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# PRICING OF THE EUROPEAN OPTIONS BY SPECTRAL THEORY

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## Abstract

We discuss the efficiency of the spectral method for computing the value of the European Call Options, which is based upon the Fourier series expansion. We propose a simple approach for computing accurate estimates. We consider the general case, in which the volatility is time dependent, but it is immediate extend our methodology at the case of constant volatility. The advantage to write the arbitrage price of the European Call Options as Fourier series, is matter of computation complexity. Infact, the methods used to evaluate options of this kind have a high value of computation complexity, furthermore, them have not the capacity to manage it. We can define, by an easy analytical relation, the computation complexity of the problem in the framework of general theory of the "Function Analysis", called The Spectral Theory.

## 1 Introduction

The Barrier-Options belong to the class of Exotic Options. These usually are traded between companies and banks and not quoted on an exchange. In this case, we usually say that them are traded in the over the counter market. Most Exotic Options are quite complicated, and their final values depend not only on the asset price at expiry but also on the asset price at previous times. They are determined by a part or the whole of the path of the asset price during the life of option. These options are called path-dependent Exotic Options. Over the time, several papers have studied the issue to evaluate the price of the Barrier Options and Double-Barrier Options; Snyder (1969) describes down-and-out stock options as limited risk special options. Merton (1973) derives a closed-form pricing formula for down-and-out calls. A down-and-out call is identical to a European call with the additional provision that the contract is canceled (knocked out) if the underlying asset price hits a prespecified lower barrier level. An up-and-out call is the same, except the contract is canceled when the underlying asset price first reaches a prespecified upper barrier level. Down-and-out and up-and-out puts are similar modifications of European put options. Knock-in options are complementary to the knock-out options: they pay off at expiration if and only if the underlying asset price does reach the prespecified barrier prior to expiration. Rubinstein and Reiner (1991) derive closed form pricing formulas for all eight types of single-barrier options. Double-barrier (double knock-out) options are canceled (knocked out) when the underlying asset first reaches either the upper or the lower barrier. Double-barrier options have been particularly popular in the OTC currency options markets over the past several years, owing in part to the significant volatility of exchange rates

experienced during this period. In response to their popularity in the marketplace, there is a growing literature on double-barrier options. Kunitomo and Ikeda (1992) derive closed-form pricing formulas expressing the prices of double-barrier knock-out calls and puts through infinite series of normal probabilities. Geman and Yor (1996) analyze the problem by probabilistic methods and derive closed-form expressions for the Laplace transform of the double-barrier option price in maturity. Schroder (2000) inverts this Laplace transform analytically using the Cauchy Residue Theorem, expresses the resulting trigonometric series in terms of Theta functions, and studies its convergence and numerical properties. Pelsser (2000) considers several variations on the basic double-barrier knock-out options, including binary double-barrier options (rebate paid at the first exit time from the corridor) and double-barrier knock-in options, and expresses their pricing formulas in terms of trigonometric series. Hui (1997) prices partial double-barrier options, including front-end and rear-end barriers. Further analysis and extensions to various versions of double-barrier contracts traded in the marketplace are given by Douady (1998), Jamshidian (1997), Hui, Lo and Yuen (2000), Schroder (2000), Sidenius (1998) and Zhang (1997). Rogers and Zane (1997) develop numerical methods for double-barrier options with time-dependent barriers. Taleb (1997) discusses practical issues of trading and hedging double-barrier options. Linetsky (1999) introduces in mathematical finance the spectral method to solve a Black-Scholes equation. The theoretical foundation of this last method is that the Black-Scholes PDE is always parabolic equation and by the Theorem of Hilbert-Schmidt, its solution can write as the sum of the eigenfunctions. Unlike of the Monte Carlo Method, we show that using the spectral expansion is possible to define the computation complexity of the problem and thus it is possible to manage it.

