98	15.94	7.02	7.02	6.96	2.56	2.58	2.53
0.2, 0.04, 0.04	16.75	16.75	16.72	7.72	7.74	7.71	2.96	2.98	2.95
0.2, 0.04, 0.08	19.02	19.03	19.03	9.54	9.54	9.54	3.99	3.99	3.99
0.2, 0.08, 0.02	15.13	15.08	15.03	5.74	5.76	5.55	1.85	1.92	1.78
0.2, 0.08, 0.04	15.40	15.34	15.27	6.26	6.29	6.13	2.14	2.20	2.09
0.2, 0.08, 0.08	16.43	16.45	16.39	7.50	7.56	7.49	2.86	2.91	2.86
0.4, 0.02, 0.02	22.99	23.00	23.00	15.60	15.61	15.61	10.38	10.39	10.38
0.4, 0.02, 0.04	23.86	23.89	23.89	16.38	16.40	16.39	11.02	11.03	11.03
0.4, 0.02, 0.08	25.84	25.85	25.84	18.11	18.11	18.11	12.43	12.43	12.43
0.4, 0.04, 0.02	21.97	21.95	21.91	14.69	14.71	14.65	9.65	9.69	9.62
0.4, 0.04, 0.04	22.69	22.71	22.68	15.36	15.41	15.36	10.21	10.25	10.21
0.4, 0.04, 0.08	24.37	24.44	24.41	16.90	16.95	16.92	11.48	11.52	11.50
0.4, 0.08, 0.02	20.30	20.27	20.11	13.14	13.19	12.99	8.39	8.49	8.29
0.4, 0.08, 0.04	20.87	20.87	20.73	13.70	13.77	13.60	8.86	8.96	8.79
0.4, 0.08, 0.08	22.14	22.22	22.11	14.93	15.04	14.92	9.90	10.01	9.89
0.6, 0.02, 0.02	29.63	29.64	29.67	23.20	23.23	23.24	18.21	18.23	18.24
0.6, 0.02, 0.04	30.35	30.40	30.40	23.89	23.94	23.94	18.84	18.88	18.87
0.6, 0.02, 0.08	31.96	31.99	31.97	25.41	25.43	25.42	20.21	20.23	20.22
0.6, 0.04, 0.02	28.57	28.58	28.58	22.23	22.27	22.24	17.34	17.40	17.35
0.6, 0.04, 0.04	29.21	29.27	29.25	22.86	22.92	22.89	17.92	17.99	17.94
0.6, 0.04, 0.08	30.61	30.72	30.67	24.20	24.30	24.25	19.16	19.24	19.20
0.6, 0.08, 0.02	26.75	26.78	26.66	20.51	20.61	20.44	15.81	15.94	15.75
0.6, 0.08, 0.04	27.30	27.37	27.25	21.06	21.18	21.02	16.32	16.46	16.28
0.6, 0.08, 0.08	28.47	28.61	28.48	22.21	22.38	22.23	17.39	17.55	17.40
AAE		0.04	0.05		0.05	0.04		0.05	0.02
APE		0.002	0.002		0.003	0.004		0.007	0.004

**Table 5**  
**Long-term American Put Option Prices in the Black-Scholes Model ( $\tau = 3$ )**

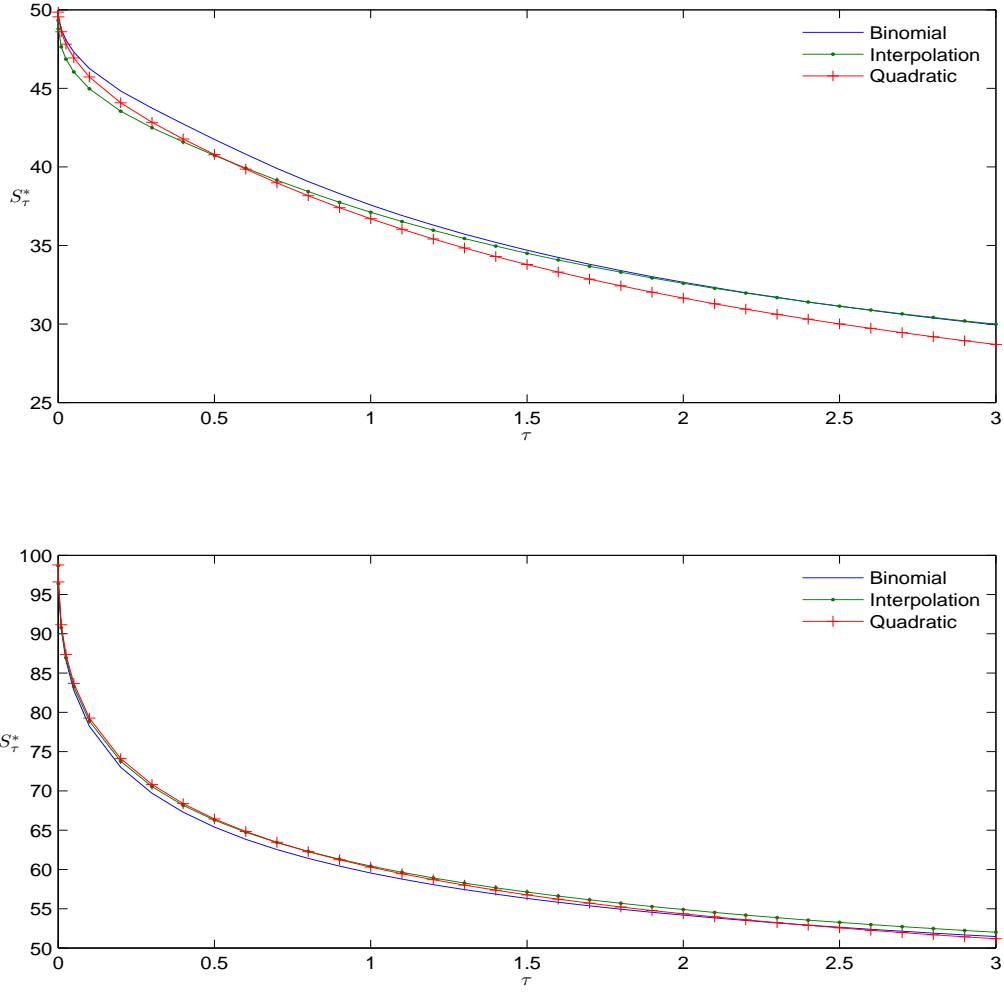
This table presents the American put option prices as a function of volatility  $\sigma$ , interest rate  $r$ , dividend rate  $\delta$ , and current stock price  $S$  in the Black-Scholes model. The interest rates and dividend rates are set to be either 0.02, 0.04, or 0.08. The volatilities are set to be either 0.2, 0.4 or 0.6. For all the options, the strike price  $K$  is fixed at 100 and time to maturity 3 years. Results from three methods are presented: the binomial tree method with 10000 time steps (treated as the true values), the quadratic approximation in Barone-Adesi and Whaley (1987), and the interpolation method in this paper. The average absolute error (AAE) and average percentage error (APE) are also presented.

