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Abstract

In this article the theoretical generalization for representation of arithmetic operations with fuzzy numbers is considered. Fuzzy numbers are generalized by means of fuzzy measures. On the basis of this generalization the new algorithm of fuzzy arithmetic which uses a principle of entropy maximum is created. As example, the summation of two fuzzy numbers is considered. The algorithm is realized in the software "Fuzzy for Microsoft Excel".

Keywords: fuzzy measure (Sugeno), fuzzy integral (Sugeno), fuzzy numbers, arithmetical operations, principle of entropy maximum

1. Introduction

In researches of various areas (for example, at modeling of complex systems), it is often necessary to realize arithmetic operations with badly certain numerical data. Uncertainty is generated by statistical data's lack, experts' fuzzy statements, by action of various factors, etc. For representation of such data fuzzy numbers are used. Subset B of real numbers set R with membership $\mu_B : R \to [0,1]$ is called a fuzzy number. In practice often there are cases when the kind of membership of fuzzy number is arbitrary, fuzzy numbers are not normal: $\max_{x \in R} \mu_B(x) \neq 1$ and not convex:

$$\mu_B(\lambda x_1 + (1-\lambda)x_2) < \mu_B(x_1) \wedge \mu_B(x_2) \forall x_1, x_2 \in R, \forall \lambda \in [0,1].$$

However, in the scientific literature [1-10] are considered basically trapezoidal and triangular approximations of fuzzy numbers. By means of these approximations it is impossible to represent fuzzy number with multimodal value, for example, "or 3 or 4 ". Moreover fuzzy arithmetic which use in researches is based on principles of extension Zadeh. It leads to occurrence of an additional artificial fuzziness and undesirable distortion of result. Multiple calculations greatly enlarge the support of fuzzy number - result. It does not allow creating the software which provides correct results at the decision of complex practical problems. Therefore appear a necessity to make theoretical generalization of fuzzy arithmetic, and also on the basis of this generalization to construct effective algorithms.

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2. Theoretical base

Fuzzy sets are a special case of fuzzy measures Sugeno [11]. The fuzzy set can be represented as a fuzzy measure of possibility Sugeno [11]. Therefore fuzzy numbers (as fuzzy sets which are assigning on set of real numbers) can be represented as fuzzy measures. Let there are two fuzzy numbers, which are represented as fuzzy measures g_1 and g_2. Measures are defined on spaces (X, B_X, g_1) and (Y, B_Y, g_2), where $X = Y = R$ - set of real numbers. The subset of Cartesian product for spaces $H \subseteq X \times Y$ is set the binary relation between these spaces. Let $Z = X \times Y$. Then $H \subseteq Z$. Binary relation H allows confronting each element $x \in X$ its image in Y. This image is defined by as:

$$im_H x =^{def} \{y \in Y \,|(x, y) \in H\} = F(x). \tag{1}$$

Similarly (1), to everyone $y \in Y$ is confronted the prototype in X:

$$coim_H y =^{def} \{x \in X \,|(y, x) \in H\} = \mathrm{E}(y). \tag{2}$$

Let's consider such reflections:

$$\Gamma_H(E) = \bigcap_{x \in E} F(x) \tag{3}$$

$$\Gamma'_H(F) = \bigcap_{y \in F} E(y) \tag{4}$$

According to Galois-conformity, these reflections translate any subsets to closed subsets. They induce reflection between the closed subsets in $P(X)$ and $P(Y)$, where $P(\cdot)$ - powerset.

These definitions allow presenting any relations $H \subseteq (X \times Y)$ as union of Cartesian products for subsets E_i and F_i, which are connected by reflections (3) and (4).

Lemma 1.
Any binary relation $H \subseteq X \times Y$ is possible to present in the form:

$$H = \bigcup_{i=1}^{\infty} [E_i \times F_i], \tag{5}$$

where $E_i \in P(X), F_i \in P(Y), F_i = \Gamma_H(E_i)$.

Using (5) it is possible to define a fuzzy measure on the Cartesian product of spaces, on which fuzzy measures are set. According to [4], the measure of Cartesian product for two sets is defined by operation "min". Therefore the fuzzy measure of any relation H will be set as follows.

Definition 1. Fuzzy measure of any relation $H \subseteq X \times Y$, which is set on the Cartesian product for spaces (X, B_X, g_1) and (Y, B_Y, g_2), is defined as:

$$g(H) = \bigvee_{i=1}^{\infty} \{g_1(E_i) \wedge g_2(F_i)\}, \tag{6}$$

where $F_i = \Gamma_H(E_i)$, and $\Gamma_H(\cdot)$ - reflection of a kind (3) between $E_i \subseteq X$, and $F_i \subseteq Y$.