## 2 The Black-Scholes Equation and its Transformation into the Canonical form of Parabolic PDE

In order to write the Black-Scholes equation let us make the following assumptions: the borrowing interest rate and the lending interest rate are equal to  $r$ , short selling is permitted, the assets and options are divisible, and there is no transaction cost. Therefore, we can conclude that the absence of arbitrage opportunities is equivalent to all risk-free portfolios having the same return rate  $r$ . Let  $f$  denote the value of an option that depends on the value of the underlying asset  $X$  and time  $t$ , i.e  $f = f(X, t)$ , where  $t \in [0, T]$ . This last can be considered as the value of a whole portfolio of various options, for simplicity, we can think to a simple call or put. Assume that in a time step  $dt$ , the underlying asset pays out a dividend  $qX(t)dt$ , where  $q$  is the dividend yield:

we suppose that  $X(t)$  satisfies a geometrical Brownian Motion in which the parameters  $r, q, \sigma$  are time dependent:

$$dX(t) = \mu X(t)dt + \sigma(t)X(t)dW \tag{1}$$

We require  $f$  to have at least one  $t$  derivative and two  $X$ .

At this point we construct a self financing portfolio consisting of one option and a number  $\alpha(t)$  of the underlying asset and a number  $\beta(t)$  of the bonds. The value of this portfolio is:

$$f(X, t) = \alpha(t)X(t) + \beta(t)B(t) \tag{2}$$

considering the quantities  $\alpha, \beta, q, r$  time dependent, we omit to write this, hence we have:

$$\beta B(t) = f(X, t) - \alpha X(t) \tag{3}$$

Besides, the owner of the portfolio receives  $qXdt$  for every asset held, the gain for the owner of the portfolio during the time  $dt$  is:

$$df = \alpha dX + \beta rB(t)dt + \alpha qXdt \quad (4)$$

Now, we can match the equation (6) with the equation (8) and choosing

$$\alpha = \frac{\partial f}{\partial X} \quad (5)$$

Thus we have the Black-Scholes equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t)X^2 \frac{\partial^2 f}{\partial X^2} + (r(t) - q(t))X \frac{\partial f}{\partial X} - r(t)f = 0 \quad (6)$$

In order that the solution exists and is unique, it is necessary to define the boundary condition and the initial condition. Also we require that when the value of the underlying asset hits the two barriers, lower and upper, the option is cancelled; we call this option "knock-out options". Furthermore, it is possible to build an option whose contract starts up when the underlying asset hits the barriers, lower and upper, or one of them, in this case we have the "knock-in options". In the present paper we study the case of the "knock-out options", but the method proposed is a general method and there would be no problem to evaluate the Double-Barrier Options of the type "knock-in options". Now we can rewrite the above Black-Scholes equations (7) adding the boundary condition and the initial condition. Being the Black-Scholes equation a type of equation for backward induction, we are interested at the value of  $f$  in the time  $T$ , and this is true because  $t \in [0, T]$ , where  $K$  is the strike price:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t)X^2 \frac{\partial^2 f}{\partial X^2} + (r(t) - q(t))X \frac{\partial f}{\partial X} - r(t)f = 0 \quad (7)$$

$$X \in [L, H]; \quad t \in [0, T] \quad (8)$$

$$f(L, t) = 0; \quad f(H, t) = 0 \quad (9)$$

$$f(X, T) = \max(\pm(X(T) - K), 0) \quad (10)$$

At this point we can introduce some transformations by which we reduce the Black-Scholes equations to the heat equation and this because, Green's function of the heat equation has an analytical expression.

The transformation that changes the Black-Scholes equation into a heat equation(Canonical form of parabolic PDE) is known, infact in literature exists more methods to do it. We have choused the following transformation of variables to turn the equation (7) into a heat equation with boundary conditions:

$$Y = \ln X + \int_t^T \left( r(s) - q(s) - \frac{1}{2}\sigma^2(s) \right) ds \quad (11)$$

$$\tau = \frac{1}{2} \int_t^T \sigma^2(s) ds \quad (12)$$

$$f(X, t) = e^{-\int_t^T r(s) ds} F(Y, \tau) \quad (13)$$

Substituting the relations (11) (12) (13) in the equation of the Black-Scholes (7), this last assumes the canonical form of PDE of the parabolic kind:

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial Y^2} \quad (14)$$

$$Y \in [A, B] \quad \tau \in \left[ 0, \frac{1}{2} \int_0^T \sigma^2(s) ds \right],$$

$$F(A, \tau) = 0 \quad F(B, \tau) = 0,$$

$$F(Y, 0) = \psi(e^{Y(T)})$$

where  $A = (\ln L + \int_t^T \theta(s) ds)$ ,  $B = (\ln H + \int_t^T \theta(s) ds)$ ,  $\theta(t) = (r(t) - q(t) - \frac{1}{2}\sigma^2(t))$  and  $\psi(e^{Y(T)}) = \max(\pm(X(T) - K), 0)$ .

First to solve the equation (14) we want to show that for constant values of the parameters, the last transformations of variables that turn the equation of the Black-Scholes (7) into a heat equation with boundary conditions, becomes:

$$Y = \ln X + \left( r - q - \frac{1}{2}\sigma^2 \right) (T - t) \quad (15)$$

$$\tau = \frac{1}{2}\sigma^2(T - t) \quad (16)$$

$$f(X, t) = e^{-r(T-t)} F(Y, \tau) \quad (17)$$

We want to remark that the transformation to convert the Black-Scholes equation into heat equation is not unique. Furthermore,

### 3 Price of Knock-out Double-Barrier Options

The value of the Knock-out Double-Barrier Options is given by the solution of the Black-Scholes equation with boundary conditions, that we have seen in the section number two. One can prove (see the appendix), that the solution of equation (14) is given by hereafter theorem:

**Pricing Theorem** In the Black Scholes framework the arbitrage-price, of the European Knock-out double barrier options, is given by relation:

$$f(X, T - t) = \int_{\ln H}^{\ln L} d\xi e^{-\int_t^T ds r(s)} (\xi - K)^+ \\ \sum_{k=-\infty}^{+\infty} \left[ P_{\ln L}^{\ln H} \left( \ln \left( \frac{X}{L} \right) - \xi + 2k \ln \left( \frac{H}{L} \right), \frac{1}{2} \int_t^T ds \sigma^2(s) \right) - \right. \\ \left. P_{\ln L}^{\ln H} \left( \ln \left( \frac{X}{L} \right) + \xi + 2k \ln \left( \frac{H}{L} \right), \frac{1}{2} \int_t^T ds \sigma^2(s) \right) \right].$$

for every underlying asset value  $X, \in [L, H]$ ; where  $H$  is the upper barrier and  $L$  is the lower barrier and  $\xi$  is a parameter belongs the interval  $[\ln L + \int_t^T \theta(s)ds, \ln H + \int_t^T ds\theta(s)ds]$ .

## 4 Numerical Implementation and Computation Complexity

In order to compute the price of a double-barrier option in two different settings, we compare our results with the prices obtained through Monte-Carlo simulations and with the prices given in Kunitomo and Ikeda (1992). The standard deviation is computed on a sample of 200 evaluations, each evaluation being performed on 5000 Monte-Carlo paths. Moreover, these two prices to lie within one standard deviation of the Monte-Carlo method. The Monte-Carlo simulations must be run using very small step sizes and many paths to make sure that one barrier is not "hit but missed". The consequence is that the Fourier expansion requires two orders of magnitude less operations than the Monte-Carlo simulations and one than the Laplace transform method used to Geman and Yor (1996)(e.g., it takes a fraction of a second to do it by using Pc or Mac). As a comparison, in the case of Asian options, Geman-Eydeland (1995) obtain a standard deviation as low as 0.001 for a sample of 50 evaluations, each of them being performed on 500 Monte-Carlo paths and it is in the context of delta hedging that the spectral expansion of the Asian option obtained in Linetsky (2002) proves definitely superior, both theoretically and computationally. A final manner to illustrate this point is to show that, as expected, the sensitivity of the option price to the step size in Monte-Carlo simulations becomes extremely high when the time remaining to maturity is short and the strike price close to one of the barriers.