$(\sigma, r, \delta)$	$S = 85$			$S = 100$			$S = 115$		
	Binom	Quad	Interp	Binom	Quad	Interp	Binom	Quad	Interp
0.2, 0.02, 0.02	20.73	20.77	20.71	13.13	13.20	13.12	8.05	8.11	8.04
0.2, 0.02, 0.04	23.37	23.45	23.41	15.51	15.57	15.53	9.93	9.97	9.94
0.2, 0.02, 0.08	29.63	29.67	29.63	21.19	21.21	21.19	14.64	14.65	14.64
0.2, 0.04, 0.02	18.36	18.36	18.14	10.89	11.00	10.72	6.30	6.44	6.19
0.2, 0.04, 0.04	20.06	20.18	20.02	12.60	12.76	12.59	7.69	7.84	7.67
0.2, 0.04, 0.08	24.98	25.22	25.10	17.21	17.38	17.28	11.46	11.57	11.50
0.2, 0.08, 0.02	15.80	15.75	15.43	7.86	7.95	7.35	3.94	4.11	3.62
0.2, 0.08, 0.04	16.57	16.58	16.24	8.94	9.11	8.60	4.80	5.02	4.58
0.2, 0.08, 0.08	19.00	19.24	18.98	11.70	12.03	11.71	7.05	7.36	7.06
0.4, 0.02, 0.02	31.83	31.94	31.88	25.89	26.02	25.93	21.16	21.30	21.20
0.4, 0.02, 0.04	33.71	33.92	33.80	27.74	27.94	27.82	22.93	23.11	22.99
0.4, 0.02, 0.08	37.99	38.20	38.00	31.95	32.12	31.95	26.92	27.07	26.93
0.4, 0.04, 0.02	29.13	29.30	29.06	23.30	23.53	23.24	18.77	19.04	18.73
0.4, 0.04, 0.04	30.67	30.95	30.70	24.85	25.17	24.89	20.26	20.59	20.29
0.4, 0.04, 0.08	34.14	34.62	34.29	28.33	28.79	28.47	23.61	24.02	23.72
0.4, 0.08, 0.02	25.07	25.32	24.70	19.27	19.64	18.93	15.02	15.46	14.72
0.4, 0.08, 0.04	26.22	26.56	25.99	20.47	20.92	20.26	16.17	16.67	15.99
0.4, 0.08, 0.08	28.76	29.33	28.79	23.09	23.73	23.14	18.70	19.36	18.75
0.6, 0.02, 0.02	42.71	42.90	42.90	37.92	38.12	38.09	33.88	34.09	34.04
0.6, 0.02, 0.04	44.19	44.47	44.36	39.43	39.71	39.58	35.39	35.67	35.53
0.6, 0.02, 0.08	47.35	47.72	47.35	42.65	43.00	42.66	38.62	38.94	38.63
0.6, 0.04, 0.02	39.84	40.16	40.00	35.09	35.47	35.25	31.15	31.55	31.29
0.6, 0.04, 0.04	41.13	41.56	41.33	36.42	36.89	36.61	32.48	32.97	32.66
0.6, 0.04, 0.08	43.84	44.47	44.03	39.21	39.86	39.40	35.29	35.93	35.47
0.6, 0.08, 0.02	35.24	35.78	35.23	30.49	31.13	30.48	26.66	27.36	26.65
0.6, 0.08, 0.04	36.30	36.94	36.37	31.59	32.32	31.67	27.77	28.55	27.84
0.6, 0.08, 0.08	38.49	39.35	38.71	33.88	34.81	34.10	30.08	31.05	30.29
AAE		0.29	0.13		0.32	0.12		0.34	0.10
APE		0.009	0.005		0.013	0.008		0.019	0.009

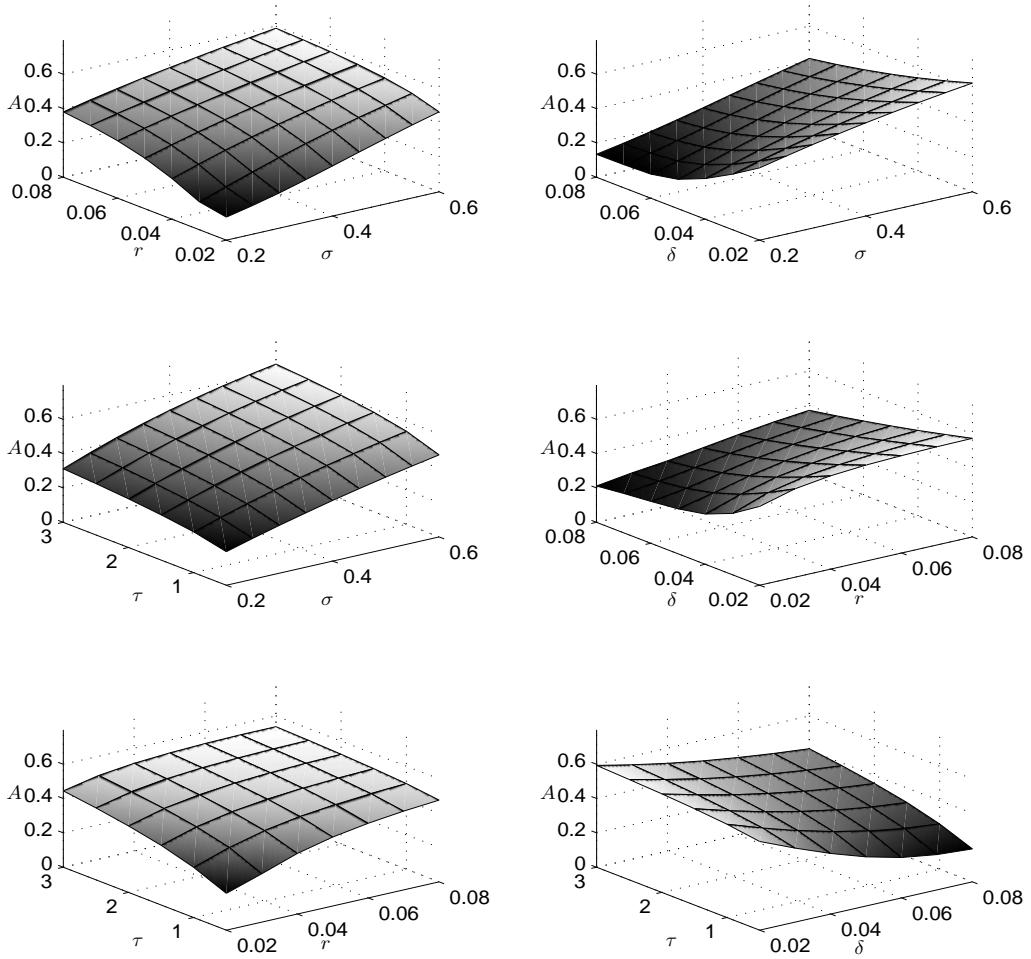
**Table 6**  
**Performance of the Interpolation Method under Stochastic Volatility Model**

