Pricing of Double Barrier Options by Spectral Theory

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Abstract

We propose to discuss the efficiency of the spectral method for computing the value of Double Barrier Options. Using this method, one may write the option price as a Fourier series, with suitable coefficients. We propose a simple approach for its computing. One consider the general case, in which the volatility is time dependent, but it is immediate extend our methodology also in the case of constant volatility. The advantage to write the arbitrage price of the Double Barrier Options as Fourier series, is matter of computation complexity. The methods used to evaluate options of this kind have a high value of computation complexity, furthermore, them have not the capacity to manage it, while using our method, one can define, through an easy analytical report, the computation complexity of the problem, and also one can choice its accuracy.

1 Introduction

The theoretical foundation that allow to use the spectral theory in mathematical finance is given from the class to which belongs the Black-Scholes PDE. Because this is ever a parabolic PDE and for the the Hilbert Schmidt theorem, its solution can be written as the sum of the eigenfunctions and it is worth noting that for the above mentioned theorem its spectrum is discrete. So that one can write the option price as a series, whose coefficients are straightforward integral functions. The convergence speed of our Fourier series, is very quick, if the following condition is verified: $\sigma^2/2 \geq l/\pi(T-t)$, but this will be clear later. Linetsky (2003) finds a series expansion of the option price similar to our, but he follows a different approach from our. In this paper we compare the spectral expansion of the option price with the Monte Carlo Method. We show that using the spectral expansion is possible to define the computation complexity of the problem and thus it is possible to manage it, unlike of Monte Carlo method.

The Barrier-Options belong to the class of Exotic Options. These usually are traded between companies and banks and not quoted on an exchange. In this case, we usually say that them are traded in the over the counter market. Most Exotic Options are quite complicated, and their final values depend not only on the asset price at expiry but also on the asset price at previous times. They are determined by a part or the whole of the path of the asset price during the life of option. These options are called path-dependent Exotic Options. Over the time, several papers have studied the issue to evaluate the price of the Barrier Options and Double-Barrier Options; Snyder (1969) describes down-and-out stock options as limited risk special options. Merton (1973) derives a closed-form pricing formula for down-and-out calls. A down-and-out call is identical to a European call with the additional provision that the contract is canceled (knocked out) if the underlying asset price hits a prespecified lower barrier level. An up-and-out call is the same, except the contract is canceled when the underlying asset price first reaches a prespecified upper barrier level. Down-and-out and up-and-out puts are similar modifications of European put options. Knock-in options are complementary to the knock-out options: they pay off at expiration if and only if the underlying asset price does reach the prespecified barrier prior to expiration. Rubinstein and Reiner (1991) derive closed form pricing formulas for all eight types of single-barrier options. Double-barrier (double knock-out) options are canceled (knocked out) when the underlying asset first reaches either the upper or the lower barrier. Double-barrier options have been particularly popular in the OTC currency options markets over the past several years, owing in part to the significant volatility of exchange rates experienced during this period. In response to their popularity in the marketplace, there is a growing literature on double-barrier options. Kunitomo and Ikeda (1992) derive closed-form pricing formulas expressing the prices of double-barrier knock-out calls and puts through infinite series of normal probabilities. Geman and Yor (1996) analyze the problem by probabilistic methods and derive closed-form expressions for the Laplace transform of the double-barrier option price in maturity. Schroder (2000) inverts this Laplace transform analytically using the Cauchy Residue Theorem, expresses the resulting trigonometric series in terms of Theta functions, and studies its convergence and numerical properties. Pelsser (2000) considers several variations on the basic double-barrier knock-out options, including binary double-barrier options (rebate paid at the first exit time from the corridor) and double-barrier knock-in options, and expresses their pricing formulas in terms of trigonometric series.Hui (1997) prices partial double-barrier options, including front-end and rear-end barriers. Further analysis and extensions to various versions of double-barrier contracts traded in the marketplace are given by Douady (1998), Jamshidian (1997), Hui, Lo and Yuen (2000), Schroder (2000), Sidenius (1998) and Zhang (1997). Rogers and Zane (1997) develop numerical methods for double-barrier options with time-dependent barriers. Taleb (1997) discusses practical issues of trading and hedging double-barrier options.

