Endogenous income taxes and indeterminacy in dynamic models: When Diamond meets Ramsey again.

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Endogenous income taxes and indeterminacy in dynamic models: When Diamond meets Ramsey again.

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Abstract

This paper introduces fiscal increasing returns, through endogenous labor income tax rates as in Schmitt-Grohe and Uribe (1997), into the overlapping generations model with endogenous labor, consumption in both periods of life and homothetic preferences (e.g., Lloyd-Braga, Nourry and Venditti, 2007). We show that under numerical calibrations of the parameters, local indeterminacy can occur for distortionary tax rates that are empirically plausible for the U.S. economy, provided that the elasticity of capital-labor substitution and the wage elasticity of the labor supply are large enough, and the elasticity of intertemporal substitution in consumption is slightly greater than unity. These indeterminacy conditions are similar to those obtained within infinite horizon models and from this point of view, Diamond meets Ramsey again.

Keywords: Indeterminacy; Endogenous labor income tax rate.

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1. Introduction

This article introduces constant government expenditure financed by labor income taxes in an aggregate overlapping generations model with endogenous labor, consumption in both periods of life and homothetic preferences. First, we show that when the share of first period consumption over the after–tax wage income is not large, local indeterminacy can occur when there are small labor income tax rates and this requires a negative stationary interest rate. Second, we show that endogenous fluctuations arise with small tax distortions, an elasticity of intertemporal substitution in consumption slightly greater than unity, a large enough elasticity of capital–labor substitution and a large enough elasticity of the labor supply .

Since Reichlin (1986), the Diamond (1965) one-sector overlapping generations model augmented to include endogenous labor supply, external effects and fiscal increasing returns has become a popular framework to analyze expectations driven business cycles.[1] Unlike those early works that focus on a particular case without first period consumption, recent works such as Cazzavillan and Pintus (2004, 2006), Lloyd-Braga et al. (2007) and Chen and Zhang (2009a, 2009b), consider a life-cycle utility function which is first, separable between consumption and leisure, and second, linearly homogenous with respect to young and old consumptions. The main contribution of the first two papers is to analyze the relationship between external effects and indeterminacy in the aggregate OLG model. Our paper differs from theirs in at least three aspects. First, we discuss the relationship between fiscal policy and indeterminacy in the very same aggregate OLG model. Particularly, we concentrate on the focal case where constant government expenditure is financed by labor income taxes and show that local indeterminacy occurs with small labor income tax rates, provided that the elasticity of

[1] For example, Cazzavillan (2001) and Gokan (2009a, 2009b).

capital-labor substitution and the elasticity of the labor supply are large enough. Second, in order to make our model analytically tractable while introducing constant government expenditure, together with current consumption, we need important assumptions on preferences: we assume that a life-cycle utility function is separable between consumption and leisure, and linearly homogenous with respect to consumptions (homothetic preferences). Lastly, we show that endogenous fluctuations can arise with a large enough elasticity of capital–labor substitution and a large enough elasticity of the labor supply. Some results have been shown by Chen and Zhang (2009 a,b) in an OLG model with endogenous labor income tax rates and totally separable preferences over young and old consumptions. By contrast, with homothetic preferences, local indeterminacy can be compatible with the empirical estimates of the elasticity of capital–labor substitution, and we find that local dynamics become much more complicated. In addition, Lloyd-Braga et al. (2007) find that local indeterminacy is associated with a large share of first period consumption over the wage income, which implies a positive interest rate. This result doesn't hold in our model.

Schmitt-Grohe and Uribe (1997) show that indeterminate equilibria may arise for empirically plausible ranges of labor income tax rates, in the context of a standard constant returns to scale neoclassical growth model, where the government's exogenous expenditures are financed solely by taxing labor (and capital) income. To be more precise, local indeterminacy arises within the range of (capital and) labor income tax rates observed for the United States, provided that the elasticity of the labor supply, the elasticity of intertemporal substitution in consumption and the elasticity of capital–labor substitution are large enough.[2] In this paper, we examine their indeterminacy result within an aggregate OLG model, and we find that our indeterminacy conditions are similar to those

[2] In the original model of Schmitt-Grohe and Uribe (1997), they find that the indeterminacy condition obtained in their model has a close correspondence with the one obtained in the increasing returns model of Benhabib and Farmer (1994) and that local indeterminacy is more likely, the higher the Frisch elasticity of labor supply with respect to wage. Pintus (2006) shows that in the one sector Ramsey model, indeterminacy occurs when externalities are small, provided that capital and labor are more substitutable than in the usual Cobb-Douglas specification, that concavity of utility for consumption is small enough and that labor supply is close to indivisible. Therefore, we have this strong conclusion.

obtained in their model and from this point of view, Diamond meets Ramsey again.