This table presents the performance of the interpolation method (coupled with a zeroth-order  $\sigma_v$ -expansion) in this paper together with other existing methods under Heston's stochastic volatility model. The other models include: the numerical method in Ito and Toivanen (2006), the two 5-th order approximations in Medvedev and Scaillet (2009), and Monte-Carlo method (taken from Medvedev and Scaillet (2009)) in Longstaff and Schwartz (2000). These are abbreviated as Interp, IT, MS1, MS2, and MC, respectively. Also reported are the critical stock price, two European put option prices  $p(K)$  and  $p(Ke^{r\tau})$ , and the interpolation weight  $\alpha$ . We use the same parameters as in Ito and Toivanen (2006) and Medvedev and Scaillet (2009). That is,  $r = 0.1$ ,  $\delta = 0$ ,  $\kappa_v = 5$  or  $2.5$ ,  $\theta_v = 0.16$ ,  $\sigma_v = 0.9$  or  $0.45$ ,  $\rho = 0.1$ . Time-to-maturity is 3 months (0.25), and strike price is set at 10. Two different instantaneous volatilities are considered:  $v_t = 0.0625$  (L) and  $v_t = 0.25$  (H). The average absolute error (AAE) and average percentage error (APE) are also presented assuming that the prices from MC are the true prices.

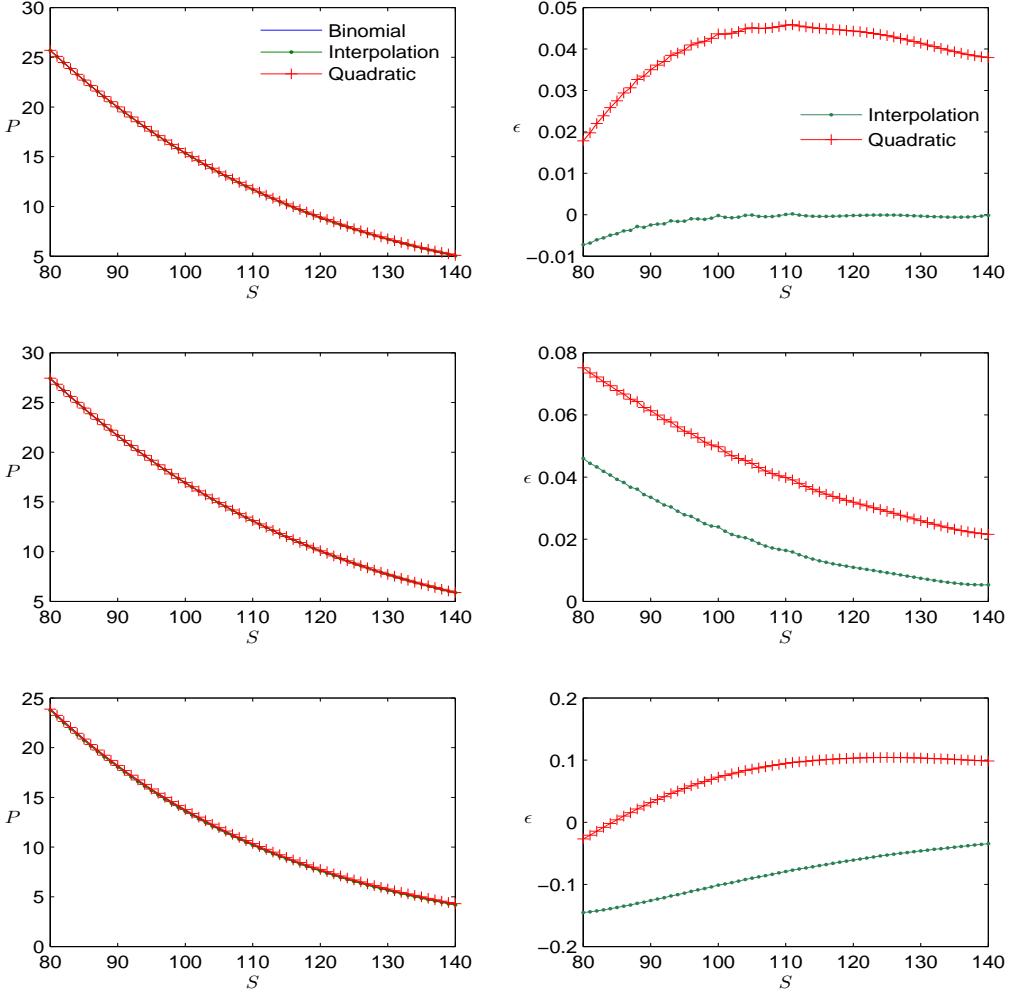
$(v_t, \sigma_v, \kappa_v, S)$	$S_\tau^*$	$p(K)$	$p(Ke^{r\tau})$	$\alpha$	Methods				
					Interp	MS1	MS2	IT	MC
L, 0.90, 5.0, 9	8.170	1.048	1.236	0.290	1.103	1.081	1.111	1.108	1.107
L, 0.45, 2.5, 9	8.371	1.009	1.198	0.333	1.072	1.072	1.077	1.077	1.075
H, 0.90, 5.0, 9	7.236	1.280	1.455	0.209	1.321	1.349	1.333	1.333	1.334
H, 0.45, 2.5, 9	7.147	1.315	1.487	0.212	1.353	1.365	1.362	1.364	1.363
L, 0.90, 5.0, 10	8.170	0.502	0.627	0.127	0.517	0.534	0.521	0.520	0.521
L, 0.45, 2.5, 10	8.371	0.459	0.582	0.132	0.475	0.475	0.478	0.479	0.478
H, 0.90, 5.0, 10	7.236	0.770	0.900	0.121	0.786	0.811	0.796	0.796	0.793
H, 0.45, 2.5, 10	7.147	0.812	0.943	0.126	0.829	0.838	0.836	0.837	0.837
L, 0.90, 5.0, 11	8.170	0.208	0.274	0.060	0.212	0.208	0.214	0.214	0.214
L, 0.45, 2.5, 11	8.371	0.173	0.235	0.057	0.177	0.174	0.178	0.178	0.177
H, 0.90, 5.0, 11	7.236	0.436	0.524	0.074	0.443	0.463	0.448	0.448	0.451
H, 0.45, 2.5, 11	7.147	0.475	0.566	0.079	0.482	0.488	0.487	0.487	0.488
AAE					0.005	0.008	0.001	0.001	
APE					0.006	0.014	0.002	0.002	