Definition 1 allows connecting measures g_1 and g_2 with a measure on the Cartesian product. But use of Galois-conformity for receiving fuzzy measure of subset $H \subseteq X \times Y$ in practice is complicated. Is possible to show, that the fuzzy measure g (H) is unequivocally defined through fuzzy integral (Sugeno) [3].

Lemma 2.
For subset $H \subseteq X \times Y$ the fuzzy measure can be set by fuzzy integral:

$$g(H) = \oint_Y g_1(E(y)) \circ g_2(\cdot) = \oint_X g_2(F(x)) \circ g_1(\cdot), \tag{7}$$

where $g_1 : 2^X \to [01], g_2 : 2^Y \to [0,1]$.

From the Lemma 2 follows:

Lemma 3.
Fuzzy measure $g_H(\cdot)$ on set $X \times Y$, $(H \subseteq X \times Y)$ is function, which assign reflection:

$$g_H(\cdot) : P(X) \times P(Y) \to [0,1], \tag{8}$$

where $P(X)$ - set of all subsets X;
$P(Y)$ - set of all subsets Y.

Let X or Y is the Cartesian product of other sets. Then there is relation $R \subseteq [X \times Z] \times Y$, for which the fuzzy measure is defined as:

$$g_R(\cdot) : [P(X) \times P(Z)] \times P(Y) \to [0,1]. \tag{9}$$

On Cartesian product P(X) of any sets $X_S, s = \overline{1, N_s}$ is possible to assign reflection $\omega : P(X_1) \times ... \times X(X_{N_s}) \to P(X)$. Therefore, exist a multibasic algebra, which is defined on powersets family of sets X_S.

Details. Let $P = \{P(X_S) | s \in S\}$ - any powersets family of sets X_S. Powersets are indexed by elements from kinds sets S. Let $O_P(P)$ is set of all operations:

$$\omega : P(X_{S_1}) \times ... \times P(X_{S_N}) \to P(X_S). \tag{10}$$

It is defined of reflection:

$$t: O_P(P) \to S \times S. \tag{11}$$

If $\Omega = \{\omega\}$ - is operations symbols set, therefore signature of algebra above kinds set S is defined as reflection:

$$\text{type}: \Omega \to S \times S. \tag{12}$$

Now we shall formulate definition the multibasic (multisortable) algebras above sets powersets $P(X_S)$.

Definition 2. Powersets family P of sets X_S is named as multisortable universal algebra of the signature Ω, if is set the reflection δ: $\Omega \to O_P(P)$, which provides diagram commutativity:

$$\begin{tikzcd} \Omega \arrow[r, "type"] \arrow[dr, "\delta"'] & S \times S \\ & O_P(P) \arrow[u, "t"'] \end{tikzcd}$$

in the sense, that $type = t \circ \delta$, where "$\circ$" - composition's operation.

If on $P(X)$ is defined binary operation $\omega : P(X) \times P(X) \to P(X)$, therefore set $P(X)$ is named as groupoid, and working in $P(X)$ binary operation ω is named as the internal composition law.

Definition 3. Binary H - operation is named as operator, which associate for fuzzy measures $g_1 : 2^X \to [0,1]$ and $g_2 : 2^Y \to [0,1]$ of measure $g : 2^Z \to [0,1]$:

$\forall C \subseteq Z,\ g(C) = g_H(H \cap \omega^{-1}(C))$,
where $g_H(\cdot)$ is defined according to (6), $H \subseteq X \times Y$
$\omega^{-1}(C) \in P(X \times Y),\ \omega : P(X) \times P(Y) \to P(Z)$ - binary operation in the three-basic algebra, which realizes such function, that:
$AND \times IN \subseteq AND' \times IN' \in P(X) \times P(Y) \Rightarrow C \subseteq C' \in P(Z)$.

If $X \equiv Y \equiv Z$, then it exists a binary operation ω, which is defined on X. Such operation can be "*min*", "*max*", or arithmetic operations from interval mathematics. If as operation ω use operation "*max*", therefore fuzzy measure $g(\cdot)$ is result of H - union operation above fuzzy measures g_1 and g_2. Let H - operations are designated so:

$$g(\cdot) = H\{g_1\ \omega\ g_2\} = H_\omega(g_1, g_2). \tag{13}$$

If ω defines binary operation in the one-basic algebra, H - operation is the operator of a kind:

$$H_\omega : F(X) \times F(X) \to F(X), \tag{14}$$

where $F(X)$ - the set of all fuzzy measures on space X, which play a role an fuzzy internal composition law.