The paper is organized as follows. Section 2 sets up the model. In Section 3, we establish the existence of a normalized steady state. Section 4 contains the derivation of the characteristic polynomial and presents the geometrical method used for the local dynamic analysis and our main results on local indeterminacy. Section 5 gathers some concluding remarks.

2. The model

As in Lloyd-Braga et al. (2007), we consider a competitive, non-monetary, overlapping generations model with production. The model involves a unique perishable good, which can be either consumed or saved as investment. Identical competitive firms all face the same technology. Identical households live for two periods. The agent consumes in both periods, supplies labor and saves when young. When old, her saved income is rented as physical capital to the firm.

The household born at time $t \geq 0$ maximizes her lifetime utility

$$\max_{c_t,\, l_t,\, \hat{c}_{t+1}} \left[u(c_t, \hat{c}_{t+1}) - v\left(l_t/B\right) \right],$$

subject to the constraints

$$c_t + K_{t+1} = (1 - \tau_t)\, w_t l_t,$$

$$\hat{c}_{t+1} = R_{t+1} K_{t+1} \tag{1}$$

$$c_t \geq 0,\ \hat{c}_{t+1} \geq 0,\ l \geq l_t \geq 0, \text{ for all } t \geq 0.$$

where l_t, c_t and K_{t+1} are labor, consumption and saving (the amount of capital), respectively, of the individual of the young generation, $\hat{c}_{t+1}$ is the consumption of the same individual when old, and

$w_t > 0$ and $R_{t+1} > 0$ are the real wage rate at time t and the gross interest rate at time $t+1$.[3] Moreover, $\tau_t \in (0,1)$, $B > 0$ and l are the labor income tax rate, a scaling parameter and the maximum amount of labor supply, respectively.

The preferences satisfy the following condition as in Lloyd-Braga et al. (2007).

Assumption 1. (i) $u(c_t, \hat{c}_{t+1})$ is C^r over R_+^2 for r large enough, increasing with respect to each argument ($u_1(c_t, \hat{c}_{t+1}) > 0$, $u_2(c_t, \hat{c}_{t+1}) > 0$), concave and homogeneous of degree one over R_{++}^2. Moreover, for all c_t, $\hat{c}_{t+1} > 0$, $lim_{\hat{c}_{t+1}/c_t \to 0} u_1/u_2 = 0$ and $lim_{\hat{c}_{t+1}/c_t \to +\infty} u_1/u_2 = +\infty$, where u_1/u_2 stands for $u_1(1, \frac{\hat{c}_{t+1}}{c_t})/u_2(1, \frac{\hat{c}_{t+1}}{c_t})$. (ii) $v\,(l_t/B)$ is C^r over $[0, l/B]$ for r large enough, increasing ($v'\,(l_t/B) > 0$) and convex ($v"\,(l_t/B) > 0$) over $(0, l/B)$. Moreover, $lim_{l_t \to 0} v'(l_t/B) = 0$ and $lim_{l_t \to l} v(l_t/B) = +\infty$.

We introduce homogeneity in order to write the capital accumulation equation as a function of the ratio between young agents' consumption and the after-tax wage income. The first order conditions can be written as follows:

$$\frac{u_1(1, \frac{\hat{c}_{t+1}}{c_t})}{u_2(1, \frac{\hat{c}_{t+1}}{c_t})} \equiv g(\frac{\hat{c}_{t+1}}{c_t}) = R_{t+1}, \tag{2}$$

$$u_1(1, \frac{\hat{c}_{t+1}}{c_t})\,(1-\tau_t)\,w_t = \frac{v'(l_t/B)}{B}, \tag{3}$$

$$c_t + \frac{\hat{c}_{t+1}}{R_{t+1}} = (1-\tau_t)\,w_t l_t, \tag{4}$$

$$K_{t+1} = (1-\tau_t)\,w_t l_t - c_t \tag{5}$$

[3] We assume total depreciation of capital.