### Remark

The method of computing of the arbitrage price of the double barrier options, through "Fourier expansion", is very efficient. Infact it's possible compile an easy algorithm, in order to have the the correct value of Double-Barrier Options and all this summing few eigenfunctions, not more of thirty. It is amazing to see the speed by which the our expansion converges at the price. In the methods in which are used the Monte Carlo Simulations is necessary compute about five thousand integrals to have the price. The difference between the two methods is clear, exists a difference of two orders. Therefore the technique that we propose, following the articles of Linetsky(2002), is more efficient than that proposed by Pelsser(1997) and Geman Yor (1996). The advantage of our method is that makes decrease the computation complexity. The compu-

tational complexity theory is a branch of the theory of computation in computer science that investigates the problems related to the resources required to run algorithms, and the inherent difficulty in providing algorithms that are efficient for both general and specific computational problems. The our idea is to evaluate the price of Barrier-Options and Double-Barrier Options like the weighted sum of the eigenfunctions of the Black-Scholes differential operator, where the coefficients  $c_k$  are the weights. these last are integrals, the which value is given in numerical way:

$$c_k = \frac{2}{l} \int_0^l d\xi (\xi - K)^+ \sin \left( \frac{k\pi\xi}{\ln \frac{H}{L}} \right) \quad (18)$$

and the price is given by following relation:

$$f(X, T - t) = e^{-\int_t^T r(s)ds} \sum_{k=-\infty}^{+\infty} e^{-\left(\frac{k\pi}{\ln \frac{H}{L}}\right)^2 \frac{1}{2} \int_t^T \sigma^2(s)ds} \left[ c_k \sin \left( \frac{k\pi\eta}{\ln \frac{H}{L}} \right) \right], \quad (19)$$

Let us note that  $e^{-\left(\frac{k\pi}{\ln \frac{H}{L}}\right)^2 \frac{1}{2} \int_t^T \sigma^2(s)ds}$  decreases quickly. Thus, choosing a small number,  $\epsilon$ , and define  $k(\epsilon)$  as hereafter:

$$\exp \left[ -\left(\frac{k\pi}{\ln \frac{H}{L}}\right)^2 \frac{1}{2} \int_t^T \sigma^2(s)ds \right] = \epsilon. \quad (20)$$

Hence, we have

$$\left(\frac{k\pi}{\ln \left(\frac{H}{L}\right)}\right)^2 = \frac{2}{\int_t^T \sigma^2(s)ds} \ln \left(\frac{1}{\epsilon}\right), \quad (21)$$

$$k(\epsilon) = \frac{1}{\pi} \ln \left(\frac{H}{L}\right) \sqrt{\frac{2}{\int_t^T \sigma^2(s)ds} \ln \left(\frac{1}{\epsilon}\right)}, \quad (22)$$

Thus we can write the computation complexity of our problem and it is shown by equation (22). Let us observe, that greater is the lifetime of the options, smaller is the value of the number  $k(\epsilon)$ . Hence fixed  $\epsilon$  like the accuracy of the problem, we can compute in approximate way the value of  $f(X, T - t)$  using the partial sum of  $k(\epsilon)$  eigenfunctions. Therefore, the contribution of the present paper is that it offers the formula by which is possible manage the accuracy, choosing the number of eigenfunctions necessary to obtain the accuracy wanted.

the results obtained are in the following tables, in which one can read the price of double knock-out call option at behavior of  $k^*$ , number of eigenfunctions used. Our results are compared to the results obtained from A. Pelsser(1997), Geman Yor(1996), Kunitomo Ikeda(1992), and reading their, we can be satisfied of our.

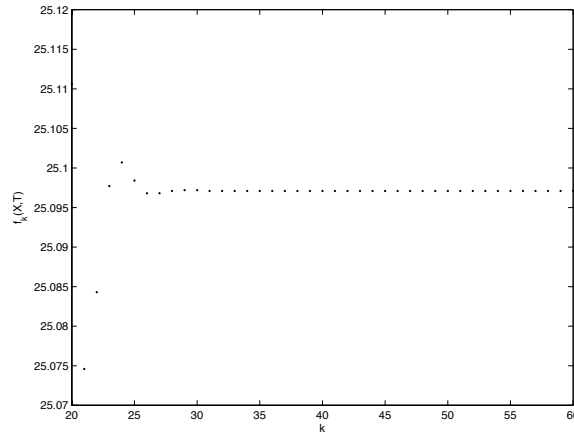


Figure 1: Approximate value of the Knock-out Double-Barrier Options respect to a  $k(\epsilon)$  number of the eigenfunctions, with expiration date to one month