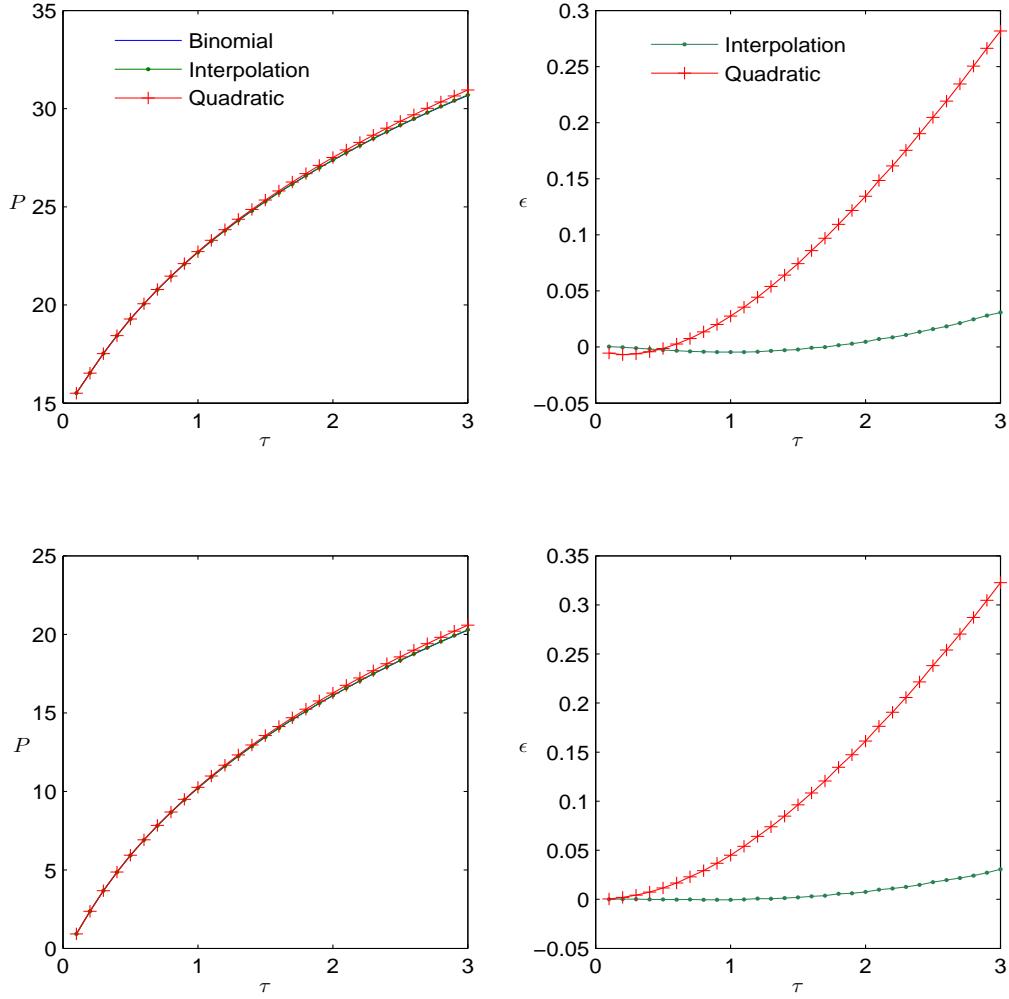
**Figure 1.** The critical stock price  $S_\tau^*$  for American put options as a function of  $\tau$  in the Black-Scholes model. In both subplots, we set  $K = 100$ , and  $\sigma = 0.4$ . The top subplot is for  $r = 0.04$ , and  $\delta = 0.08$ , while the bottom subplot uses  $r = 0.08$  and  $\delta = 0.04$ . The horizontal axis is measured in years. The results from three methods are plotted: the binomial tree model with 10000 time steps which is treated as the true value, the interpolation method proposed in this paper, and the quadratic approximation used in Barone-Adesi and Whaley (1987) and Ju and Zhong (1999).