with $g\prime(\frac{\hat{c}_{t+1}}{c_t}) > 0$. It is easy to know that

$$\frac{\hat{c}_{t+1}}{c_t} = g^{-1}(R_{t+1}) \equiv h(R_{t+1}). \tag{6}$$

Using (2), (4), (6) and the homogeneity property, we can have

$$c_t = \frac{u_1(1, h(R_{t+1}))}{u(1, h(R_{t+1}))} (1 - \tau_t) w_t l_t \equiv \alpha(R_{t+1}) (1 - \tau_t) w_t l_t, \tag{7}$$

where $\alpha(R) \in (0, 1)$ is the share of first period consumption over the after-tax wage income. Moreover, equation (5) becomes

$$K_{t+1} = (1 - \alpha(R_{t+1})) (1 - \tau_t) w_t l_t. \tag{8}$$

As in Lloyd-Braga et al. (2007, p.516), the elasticity of intertemporal substitution in consumption $\gamma(R)$ and the wage elasticity of the labor supply are given by

$$\gamma(R) = \frac{R}{g'(h(R))h(R)} = -(\frac{u_{11}(1, h(R))}{u_1(1, h(R))} + \frac{u_{22}(1, h(R))h(R)}{u_2(1, h(R))})^{-1} > 0. \tag{9}$$

$$\varepsilon_l(l_t/B) = \frac{v'(l_t/B)}{(l_t/B)v''(l_t/B)} > 0. \tag{10}$$

It is easy for us to have the identity $\alpha(R) \equiv \frac{1}{1+\frac{h(R)}{R}}$ and the elasticity of the propensity to consume $\alpha(R)$: $\alpha'(R)\frac{R}{\alpha(R)} = (1 - \gamma(R))(1 - \alpha(R))$. The saving function is an increasing function of R iff $\gamma(R) > 1$.

The perishable output (y_t) is produced using capital (K_t) and labor (l_t),

$$y_t = AF(K_t, l_t) = Al_t f(a_t), \tag{11}$$

where $a_t = K_t/l_t$ and $A > 0$ is a scaling factor.

Assumption 2. The reduced production function $y_t/l_t = Af(a_t)$ is a continuous function of the capital-labor ratio $a_t = K_t/l_t \geq 0$ and has continuous derivatives of all required orders for $a_t > 0$, with $f'(a_t) > 0$, $f"(a_t) < 0$.

The competitive factor market implies that the real wage rate and the real gross rate of return on capital stock are

$$w_t = A\left[f\left(a_t\right) - a_t f'(a_t)\right] = Aw\left(a_t\right), \tag{12}$$

$$R_t = Af'\left(a_t\right). \tag{13}$$

As usual, the share of capital in total income and the elasticity of capital-labor substitution can be expressed as follows:

$$s\left(a\right) = \frac{af'\left(a\right)}{f\left(a\right)} \in \left(0, 1\right), \text{ and } \sigma(a) = -\frac{\left(1 - s(a)\right) f'(a)}{af"\left(a\right)} > 0. \tag{14}$$

As in Schmitt-Grohe and Uribe (1997), at each point in time, the government finances its **constant** expenditure through labor income taxes, i.e.,

$$g = \tau_t w_t l_t > 0. \tag{15}$$

We can easily derive the dynamic system characterizing equilibrium paths of $(K_t,\, l_t)$.

$$K_{t+1} = \left[1 - \alpha\left(Af'\left(a_{t+1}\right)\right)\right]\left[Aw\left(a_t\right) l_t - g\right], \tag{16-1}$$

$$\frac{l_t}{B} v'\left(\frac{l_t}{B}\right) = u_1\left[1, h\left(Af'\left(a_{t+1}\right)\right)\right]\left[Aw\left(a_t\right) l_t - g\right]. \tag{16-2}$$

with $a_t = K_t/l_t$, $g = \tau_t w_t l_t$ and K_0 given.