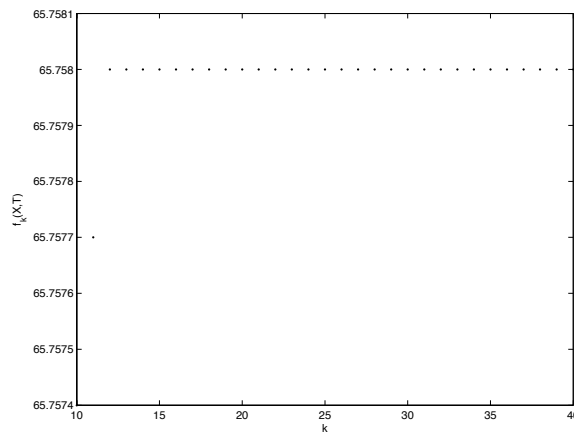


Figure 2: Approximate value of the Knock-out Double-Barrier Options respect to a  $k(\epsilon)$  number of the eigenfunctions, with expiration date to six months



Table 1: Value of Double Knock-Out Call Option if  $S = 1000$ ,  $K = 1000$ ,  $r = 0.05$ ,  $(T-t) = 1/12$

$\sigma$	H	L	$V(x, T-t)$	$k^*$
0.2	1500	500	25.1207	12
0.2	1200	800	24.7568	12
0.2	1050	950	2.5970	3
0.3	1500	500	36.5842	22
0.3	1200	800	29.4473	8
0.3	1050	950	0.9583	2
0.4	1500	500	47.8475	16
0.4	1200	800	25.8442	6
0.4	1050	950	0.4381	2

Table 2: Value of Double Knock-out Call Option if  $S = 1000$ ,  $K = 1000$ ,  $r = 0.05$ ,  $(T-t) = 1/2$

$\sigma$	H	L	$f(X, T-t)$	$k^*$
0.2	1500	500	66.1289	13
0.2	1200	800	22.1128	5
0.2	1050	950	0.2468	2
0.3	1500	500	67.8773	9
0.3	1200	800	10.0795	4
0.3	1050	950	0.0754	1
0.4	1500	500	53.3454	7
0.4	1200	800	5.0195	3
0.4	1050	950	0.0321	1

## 5 Conclusions

In this article, we have shown a simple and easy-to-use method in terms of the eigenfunctions series for computing accurate estimates of Black-Scholes Double-Barrier Options prices, with constant and time-dependent parameters, for an underlying asset driven by classical geometric Brownian motion. This approach is also able to provide the price of the Double-Barrier Options for values very tight of the bounds upper and lower in which can be efficiently further improved in a systematic manner by means increasing the number of eigenfunctions to use in the approximation scheme. Several articles, with the aim evaluate the Double-Barrier Options, use numerical methods. In order to name a few of most the important articles on this problem, we indicate: "PRICING AND HEDGING DOUBLE BARRIER OPTIONS:A PROBABILISTIC APPROACH" H. Geman and M. Yor(1996), and "PRICING DOUBLE BARRIER OPTIONS: AN ANALYTICAL APPROACH" A. Pelsser(1997), "STRUCTURING, PRICING AND HEDGING DOUBLE-BARRIER STEP OPRIONS" V. Linetsky and D. Davydov(2002). The main goal of the present work is to study the computation complexity of algorithm, offering an explicit formula for it. This result is very important because often is used the Monte Carlo method and Laplace transform method to evaluate options of this kind, and for this former method there is no possible to manage the computation complexity because it is impossible to write an analytical formula that shows the its computation complexity. Therefore, given the power of this method, it is straightforward to generalize the approach to more complicated situ-

ations, because it is a general method to solve Black-Scholes equations with boundary conditions with constant parameters and time dependent parameters. Furthermore, our method can also be extended to pricing American options with time-dependent parameters and to the SABR model with stochastic parameters. This research is now in progress and will be published early.