Definition 4. The ordered pair, which formed from $F(X)$ and fuzzy internal composition law H_ω on $F(X)$ for fuzzy measures is named as fuzzy groupoid $(F(X), H_\omega)$.

According to definition 3, result H_ω- operation can be specified under the theorem.

The theorem 1.
The fuzzy measure of subset $C \subseteq Z$, which is result of H_ω - operation above fuzzy measures $g_1 : 2^X \to [0,1]$ and $g_2 : 2^Y \to [0,1]$, satisfies to formula:

$$g(C) = H_{\omega}(g_1, g_2) = \int_B g_1(E(y) \cap A) \circ g_2 = \int_A g_2(F(x) \cap B) \circ g_1, \quad (15)$$

where $y \in Y$, $x \in X$, $C \subseteq Z$, $H \subseteq X \times Y$, $C = \omega(A, B)$, $A \subset X$, $B \subseteq Y$, $E(y)$, $F(x)$ - are defined on (1), (2).

The proof.

According to definition 3, fuzzy measure $g(C)$ is defined from expression: $g(C) = g_H(H \cap \omega^{-1}(C))$. As $\omega^{-1}(C) \in P(X \times Y)$, that on $X \times Y$ is exist a correlation $Q \subseteq X \times Y$ such, that $Q = \{(x, y) \mid x \in A, y \in B\}$. Then $E^Q(y) = \{X \mid (x, y) \in Q\}$, and therefore $\forall y \notin B, E^Q(y) = \varnothing$, and $\forall y \in B, E^Q(y) = A$.

Then we can write: $g(c) = g_H(H \cap Q)$. We shall designate $H' = H \cap Q$. The fuzzy measure $g_{H'}$, according to (7), is defined by expression:

$$g(c) = \int_Y g_1(E^{H'}(y)) \circ g_2 = \int_{B \cup (Y \setminus B)} g_1(E^{H'}(y)) \circ g_2 = \int_Y \{[\chi_B(y) \vee \chi_{\overline{B}}] \wedge g_1(E^{H'}(y))\} \circ g_2,$$

where $\chi_e, \chi_{\bar{e}}$ - characteristic functions of sets B and $\overline{B} = Y \setminus B$.

Square bracket's disclosing in subintegral expression gives:

$$g(C) = \int_Y \{(\chi_B(y) \wedge g_1(E^{H'}(y))) \vee (\chi_{\overline{B}}(y) \wedge g_1(E^{H'}(y)))\} \circ g_2 .$$

Let's consider expression under integral.

$$g_1(E^{H'}(y)) = g_1(E^H(y) \cap E^Q(y)) = \begin{cases} g_1(E^H(y) \cap A), y \notin B \\ g_1(E^H(y) \cap \varnothing) = 0, y \in B. \end{cases}$$

Then $\chi_{\overline{B}}(y) \wedge g_1(E^{H'}(y)) = 0$, and therefore:

$$g(C) = \int_Y \{\chi_B(y) \wedge g_1(E^{H'}(y))\} \circ g_2 = \int_B g_1(E^{H'}(y)) \circ g_2 = \int_B g_1(E^{H'}(y) \cap A) \circ g_2.$$

For the second integral in (15) reasoning are similar. The theorem is proved.

It is possible to show, that H_ω - operation, which defines fuzzy groupoid ($F(X)$, H_ω), has commutativity, associativity and distributivity (under some conditions) properties.

Formulas for H_ω - operations allow to consider arithmetic operations above fuzzy numbers, which are presented as distributions of fuzzy measures g_1 and g_2 on the space of real numbers.

3. Algorithm

For calculations with fuzzy numbers, operation $\omega \in \Omega$ is defined as one from set of arithmetic operations (+,-,×,/). Let fuzzy numbers are defined as fuzzy measures g_1 and

g_2 on discrete supports A and B. Then as relation $H \subseteq A\times B$ the cross-set can be used. This set is operation result $A\ \omega\ B$ ($a_i\ \omega\ b_i$, $\forall\ i, j$), which is represented as matrix. On crossing of a line and a column is disposed matrix element $a^*\omega\ b^*$, which has the maximal density of fuzzy measure:

$$a\,\omega\, b = \arg\max_{(a_i\,\omega\, b_i)} g_{A\times B}\,(a_i\,\omega\, b_i)\ ,$$

where $g_{A\times B}\,(\cdot)$ - fuzzy measure of all matrix $A\ \omega\ B$.

Then as result H_ω - operations (for example, as H_+ - summation) the fuzzy number will be defined by the formula (15). Let fuzzy numbers are defined as possibility measure distributions. Then we have a variant of fuzzy arithmetic, which provides calculations with fuzzy numbers.