2 The Black-Scholes Equation and its Transformation into the Canonical form of Parabolic PDE

In order to write the Black-Scholes equation let us make the following assumptions: the borrowing interest rate and the lending interest rate are equal to r, short selling is permitted, the assets and options are divisible, and there is no transaction cost. Therefore, we can conclude that the absence of arbitrage opportunities is equivalent to all risk-free portfolios having the same return rate r. Let f denote the value of an option that depends on the value of the underlying asset X and time t, i.e $f = f(X,t)$, where $t \in [0,T]$. This last can be considered as the value of a whole portfolio of various options, for simplicity, we can think to a simple call or put. Assume that in a time step dt, the underlying asset pays out a dividend $qX(t)dt$, where q is the dividend yield:

we suppose that $X(t)$ satisfies a geometrical Brownian Motion in which the parameters r, q, σ are time dependent:

$$dX(t) = \mu X(t)dt + \sigma(t)X(t)dW \tag{1}$$

We require f to have at least one t derivative and two X.

At this point we construct a self financing portfolio consisting of one option and a number $\alpha(t)$ of the underlying asset and a number $\beta(t)$ of the bonds. The value of this portfolio is:

$$f(X,t) = \alpha(t)X(t) + \beta(t)B(t) \tag{2}$$

considering the quantities α, β, q, r time dependent, we omit to write this, hence we have:

$$\beta B(t) = f(X,t) - \alpha X(t) \tag{3}$$

Besides, the owner of the portfolio receives $qXdt$ for every asset held, the gain for the owner of the portfolio during the time dt is:

$$df = \alpha dX + \beta r B(t)dt + \alpha qXdt \tag{4}$$

Now, we can match the equation (6) with the equation (8) and choosing

$$\alpha = \frac{\partial f}{\partial X} \tag{5}$$

Thus we have the Black-Scholes equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t)X^2\frac{\partial^2 f}{\partial X^2} + (r(t) - q(t))X\frac{\partial f}{\partial X} - r(t)f = 0 \tag{6}$$

In order to grant the existence and the uniqueness of the solution, it is necessary to define the boundary condition and the initial condition. Also we require that when the value of the underlying asset hits the two barriers, lower and upper, the option is cancelled; being the Black-Scholes equation a type of equation for backward induction, we are interested at the value of f in the time T, and this is true because $t \in [0,T]$, where K is the strike price:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t)X^2\frac{\partial^2 f}{\partial X^2} + (r(t) - q(t))X\frac{\partial f}{\partial X} - r(t)f = 0 \tag{7}$$

$$X \in [L,H]; \qquad t \in [0,T] \tag{8}$$

$$f(L,t) = 0; \qquad f(H,t) = 0 \tag{9}$$

$$f(X,T) = max(\pm(X(T) - K), 0)\mathbb{I}_{(L<X(t)<H:t\in[0,T])} \tag{10}$$

At this point we can introduce same transformations by which we reduce the Black-Scholes equations to the heat equation and this because, Green's function of the heat equation has an analytical expression.

The transformation that changes the Black-Scholes equation into a heat equation(Canonical form of parabolic PDE) is known, infact in literature exists more methods to do it. We have choused the following transformation of variables to turn the equation (7) into a heat equation with boundary conditions:

$$Y = lnX + \int_t^T \left(r(s) - q(s) - \frac{1}{2}\sigma^2(s)\right) ds \tag{11}$$

$$\tau = \frac{1}{2}\int_t^T \sigma^2(s)ds \tag{12}$$

$$f(X,t) = e^{-\int_t^T r(s)ds}F(Y,\tau) \tag{13}$$

Substituting the relations (11) (12) (13) in the equation of the Black-Scholes (7), this last assumes the canonical form of PDE of the parabolic kind:

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial Y^2} \tag{14}$$

$$Y \in [A,B] \qquad \tau \in \left[0, \frac{1}{2}\int_0^T \sigma^2(s)ds\right],$$

$$F(A,\tau) = 0 \qquad F(B,\tau) = 0,$$

$$F(Y,0) = \psi(e^{Y(T)})$$

where $A(t) = (\ln L + \int_t^T \theta(s)ds)$, $B(t) = (\ln H + \int_t^T \theta(s)ds)$, $\theta(t) = (r(t) - q(t) - \frac{1}{2}\sigma^2(t))$ and $\psi(e^{Y(T)}) = max(\pm(X(T) - K), 0)\mathbb{I}_{(L<X(t)<H:t\in[0,T])}$.