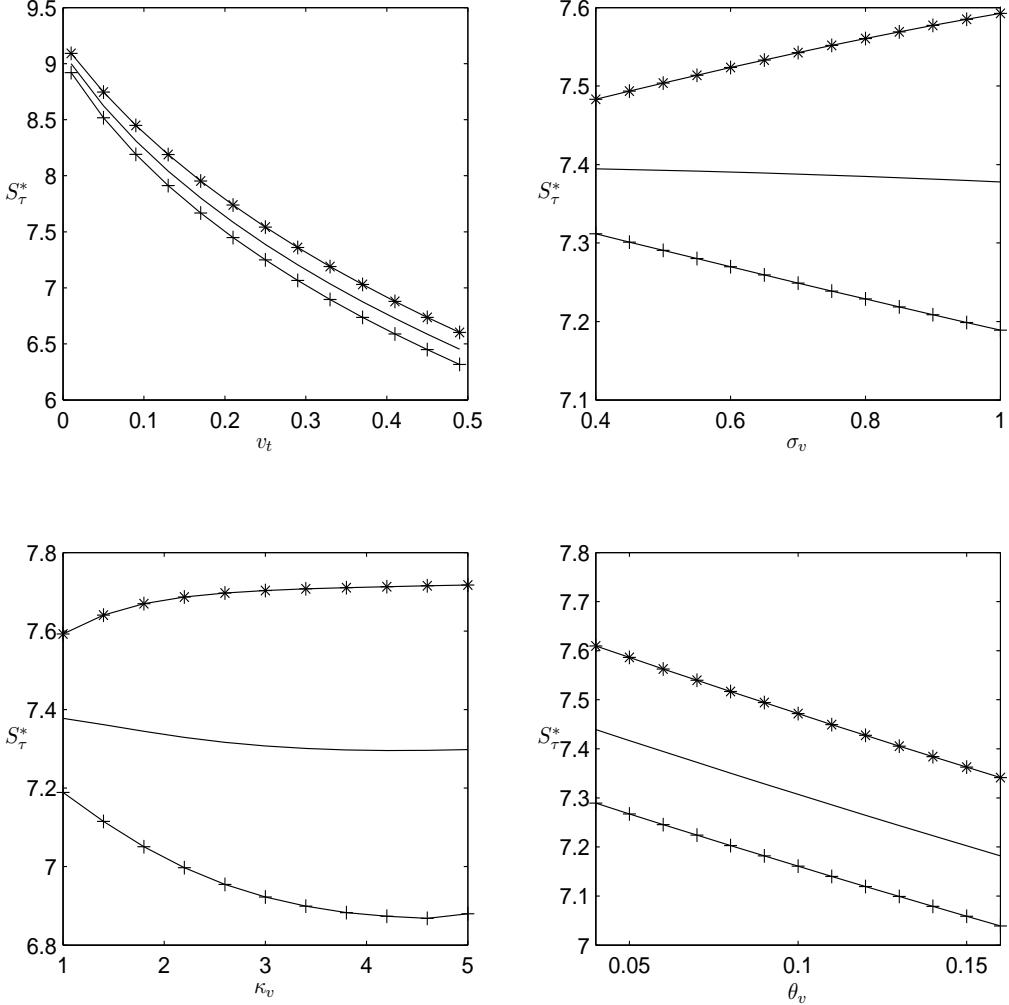
**Figure 2.** The interpolation weight  $A$  as a function of  $\sigma, r, \delta$  and  $\tau$  in the Black-Scholes model. In all six subplots, the interpolation weight from the interpolation method is plotted. The benchmark parameter vector is  $\sigma = 0.4, r = \delta = 0.04$  and  $\tau = 1$ . In each of the six subplots, two parameters are relaxed while fixing the other two parameters. The relaxed parameters are drawn in the subplots as  $x$ -axis and  $y$ -axis.



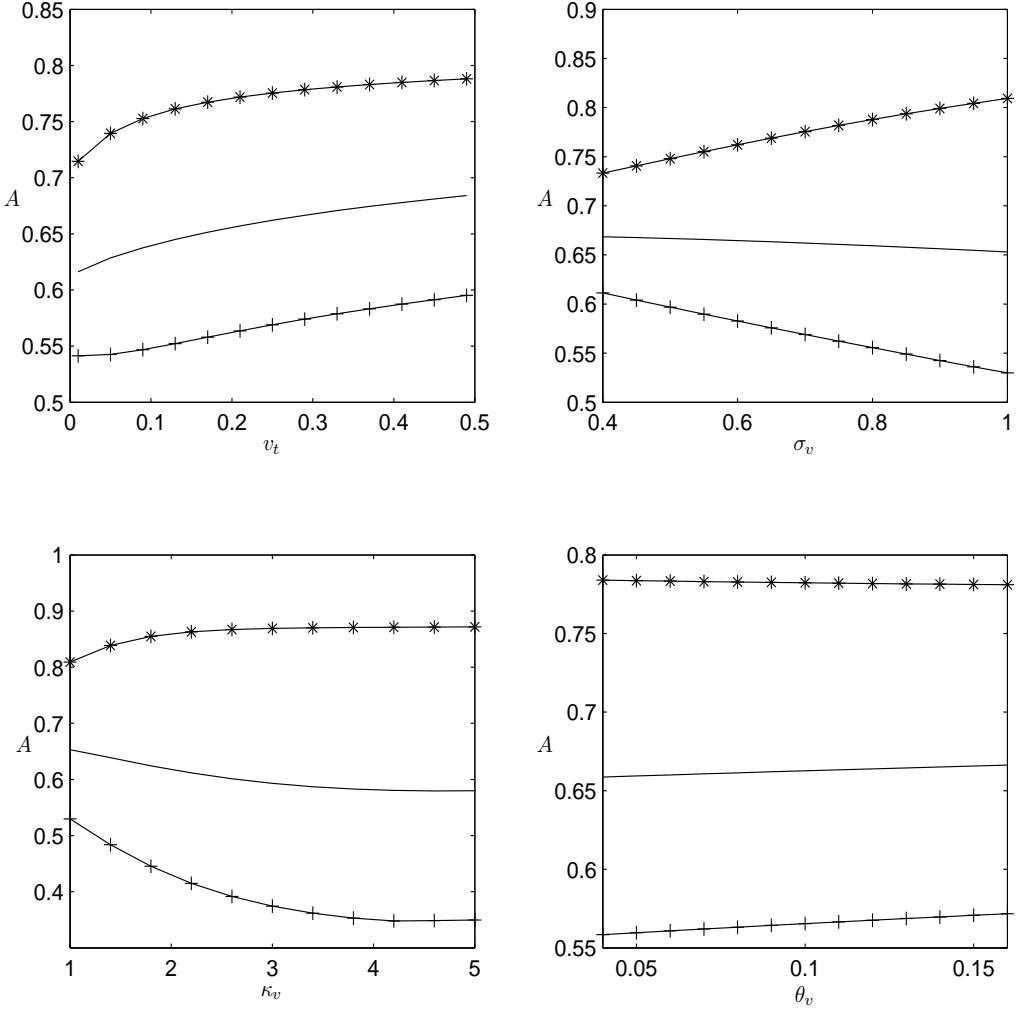
**Figure 3.** The American put option prices and absolute approximation errors as functions of the current stock price  $S$  in the Black-Scholes model. In all the subplots,  $K = 100$ ,  $\sigma = 0.4$ , and  $\tau = 1$ . The pairs of interest rate  $r$  and dividend rate  $\delta$  used in the top, middle and bottom panels are  $(0.04, 0.04)$ ,  $(0.04, 0.08)$ , and  $(0.08, 0.04)$ , respectively. Three methods are used: the binomial tree model with 10000 time steps which is treated as the true value, the interpolation method proposed in this paper, and the quadratic approximation used in Barone-Adesi and Whaley (1987). The left three subplots plot the option prices from the three methods, while the right subplots plot the pricing errors of the last two methods.