According to [2], fuzzy number B is named as the fuzzy set, which is defined on set of real numbers R:

$$B : R \to [0,1]\,. \tag{16}$$

For computer memory economy, fuzzy numbers are defined on the limited interval $S^B \subset R,\ S^B = [\min S^B, \max S^B]$. This interval can be broken into smaller intervals by quantity N:

$$S^B = \bigcup_{k=\overline{1,N-1}} [s_k^B, s_{k+1}^B], \tag{17}$$

where $s_{k+1}^B - s_k^B = \dfrac{\max S^B - \min S^B}{N}$ - is subintervals length, which is designated as Δs^B.

Initial subintervals lengths together with membership values of fuzzy number in points $s_k^B : \mu_k^B = \begin{cases} B\,(s_k^B)\ \forall\ s_k^B \in S^B \\ 0\ \forall\ s_k^B \notin S^B \end{cases}$ form representation of fuzzy numbers in a computer:

$$B = \{(s_k^B, \mu_k^B), k = \overline{1,N}\}\,. \tag{18}$$

Membership values set of fuzzy numbers form assurance distribution:

$$M^B = \{\mu_k^B, k = \overline{1,N}\}, M^B : S^B \to [0,1]\,. \tag{19}$$

Set of initial subintervals lengths form support:

$$S^B = \{s_k^B, k = \overline{1,N}\}\,. \tag{20}$$

Fuzzy numbers have characteristics, which are used for the analysis:

- the gravities centre or the most possible value:

$$s_k^{B^*} = \sum_{i=1}^{N} s_k^B \bullet \mu_i^B \Big/ \sum_{i=k}^{N} \mu_i^B ; \qquad (21)$$

- number value with the maximal assurance degree:

$$MaxB = \{s_k^B \mid \mu_k^B = \max_i \mu_i^B\} ; \qquad (22)$$

- the minimal value on assurance level α:

$$MinB_\alpha = \{s_k^B \mid s_k^B = \min_i s_i^B, \mu_k^B \geq \alpha\} ; \qquad (23)$$

- the maximal value on assurance level α:

$$MaxB_\alpha = \{s_k^B \mid s_k^B = \max_i s_i^B, \mu_k^B \geq \alpha\} . \qquad (24)$$

The operations algorithm with fuzzy numbers is based on the stated theoretical positions. It is realized on software "Fuzzy for Excel 7.0" It consists of following steps.

Let we have two fuzzy numbers *C* and *B*. It is necessary to calculate number $D = C\,\omega\,B$, where ω-one of two-place arithmetic operations: summation, subtraction, multiplication, division.

Step 1. Decisions matrix reception.

Result of operation ω above fuzzy numbers *C* and *B* is represented as square matrix (at identical operands digitization). The matrix is formed on the Cartesian product of supports *C* and *B*:

$$V = (S^\omega, M^\omega); S^\omega = S^C \times S^B = \{s_{ij}^\omega \mid s_{ij}^\omega = s_i^C \omega \;\; s_j^B ; i, j = \overline{1, N}\} ,$$

$$M^\omega = \{\mu_{ij}^\omega \mid \mu_{ij}^\omega = \min(\mu_i^C, \mu_j^B); i, j = \overline{1, N}\} \qquad (25)$$

Support elements s_{ij}^ω are formed as operation result ω above supports of fuzzy numbers *C* and *B*. Membership values μ_{ij}^ω are formed as the minimal from memberships value *C* and *B*.

Step 2. Reception of potential decisions.

In matrix *V* defines line index *m* and column index *n* of element, which has the maximal membership value:

$$\mu_{mn}^\omega = \max_{i,j=\overline{1,N}} \mu_{ij}^\omega . \qquad (26)$$

There are two fuzzy numbers V_m and V_n, which are located in the line and the column accordingly, corresponds to this element:

$$V_m = (S^{V_m}, M^{V_m}), V_n = (S^{V_n}, M^{V_n}) . \qquad (27)$$

Then numbers are united:

$$X = V_m \bigcup V_n, X = (S^X, M^X). \quad (28)$$

The support of fuzzy number *X* calculate similarly (17) as division of united interval $S^X = [\min(S^{V_m}, S^{V_n}), \max(S^{V_m}, S^{V_n})]$ on *N* subintervals.