First to solve the equation (14) we want to show that for constant values of the parameters, the last transformations of variables that turn the equation of the Black-Scholes (7) into a heat equation with boundary conditions, becomes:

$$Y = lnX + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t) \tag{15}$$

$$\tau = \frac{1}{2}\sigma^2(T-t) \tag{16}$$

$$f(X,t) = e^{-r(T-t)}F(Y,\tau) \tag{17}$$

We want to remark that the transformation to convert the Black-Scholes equation into heat equation is not unique.

3 Pricing Double Barrier Options

The value of a Knock-out down and out Call or Put options is given by the solution of the Black-Scholes equation with boundary conditions, that we have seen in the section number two. It is clear that in analogous way we can write the price of a Knock-in Option, but here we have chosen to study only the case of a Knock-Out. One can prove (see the appendix), that the solution of equation (14) is given by hereafter theorem:

Pricing Theorem In Black Scholes framework the arbitrage-price, of a Knock-out Call Double Barrier Options, is given by relation:

$$\begin{aligned} f(X,t) = \int_0^{\ln(H/L)} & d\xi e^{-\int_t^T dsr(s)}(e^{\xi}L-K)^{+}\mathbb{I}_{[A(t)<\xi<B(t):t\in[0,T]]}G(X,\xi,t) \\ &= \int_0^{\ln(H/L)} d\xi e^{-\int_t^T dsr(s)}(e^{\xi}L-K)^{+}\mathbb{I}_{[A(t)<\xi<B(t):t\in[0,T]]} \\ \frac{2}{\ln(H/L)} \sum_{n=-\infty}^{\infty} & e^{-\frac{(n\pi)^2}{(\ln(H/L))^2}\frac{1}{2}\int_t^T\sigma^2(s)ds}\sin n\pi\left(\frac{\xi}{\ln(H/L)}\right)\sin n\pi\left(\frac{\ln(X/L)}{\ln(H/L)}\right) \end{aligned} \tag{18}$$

for every underlying asset value $X, \in [L,H]$; where H is the upper barrier and L is the lower barrier.

4 Numerical Implementation and Computation Complexity

In order to compute the price of a Knock-out Call Double Barrier option in two different settings, we compare our results with the prices obtained through Monte-Carlo simulations and with the prices given in Kunitomo and Ikeda (1992). The standard deviation is computed on a sample of 200 evaluations, each evaluation being performed on 5000 Monte-Carlo paths. Moreover, these two prices are within one standard deviation of the Monte-Carlo method. The Monte-Carlo method must be run using very small step sizes and many paths to be sure that the barriers are not touched. The consequence is that the Fourier expansion requires two orders of magnitude less operations than the Monte-Carlo method and one than the Laplace transform method used to Geman and Yor (1996)(e.g., to do it, using a standard Pc or Mac, it is necessary a fraction of a second). As a comparison, in the case of Asian options, Geman-Eydeland (1995) obtain a standard deviation as low as 0.001 for a sample of 50 evaluations, each of them being performed on 500 Monte-Carlo paths. It is in the context of delta hedging that the spectral expansion of the Asian option obtained in linetsky (2002) proves definitely superior, both theoretically and computationally. A final manner to illustrate this point is to show that, as expected, the sensitivity of the option price to the step size in Monte-Carlo simulations becomes extremely high when the time remaining to maturity is short and the strike price near to one of the two barriers H or L.