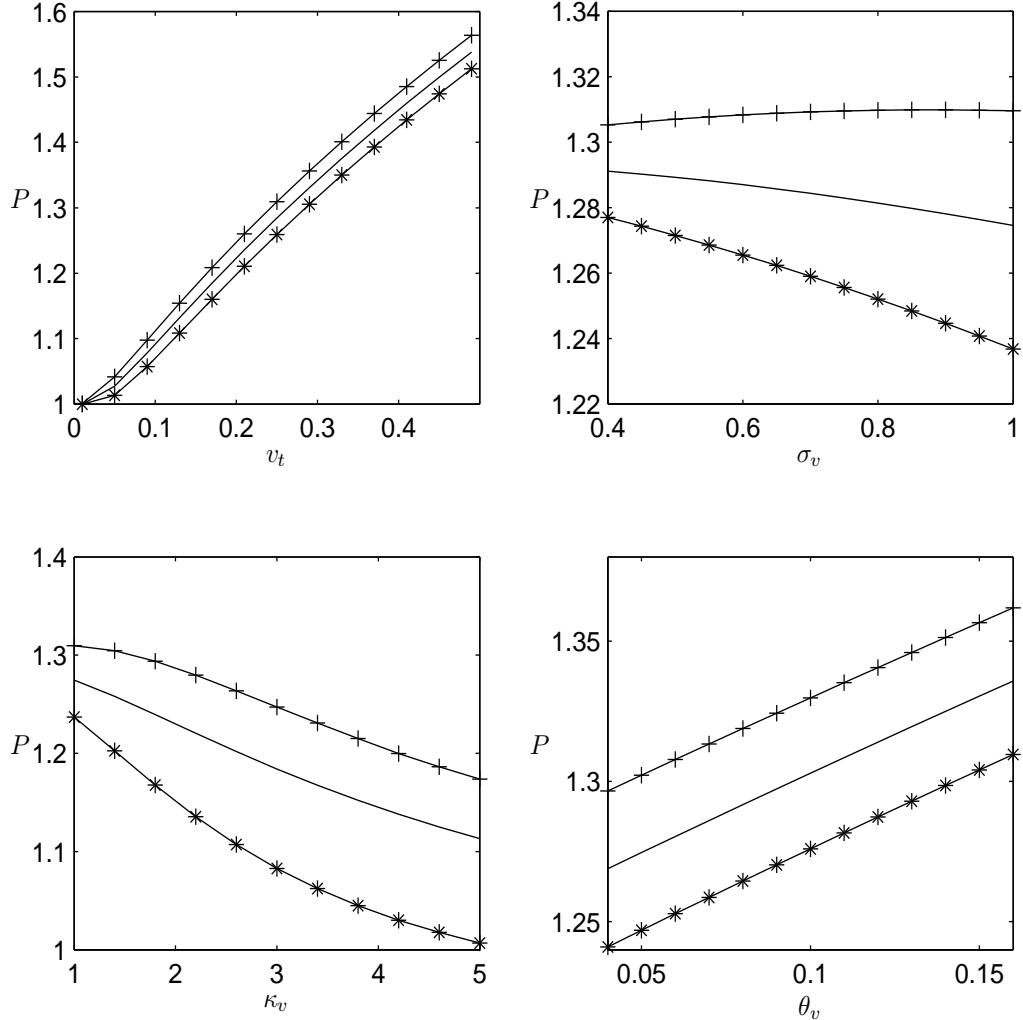
**Figure 4. The American put option prices and absolute approximation errors as functions of the time to maturity  $\tau$  in the Black-Scholes model.** In all the subplots,  $K = 100$ ,  $\sigma = 0.4$ ,  $r = 0.04$  and  $\delta = 0.04$ . The current stock prices used in the top and bottom subplots are  $S = 85$  and  $S = 115$ , respectively. These two approximately represent out-of-the-money and in-the-money options, respectively. Three methods are used: the binomial tree model with 10000 time steps which is treated as the true value, the interpolation method proposed in this paper, and the quadratic approximation used in Barone-Adesi and Whaley (1987). The left three subplots plot the option prices from the three methods, while the right subplots plot the pricing errors of the last two methods.



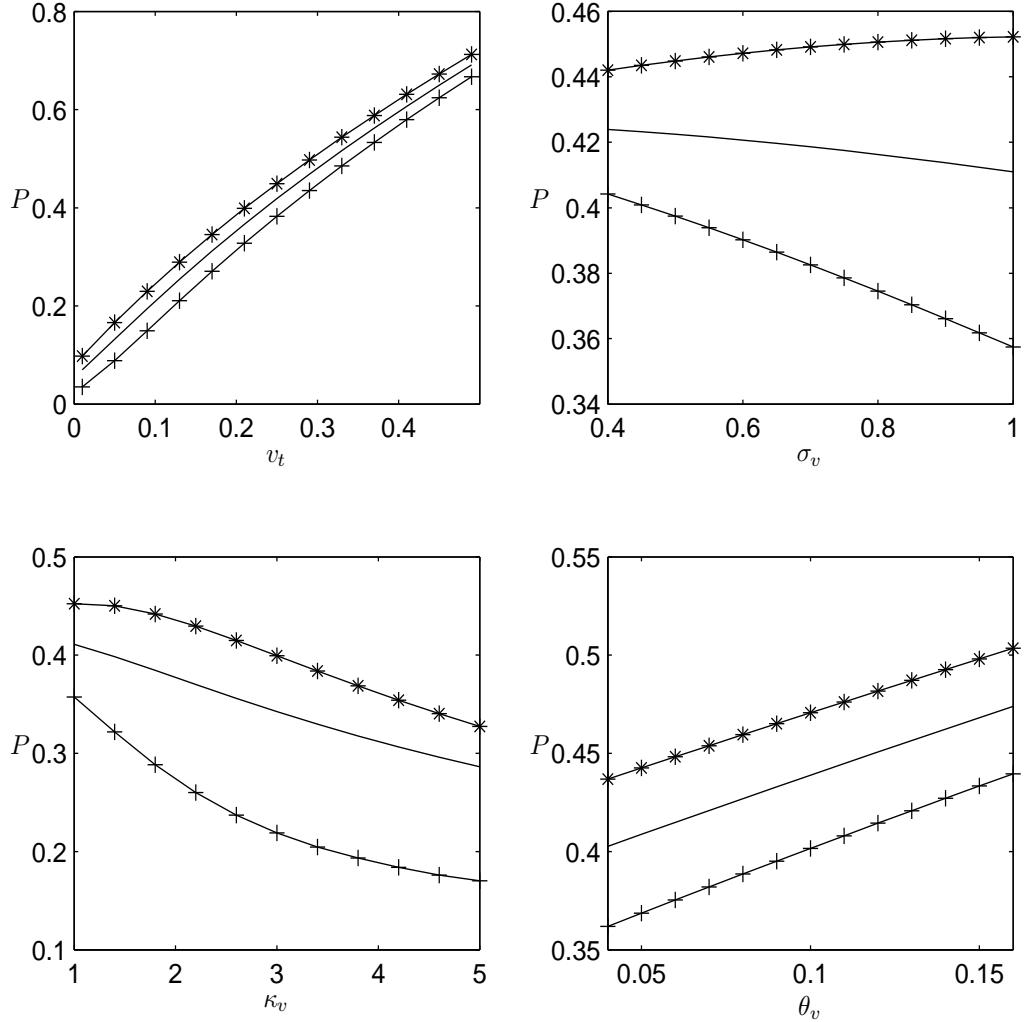
**Figure 5. The critical stock price in Heston's stochastic volatility model.** The critical stock prices for American put options under Heston's stochastic volatility model are plotted as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . Three different values of shock correlation  $\rho$  are plotted on each of the four subplots:  $-0.6$  (the curves marked with  $*$ ),  $0$  (the curves with no markers), and  $0.6$  (the curves marked with  $+$ ). The base parameter values are:  $K = 10$ ,  $r = 0.1$ ,  $\delta = 0$ ,  $v_t = 0.25$ ,  $\sigma_v = 0.7$ ,  $\kappa_v = 3$ , and  $\theta_v = 0.10$ . The subplots have no dependence on the current stock price  $S$ . On each of the four subplots, one parameter (the  $x$ -axis) is relaxed to vary in a fixed region.