## Appendix

### Green's Function of Heat Equation with Boundary Conditions

Let be given the PDE in canonical form of the parabolic kind of the second order, with following boundary conditions:

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial Y^2} \quad (23)$$

$$Y \in [A, B] \quad \tau \in [0, \bar{T}], \quad \bar{T} = \frac{1}{2} \int_0^T \sigma^2(s) ds;$$

$$F(A, \tau) = 0 \quad F(B, \tau) = 0,$$

$$F(Y, 0) = \psi(e^Y)$$

we can use the separable variable method and we rewrite the function  $F(Y, \tau)$  as the product between  $U(Y)$  and  $W(\tau)$ , in this way the PDE (27) becomes a system of two ODE in which one is a linear differential equation of the first order respect to  $t$ , and the remainder is a Sturm-Liouville problem of the second order:

$$\frac{\partial U(Y)W(\tau)}{\partial \tau} = \frac{\partial^2 U(Y)W(\tau)}{\partial Y^2} \quad (24)$$

thus we have

$$U(Y) \frac{\partial W(\tau)}{\partial \tau} = W(\tau) \frac{\partial^2 U(Y)}{\partial Y^2} \quad (25)$$

$$\frac{1}{W(\tau)} \frac{\partial W(\tau)}{\partial \tau} = \frac{1}{U(Y)} \frac{\partial^2 U(Y)}{\partial Y^2}. \quad (26)$$

Therefore the left hand side depends only of the variable  $t$  and the right hand side depends only of the variable  $Y$ ; then we can match the left hand side and the right hand side equal to a constant:

$$\frac{1}{W(\tau)} \frac{dW(\tau)}{d\tau} = -\lambda^2 \quad (27)$$

$$\frac{1}{U(Y)} \frac{d^2 U(Y)}{dY^2} = -\lambda^2 \quad (28)$$

note that we have choused like constant  $-\lambda^2$ , because it makes bounded the function  $F(Y, \tau)$ . Solving the above system of ODE, we have:

$$W(\tau) = W(0)e^{-\lambda^2 \tau} \quad (29)$$

$$\frac{d^2 U(Y)}{dY^2} + \lambda^2 U(Y) = 0. \quad (30)$$

The equation (27) is solved and its solution is done from equation (29). The equation (30) plus the boundary conditions is a Sturm-Liouville problem, the which solution is offered hereafter:

$$\frac{d^2 U(Y)}{dY^2} + \lambda^2 U(Y) = 0 \quad Y \in [A, B]$$

$$U(A) = 0, \quad U(B) = 0;$$

In order to change the interval of the definition and thus to simplify the computation, we introduce the subsequent variable:

$$Y = \eta + A \quad \Longrightarrow \quad \eta = Y - A$$

Hence, we have  $U(Y) = U(\eta + L) = \aleph(\eta)$ , where  $\eta \in [0, l]$  and  $l = B - A = \ln H - \ln L$

$$\frac{dU(Y)}{dy} = \frac{d\aleph}{d\eta}, \quad \frac{d^2U(Y)}{dY^2} = \frac{d^2\aleph}{d\eta^2}.$$

The equation (34) is now defined in the interval  $[0, l]$

$$\begin{aligned} \frac{d^2\aleph(\eta)}{d\eta^2} + \lambda^2\aleph(\eta) &= 0 \quad \eta \in [0, l] \\ \aleph(0) &= 0, \quad \aleph(l) = 0; \end{aligned}$$