Remark
The method of computing of the arbitrage price of the double barrier options, through "Fourier expansion", is very efficient. Infact it's possible compile an easy algorithm, in order to have the the correct value of Double-Barrier Options and all this summing few eigenfunctions, not more of thirty. It is amazing to see the speed by which the our expansion converges at the price. In the methods in which are used the Monte Carlo Simulations

is necessary compute about five thousand integrals to have the price. The difference between the two methods is clear, exists a difference of two orders. Therefore the technique that we propose, following the articles of Linetsky(2002), is more efficient than that proposed by Pelsser(1997) and Geman Yor (1996). The advantage of our method is that makes decrease the computation complexity. The computational complexity theory is a branch of the theory of computation in computer science that investigates the problems related to the resources required to run algorithms, and the inherent difficulty in providing algorithms that are efficient for both general and specific computational problems. The our idea is to evaluate the price of Barrier-Options and Double-Barrier Options like the weighted sum of the eigenfunctions of the Black-Scholes differential operator, where the coefficients c_k are the weights. these last are integrals, the which value is given in numerical way:

$$c_n = \frac{2}{\ln(H/L)} \int_0^{\ln(H/L)} d\xi (e^{\xi} L - K)^+ \sin n\pi \left(\frac{\xi}{\ln \frac{H}{L}} \right) \tag{19}$$

and the price is given by following relation:

$$f(X,t) = e^{-\int_t^T r(s)ds} \sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n\pi}{\ln\frac{H}{L}}\right)^2 \frac{1}{2}\int_t^T \sigma^2(s)ds} \left[c_n \sin n\pi \left(\frac{\ln(X/L)}{\ln(H/L)} \right) \right], \tag{20}$$

Let us note that $e^{-\left(\frac{n\pi}{\ln\frac{H}{L}}\right)^2 \frac{1}{2}\int_t^T \sigma^2(s)ds}$ decreases quickly. Thus, choosing a small number, ϵ, and define $n(\epsilon)$ as hereafter:

$$exp\left[-\left(\frac{n\pi}{\ln\frac{H}{L}}\right)^2 \frac{1}{2}\int_t^T \sigma^2(s)ds \right] = \epsilon. \tag{21}$$

Hence, we have

$$\left(\frac{n\pi}{\ln\left(\frac{H}{L}\right)}\right)^2 = \frac{2}{\int_t^T \sigma^2(s)ds} \ln\left(\frac{1}{\epsilon}\right), \tag{22}$$

$$n(\epsilon) = \frac{1}{\pi} \ln\left(\frac{H}{L}\right) \sqrt{\frac{2}{\int_t^T \sigma^2(s)ds} \ln\left(\frac{1}{\epsilon}\right)}, \tag{23}$$

Thus, now, we can study the computation complexity of our problem, we define

$$a_n = e^{-\left(\frac{n\pi}{\ln(H/L)}\right)^2 \int_t^T \sigma^2(s)ds} c_n \tag{24}$$

so that we have

$$a_n = e^{-\left(\frac{n\pi}{\ln(H/L)}\right)^2 \int_t^T \sigma^2(s)ds} \frac{2}{\ln(H/L)} \int_0^{\ln(H/L)} d\xi (e^{\xi} L - K)^+ \sin n\pi \left(\frac{\xi}{\ln H/L)} \right) \tag{25}$$

the value of the integral is a function of n and it results to be

$$\int_0^{\ln(H/L)} d\xi (e^{\xi} L - K)^+ \sin n\pi \left(\frac{\xi}{\ln(H/L)} \right) = \int_{\ln(K/L)}^{\ln(H/L)} d\xi (e^{\xi} L - K) \sin n\pi \left(\frac{\xi}{\ln(H/L)} \right)$$

$$= (-1)^{n+1} K \left(\frac{n^2\pi^2(\ln(H/L) - n\pi)}{L \ln(H/L) \left[(\ln H/L)^2 + (n\pi)^2 \right]} \right)$$

thus we have

$$c_n = (-1)^{n+1} K \left(\frac{2n^2\pi^2(\ln(H/L) - n\pi)}{L (\ln H/L)^2 \left[(\ln H/L)^2 + (n\pi)^2 \right]} \right)$$

and

$$a_n = (-1)^{n+1} K \left(\frac{2n^2\pi^2(\ln(H/L) - n\pi)}{L (\ln H/L)^2 \left[(\ln H/L)^2 + (n\pi)^2 \right]} \right) e^{-\left(\frac{n\pi}{\ln(H/L)}\right)^2 \int_t^T \sigma^2(s)ds}$$