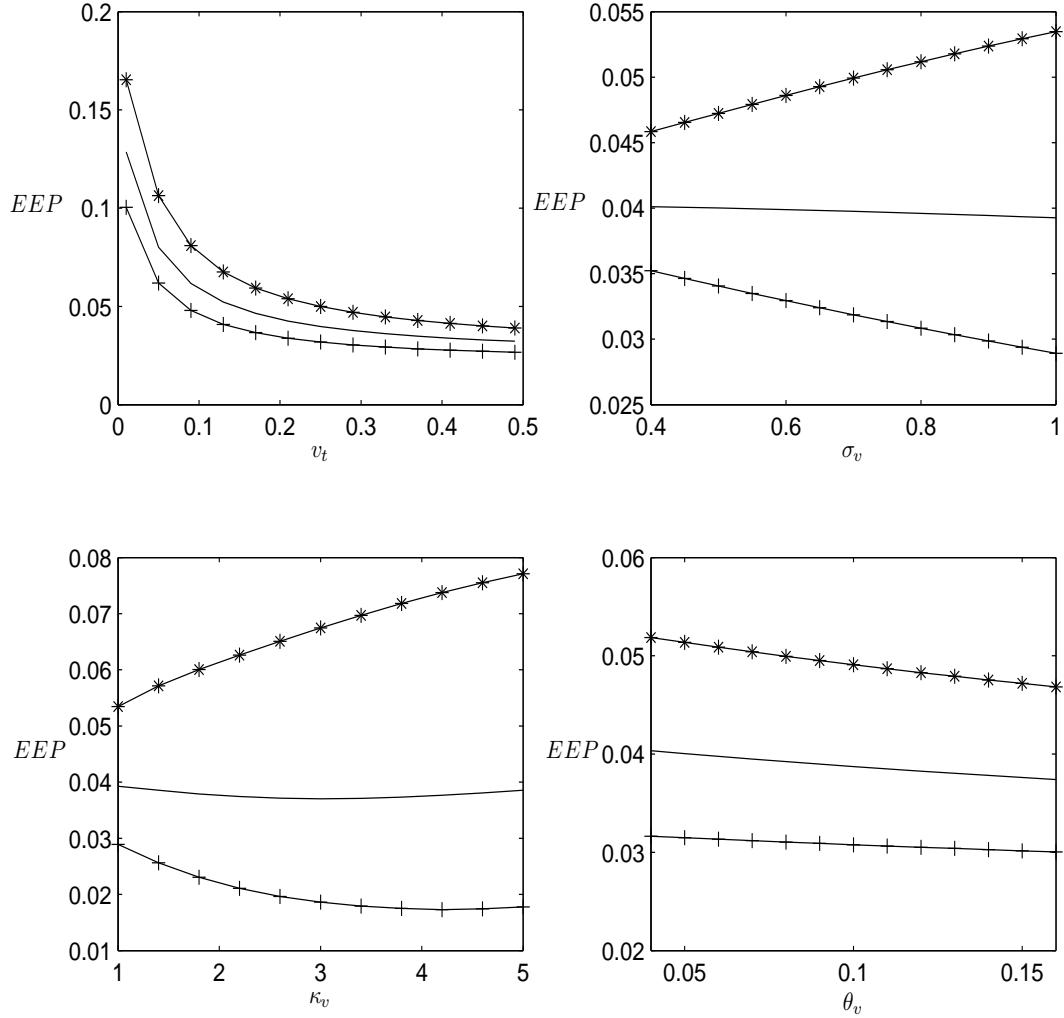
**Figure 6. The interpolation parameter in Heston's stochastic volatility model.** The interpolation parameter  $A$  in our interpolation method for American put options under Heston's stochastic volatility model are plotted as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . Three different values of shock correlation  $\rho$  are plotted on each of the four subplots:  $-0.6$  (the curves marked with  $*$ ),  $0$  (the curves with no markers), and  $0.6$  (the curves marked with  $+$ ). The base parameter values are:  $K = 10$ ,  $r = 0.1$ ,  $\delta = 0$ ,  $v_t = 0.25$ ,  $\sigma_v = 0.7$ ,  $\kappa_v = 3$ , and  $\theta_v = 0.10$ . Since  $A$  is the interpolation weight exactly when  $S = S_\tau^*$ , the subplots have no dependence on the current stock price  $S$ . On each of the four subplots, one parameter (the  $x$ -axis) is relaxed to vary in a fixed region.