The membership value of number *X* calculates as memberships maximum V_m and V_n in the points for united interval. As supports of fuzzy numbers V_m , V_n and *X* not always can coincide, linear approximation of membership values in support points is used:

$$\forall s^Y \in [s_k^Y, s_{k+1}^Y] : M'^Y(s^Y) = M^Y(s_k^Y) - \frac{s^Y + s_k^Y}{\Delta s^Y} \bullet (M^Y(s_k^Y) - M^Y(s_{k+1}^Y)). \quad (29)$$

Then number membership *X* is defined as:

$$M^X(s_k^X) = \max(M'^{V_m}(s_k^X), M'^{V_n}(s_k^X)) \forall s_k^X \in S^X. \quad (30)$$

However the numbers quantity *X* in matrix *V* can be more than one. Therefore the current step repeats and creates set $\Psi = \{X_l, l = \overline{1, Card\Psi}\}$.

Step 3. Reception of final decision.

From all set Ψ get out fuzzy number, which has the maximal area of confidence distribution (principle of entropy maximum). It is accepted as operation result above two fuzzy numbers:

$$D = X_l, X_l \in \Psi \ : \ \sum_{k=1}^{N-1} \left| s_{k+1}^{X_l} - s_k^{X_l} \right| \bullet \mu_k^{X_l} \to \max_l . \quad (31)$$

At a high digitization degree this expression yields admissible results. The condition of maximal membership area can be considered as a condition of maximal energy. It provides the minimal information losses in arithmetic operations with fuzzy numbers.

On the one hand, the principle of entropy maximum allows preventing excessive enlargement in the support of fuzzy number at performance of arithmetic operations. On the other hand, it provides preservation of self-descriptiveness of fuzzy number - result.

4. An example

Summation of two fuzzy numbers *C* and *B*.

Let fuzzy number *C* is described by fuzzy set

$C = \{(1/0.1), (2/0.3), (3/0.8), (4/0.4), (5/0.2)\}$, and *B* is described by fuzzy set

$B = \{(7/0.9), (8/0.9), (9/0.5), (10/0.3), (11/0.1)\}$.

Step 1.

Using (25), we come in a matrix:

$$\begin{pmatrix} 8/0.1 & 9/0.1 & 10/0.1 & 11/0.1 & 12/0.1 \\ 9/0.3 & 10/0.3 & 11/0.3 & 12/0.3 & 13/0.1 \\ 10/0.8 & 11/0.8 & 12/0.5 & 13/0.3 & 14/0.1 \\ 11/0.4 & 12/0.4 & 13/0.4 & 14/0.3 & 15/0.1 \\ 12/0.2 & 13/0.2 & 14/0.2 & 15/0.2 & 16/0.1 \end{pmatrix}$$

Step 2.

The matrix has two elements with the maximal membership values. These elements are placed in a line 3 with columns numbers 1 and 2.

Using (26) - (28), we find two fuzzy numbers, which form set of potential decisions (for simplification, numbers will not be transformed to the homogeneous support):

$$X_1 = \{(8/0.1), (9/0.3), (10/0.8), (11/0.8), (12/0.5), (13/0.3), (14/0.1)\}$$
$$X_2 = \{(9/0.1), (10/0.8), (11/0.8), (12/0.5), (13/0.3), (14/0.1)\}$$

Step 3.

By formula (31) we calculate confidence areas for these numbers (values is 2.8 and 2.5 accordingly). First number is summation result. Numbers also are shown on fig. 1.

If to use algorithm with a principle of extension Zadeh, the summation result will have the support [8, 16]. If to use the considered algorithm, fuzzy number - result of summation has the support [8, 14]. Small extension of the support is especially important for problems with a big quantity of consecutive arithmetic operations.

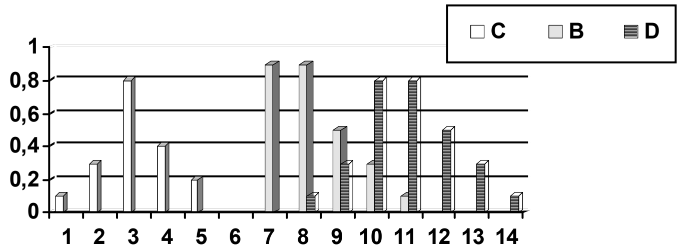

Figure 1 - Fuzzy numbers' summation

5. Conclusions

Thus, the algorithm provides performance of arithmetic operations with two fuzzy numbers, which are defined as fuzzy possibilities measure. The considered theoretical positions allow building the effective practical algorithms for processing fuzzy numerical data on the basis of new concept *H*-operations. On this theoretical basis the algorithm of arithmetic operations with fuzzy numbers is developed and approved. The algorithm is realized in software "Fuzzy for Microsoft Excel", which expands Microsoft Excel capacity for calculations with fuzzy numbers.

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