we can increase a_n with b_n, in other words $a_n < b_n$

$$b_n = (-1)^n K \left(\frac{2n\pi}{L(\ln H/L)^2} \right) e^{-\left(\frac{n\pi}{\ln(H/L)}\right)^2 \int_t^T \sigma^2(s)ds}$$

so that if the following report is true $\left(\frac{\pi}{\ln(H/L)}\right)^2 \int_t^T \sigma^2(s)ds \geq 1$
b_n converge very quickly and are sufficient to compute only three terms, to have an accurate solution , indeed it results $b_1 \sim 10^{-1}$, $b_2 \sim 10^{-4}$, $b_3 \sim 10^{-6}$, $b_4 \sim 10^{-9}$. If unfortunatly the above condition is not verified, it is sufficient to sum more coefficients, to obtain the option price. At this point we can state that the computation complexity of our problem is at least "10"(but it can be different, it depends from the accurancy), i.e, the number of operation necessary to calculate the price: in fact a PC must to compute three coefficients a_n, at these must be multiplied the eigenfunctions and hence summed. Let us observe, that greater is the lifetime of the options, smaller is the value of the number $n(\epsilon)$. Hence fixed ϵ like the accuracy of the problem, we can compute in approximate way the value of $f(X,t)$ using the partial sum of $n(\epsilon)$ eigenfunctions. Therefore, the contribution of the present paper is that it offers the formula by which is possible manage the accuracy, choosing the number of eigenfunctions necessary to obtain the accuracy wanted. The results obtained are in the tables shown in the last page, in which one can read the price of double knock-out call option at behavior of n^*, number of eigenfunctions used. Our results are compared to the results obtained from A. Pelsser(1997), Geman Yor(1996), Kunitomo Ikeda(1992), and reading their, we can be satisfied of our.

5 Conclusions

In this article, we have shown a simple and easy-to-use method for pricing Double Barrier Options. This approach is also able to provide the price of the Double-Barrier Options for values very tight of the bounds upper and lower in which can be efficiently further improved in a systematic manner by means increasing the number of eigenfunctions to use in the approximation scheme. Also we have shown that if the volatility verifies the condition $\sigma^2/2 \geq l/\pi(T-t)$, the number of eigenfunctions necessary to have a good price are very small. In order to name a few of most the important articles on this problem, we indicate: H. Geman and M. Yor(1996), and A. Pelsser(1997), V.Linetsky and D. Davydov(2002). The main goal of the present work is to study the computation complexity of algorithm, offering an explicit formula for it. This result is very important because often is used the Monte Carlo method or Laplace transform method to evaluate options of this kind, and for this former method there is no possible to manage the computation complexity because it is impossible to write an analytical formula that shows the its computation complexity. Therefore, given the power of this method, it is straightforward to generalize the approach to more complicated situations, because it is a general method to solve Black-Scholes equations with boundary conditions with constant parameters and time dependent parameters.

Table 1: Value of Double Knock-Out Call Option if $S = 1000$, $K = 1000$, $r = 0.05$, $(T - t) = 1/12$

σ	H	L	f(x, T-t)	k*
0.2	1500	500	25.1207	12
0.2	1200	800	24.7568	12
0.2	1050	950	2.5970	3
0.3	1500	500	36.5842	22
0.3	1200	800	29.4473	8
0.3	1050	950	0.9583	2
0.4	1500	500	47.8475	16
0.4	1200	800	25.8442	6
0.4	1050	950	0.4381	2

Table 2: Value of Double Knock-out Call Option if $S = 1000$, $K = 1000$, $r = 0.05$, $(T - t) = 1/2$

σ	H	L	f(X, T-t)	k*
0.2	1500	500	66.1289	13
0.2	1200	800	22.1128	5
0.2	1050	950	0.2468	2
0.3	1500	500	67.8773	9
0.3	1200	800	10.0795	4
0.3	1050	950	0.0754	1
0.4	1500	500	53.3454	7
0.4	1200	800	5.0195	3
0.4	1050	950	0.0321	1