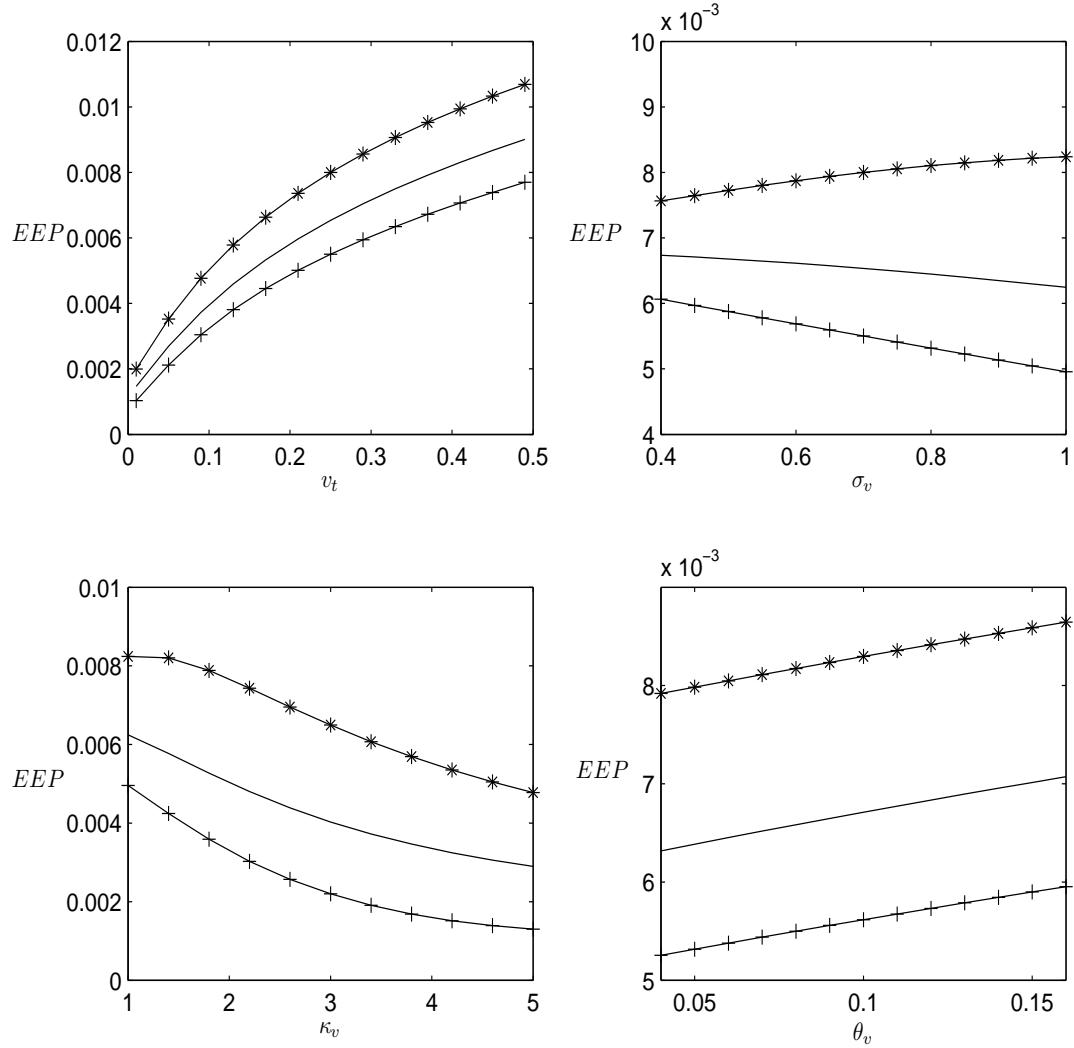
**Figure 7. The American put option price in Heston’s stochastic volatility model when  $S < K$  (in-the-money).** The American put option price  $P$  from our interpolation method under Heston’s stochastic volatility model are plotted as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . Three different values of shock correlation  $\rho$  are plotted on each of the four subplots:  $-0.6$  (the curves marked with  $*$ ),  $0$  (the curves with no markers), and  $0.6$  (the curves marked with  $+$ ). The base parameter values are:  $K = 10$ ,  $r = 0.1$ ,  $\delta = 0$ ,  $v_t = 0.25$ ,  $\sigma_v = 0.7$ ,  $\kappa_v = 3$ , and  $\theta_v = 0.10$ . The current stock price is fixed at  $S = 9$  so the options are “in-the-money”. On each of the four subplots, one parameter (the  $x$ -axis) is relaxed to vary in a fixed region.



**Figure 8. The American put option price in Heston’s stochastic volatility model when  $S > K$  (out-of-the-money).** The American put option price  $P$  from our interpolation method under Heston’s stochastic volatility model are plotted as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . Three different values of shock correlation  $\rho$  are plotted on each of the four subplots:  $-0.6$  (the curves marked with  $*$ ),  $0$  (the curves with no markers), and  $0.6$  (the curves marked with  $+$ ). The base parameter values are:  $K = 10$ ,  $r = 0.1$ ,  $\delta = 0$ ,  $v_t = 0.25$ ,  $\sigma_v = 0.7$ ,  $\kappa_v = 3$ , and  $\theta_v = 0.10$ . The current stock price is fixed at  $S = 11$  so the options are “out-of-the-money”. On each of the four subplots, one parameter (the  $x$ -axis) is relaxed to vary in a fixed region.



**Figure 9. The early exercise premium in Heston’s stochastic volatility model when  $S < K$  (in-the-money).** The early exercise premiums from our interpolation method under Heston’s stochastic volatility model are plotted as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . Three different values of shock correlation  $\rho$  are plotted on each of the four subplots:  $-0.6$  (the curves marked with  $*$ ),  $0$  (the curves with no markers), and  $0.6$  (the curves marked with  $+$ ). The base parameter values are:  $K = 10$ ,  $r = 0.1$ ,  $\delta = 0$ ,  $v_t = 0.25$ ,  $\sigma_v = 0.7$ ,  $\kappa_v = 3$ , and  $\theta_v = 0.10$ . The current stock price is fixed at  $S = 9$  so the options are “in-the-money”. On each of the four subplots, one parameter (the  $x$ -axis) is relaxed to vary in a fixed region.



**Figure 10.** The early exercise premiums in Heston’s stochastic volatility model when  $S > K$  (out-of-the-money). The early exercise premiums  $EEP$  from our interpolation method under Heston’s stochastic volatility model are plotted as a function of the instantaneous variance  $v_t$ , the volatility of variance  $\sigma_v$ , the mean-reverting strength  $\kappa_v$ , and the long-run mean instantaneous variance  $\theta_v$ . Three different values of shock correlation  $\rho$  are plotted on each of the four subplots:  $-0.6$  (the curves marked with  $*$ ),  $0$  (the curves with no markers), and  $0.6$  (the curves marked with  $+$ ). The base parameter values are:  $K = 10$ ,  $r = 0.1$ ,  $\delta = 0$ ,  $v_t = 0.25$ ,  $\sigma_v = 0.7$ ,  $\kappa_v = 3$ , and  $\theta_v = 0.10$ . The current stock price is fixed at  $S = 11$  so the options are “out-of-the-money”. On each of the four subplots, one parameter (the  $x$ -axis) is relaxed to vary in a fixed region.