The solution of the equation is given by following relation:

$$\aleph(\eta) = \sum_{k=-\infty}^{+\infty} \left[ c_k \sin \frac{k\pi\eta}{l} \right]. \quad (31)$$

where  $\alpha_k$  is equal to zero for the boundary condition  $\aleph(0) = 0$ . At this point, after we have substituted the variable  $Y$  with  $\eta$ , thus  $F(Y, \tau) = \bar{F}(\eta, \tau)$  and we can write the solution of the heat equation (27) as follows:

$$\frac{\partial \bar{F}}{\partial \tau} = \frac{\partial^2 \bar{F}}{\partial \eta^2} \quad (32)$$

$$\eta \in [0, l] \quad \tau \in [0, \bar{T}], \quad \bar{T} = \frac{1}{2} \int_0^T \sigma^2(s) ds;$$

$$\bar{F}(0, \tau) = 0 \quad F(l, \tau) = 0,$$

$$\bar{F}(\eta, 0) = \psi(e^{\eta(T)+A}) = (X(T) - K)^+$$

Remembering that  $\bar{F}(\eta, \tau) = \aleph(\eta)W(\tau)$ , hence we have:

$$\bar{F}(\eta, \tau) = \sum_{k=-\infty}^{+\infty} e^{-\left(\frac{k\pi}{l}\right)^2 \tau} \left[ c_k \sin \left( \frac{k\pi\eta}{l} \right) \right], \quad (33)$$

and this is true if and only

$$c_k = \frac{2}{l} \int_0^l d\xi (\xi - K)^+ \sin \left( \frac{k\pi\xi}{l} \right) \quad (34)$$

$$\bar{F}(\eta, \tau) = \sum_{k=-\infty}^{+\infty} e^{-\left(\frac{k\pi}{l}\right)^2 \tau} \left[ \frac{2}{l} \int_0^l d\xi (\xi - K)^+ \sin \left( \frac{k\pi\xi}{l} \right) \sin \left( \frac{k\pi\eta}{l} \right) \right], \quad (35)$$

$$\bar{F}(\eta, \tau) = \int_0^l d\xi(\xi - K)^+ \left[ \frac{2}{l} \sum_{k=-\infty}^{+\infty} e^{-\left(\frac{k\pi}{l}\right)^2 \tau} \sin\left(\frac{k\pi\xi}{l}\right) \sin\left(\frac{k\pi\eta}{l}\right) \right]. \quad (36)$$

In order to simplify the above relation we introduce the Green's function:

$$G(\eta, \xi) = \left[ \frac{2}{l} \sum_{k=-\infty}^{+\infty} e^{-\left(\frac{k\pi}{l}\right)^2 \tau} \sin\left(\frac{k\pi\xi}{l}\right) \sin\left(\frac{k\pi\eta}{l}\right) \right], \quad \eta, \xi \in [0, l]$$

so that we may write, in very elegant way, the solution of the parabolic PDE in canonical form of the second order, as follows:

$$\bar{F}(\eta, \tau) = \int_0^l d\xi(\xi - K)^+ G(\eta, \xi). \quad (37)$$

and using the Poisson's transform, we can write the Green's function in the form of the difference between two normal distributions:

$$\begin{aligned} G(\eta, \xi) &= \left[ \frac{2}{l} \sum_{k=-\infty}^{+\infty} e^{-\left(\frac{k\pi}{l}\right)^2 \tau} \sin\left(\frac{k\pi\xi}{l}\right) \sin\left(\frac{k\pi\eta}{l}\right) \right] \\ &= \frac{1}{2\sqrt{\pi\tau}} \sum_{k=-\infty}^{+\infty} \left[ e^{-\frac{(\eta-\xi+2kl)^2}{4\tau}} - e^{-\frac{(\eta+\xi+2kl)^2}{4\tau}} \right] \end{aligned} \quad (38)$$

Therefore it is corrected to write:

$$G(\eta, \xi) = \sum_{k=-\infty}^{+\infty} [P_0^l(\eta - \xi + 2kl, \tau) - P_0^l(\eta + \xi + 2kl, \tau)] \quad (39)$$

and finally we can read the solution in compact way:

$$\begin{aligned} \bar{F}(\eta, \tau) &= \int_0^l d\xi(\xi - K)^+ G(\eta, \xi) \\ &= \int_0^l d\xi(\xi - K)^+ \sum_{k=-\infty}^{+\infty} [P_0^l(\eta - \xi + 2kl, \tau) - P_0^l(\eta + \xi + 2kl, \tau)]. \end{aligned} \quad (40)$$

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