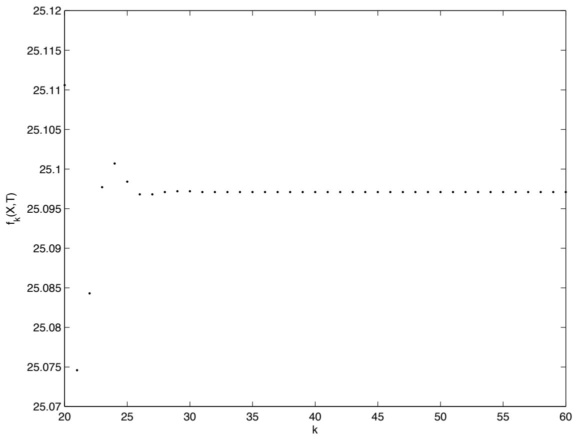

Figure 1: Approximate value of the Knock-out Double-Barrier Options respect to a $n(\epsilon)$ number of the eigenfunctions, with expiration date to one month

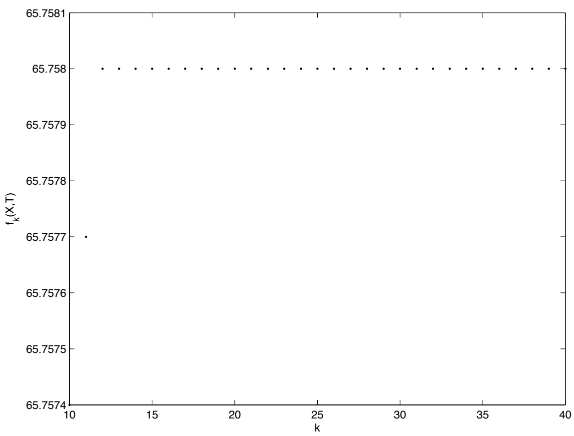

Figure 2: Approximate value of the Knock-out Double-Barrier Options respect to a $n(\epsilon)$ number of the eigenfunctions, with expiration date to six months

Appendix

Green's Function of Heat Equation with Boundary Conditions

Let be given the PDE in canonical form of the parabolic kind of the second order, with following boundary conditions:

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial Y^2} \tag{26}$$

$$Y \in [A, B] \qquad \tau \in \left[0, \overline{T}\right], \qquad \overline{T} = \frac{1}{2}\int_0^T \sigma^2(s)ds;$$

$$F(A, \tau) = 0 \qquad F(B, \tau) = 0,$$

$$F(Y, 0) = \psi(e^Y)$$

we can use the separable variable method and we rewrite the function $F(Y, \tau)$ as the product between $U(Y)$ and $W(\tau)$, in this way the PDE (27) becomes a system of two ODE in which one is a linear differential equation of the first order respect to t, and the remainder is a Sturm-Liouville problem of the second order:

$$\frac{\partial U(Y)W(\tau)}{\partial \tau} = \frac{\partial^2 U(Y)W(\tau)}{\partial Y^2} \tag{27}$$

thus we have

$$U(Y)\frac{\partial W(\tau)}{\partial \tau} = W(\tau)\frac{\partial^2 U(Y)}{\partial Y^2} \tag{28}$$

$$\frac{1}{W(\tau)}\frac{\partial W(\tau)}{\partial \tau} = \frac{1}{U(Y)}\frac{\partial^2 U(Y)}{\partial Y^2}. \tag{29}$$

Therefore the left hand side depends only of the variable t and the right hand side depends only of the variable Y; then we can match the left hand side and the right hand side equal to a constant:

$$\frac{1}{W(\tau)}\frac{dW(\tau)}{d\tau} = -\lambda^2 \tag{30}$$

$$\frac{1}{U(Y)}\frac{d^2 U(Y)}{dY^2} = -\lambda^2 \tag{31}$$

note that we have choused like constant $-\lambda^2$, because it makes bounded the function $F(Y, \tau)$. Solving the above system of ODE, we have:

$$W(\tau) = W(0)e^{-\lambda^2 \tau} \tag{32}$$

$$\frac{d^2 U(Y)}{dY^2} + \lambda^2 U(Y) = 0. \tag{33}$$

The equation (27) is solved and its solution is done from equation (29). The equation (30) plus the boundary conditions is a Sturm-Liouville problem, the which solution is offered hereafter:

$$\frac{d^2 U(Y)}{dY^2} + \lambda^2 U(Y) = 0 \qquad Y \in [A, B]$$

$$U(A) = 0, \qquad U(B) = 0;$$

In order to change the interval of the definition and thus to simplify the computation, we introduce the subsequent variable:

$$Y = \eta + A \qquad \Longrightarrow \qquad \eta = Y - A$$

Hence, we have $U(Y) = U(\eta + L) = \aleph(\eta)$, where $\eta \in [0, l]$ and $l = B - A = \ln H - \ln L$

$$\frac{dU(Y)}{dy} = \frac{d\aleph}{d\eta}, \qquad \frac{d^2 U(Y)}{dY^2} = \frac{d^2\aleph}{d\eta^2}.$$

The equation (34) is now defined in the interval $[0, l]$

$$\frac{d^2\aleph(\eta)}{d\eta^2} + \lambda^2\aleph(\eta) = 0 \qquad \eta \in [0, l]$$

$$\aleph(0) = 0, \qquad \aleph(l) = 0;$$

The solution of the equation is given by following relation:

$$\aleph(\eta) = \sum_{n=-\infty}^{+\infty} \left[c_n sin \frac{n\pi\eta}{l} \right]. \tag{34}$$

where α_n is equal to zero for the boundary condition $\aleph(0) = 0$. At this point, after we have substituted the variable Y with η, thus $F(Y,\tau) = \overline{F}(\eta,\tau)$ and we can write the solution of the heat equation (27) as follows:

$$\frac{\partial \overline{F}}{\partial \tau} = \frac{\partial^2 \overline{F}}{\partial \eta^2} \tag{35}$$

$$\eta \in [0,l] \qquad \tau \in \left[0, \overline{T}\right], \qquad \overline{T} = \frac{1}{2}\int_0^T \sigma^2(s)ds;$$

$$\overline{F}(0,\tau) = 0 \qquad F(l,\tau) = 0,$$

$$\overline{F}(\eta,0) = \psi(e^{\eta(T)+A}) = (X(T) - K)^+$$

Remembering that $\overline{F}(\eta,\tau) = \aleph(\eta)W(\tau)$, hence we have:

$$\overline{F}(\eta,\tau) = \sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \left[c_n \sin\left(\frac{n\pi\eta}{l}\right)\right], \tag{36}$$

and this is true if and only

$$c_n = \frac{2}{l}\int_0^l d\xi(\xi - K)^+ \sin\left(\frac{n\pi\xi}{l}\right) \tag{37}$$

$$\overline{F}(\eta,\tau) = \sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \left[\frac{2}{l}\int_0^l d\xi(\xi - K)^+ sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right)\right], \tag{38}$$

$$\overline{F}(\eta,\tau) = \int_0^l d\xi(\xi - K)^+ \left[\frac{2}{l}\sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right)\right]. \tag{39}$$

In order to simplify the above relation we introduce the Green's function:

$$G(\eta,\xi) = \left[\frac{2}{l}\sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right)\right], \qquad \eta,\xi \in [0,l]$$

so that we may write, in very elegant way, the solution of the parabolic PDE in canonical form of the second order, as follows:

$$\overline{F}(\eta,\tau) = \int_0^l d\xi(\xi - K)^+ G(\eta,\xi). \tag{40}$$

and using the Poisson's transform, we can write the Green's function in the form of the difference between two normal distributions:

$$G(\eta,\xi) = \left[\frac{2}{l}\sum_{n=-\infty}^{+\infty} e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\eta}{l}\right)\right]$$

$$= \frac{1}{2\sqrt{\pi\tau}}\sum_{n=-\infty}^{+\infty} \left[e^{-\frac{(\eta-\xi+2nl)^2}{4\tau}} - e^{-\frac{(\eta+\xi+2nl)^2}{4\tau}} \right] \tag{41}$$

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