3. Steady state existence

A steady state is a pair $(K^*,\ l^*)$ such that.

$$K^* = \left[1 - \alpha\left(Af'\left(K^*/l^*\right)\right)\right]\left[Aw\left(K^*/l^*\right)l^* - g\right], \tag{17-1}$$

$$\frac{l^*}{B}v'\left(\frac{l^*}{B}\right) = u_1\left[1, h\left(Af'\left(K^*/l^*\right)\right)\right]\left[Aw\left(K^*/l^*\right)l^* - g\right]. \tag{17-2}$$

To simplify the algebra, we follow the procedure used in Lloyd-Braga et al. (2007) and use the parameters A and B to normalize the steady state.

Proposition 1. *Under those assumptions on the utility and production functions, let* $V(B) = v'(\frac{1}{B})/B$. *Then* $(K^*, l^*) = (1, 1)$ *is a normalized steady state (NSS) of the dynamic system (16) if and only if* $\lim_{A\to+\infty} G(A) > 1$, *where* $G(A) \equiv [1 - \alpha(Af'(1))][Aw(1) - g]$. *The scaling parameters are set at* $A^* > 0$ *and* $B^* > 0$ *that satisfy the following equations:*

$$1 = \left[1 - \alpha\left(Af'(1)\right)\right]\left[Aw(1) - g\right],$$

$$B = V^{-1}\{u_1\left[1, h\left(Af'(1)\right)\right]\left[Aw(1) - g\right]\}.$$

Proof. See Appendix A.1. ■

Multiplicity of steady states may arise in our model. For brevity, we just analyze the local dynamics around the NSS.

Assumption 3. $\lim_{A\to+\infty} G(A) > 1$, $A = A^*$ and $B = B^*$.

Before we study the local dynamics around the NSS, we evaluate all the shares and elasticities at the NSS. We set $\alpha(A^*f'(1)) = \alpha$, $\gamma(A^*f'(1)) = \gamma$, $\varepsilon_l(1/B^*) = \varepsilon_l$, $s(1) = s$, $\sigma(1) = \sigma$ and $g = \tau^{NSS}A^*w(1)$, where τ^{NSS} is the steady state labor income tax rate evaluated at the NSS.

Proposition 2. *Under Assumptions 1-3, $R^* \geq 1$ iff $\alpha \geq \frac{(1-\tau^{NSS})(1-s)-s}{(1-\tau^{NSS})(1-s)} \equiv \alpha_1$. In this case, the interest rate around the NSS is positive.*[4]

Proof. *See Appendix A.2.* ■

We assume that the share of capital in total income is less than $\frac{1-\tau^{NSS}}{2-\tau^{NSS}}$, which makes the lower bound α_1 positive. It implies that if a positive interest rate exists, a large share of first period consumption over the wage income is required.

4. Local dynamics analysis

First, we linearize the dynamic system around the NSS $(1, 1)$.

Proposition 3. *The two-dimensional system (16) defines uniquely a local dynamics near the NSS $(K^*, l^*) = (1, 1)$. The linearized dynamics for the deviations $dK_t = K_t - K^*$, $dl_t = l_t - l^*$ are determined by the determinant D and the trace T of the Jacobian matrix. And the expressions of D and T are given by*

$$D = \frac{s}{(1-s)(1-\alpha)} \frac{1}{1-\tau^{NSS}} (\frac{1+\varepsilon_l}{\varepsilon_l}),$$

$$T = \frac{1}{(1-s)(1-\alpha)} \{ \frac{1-\sigma-\alpha\gamma(1-s)}{1-\tau^{NSS}} + (\frac{1+\varepsilon_l}{\varepsilon_l}) [\sigma - \alpha(1-\gamma)(1-s)] \}.$$

Proof. *See Appendix A.3.* ■

A simple way to analyze the local dynamics of the normalized steady state is to observe the variation of the trace T and the determinant D in the (T, D) plane as some parameters are made vary continuously. In particular, we are interested in the two roots of the characteristic polynomial $Q(\pi) = \pi^2 - T\pi + D$. There is a local eigenvalue which is equal to $+1$ when $1 - T + D = 0$. It is represented by the line (AC) in Fig. 1. Moreover, one eigenvalue is -1 when $1 + T + D = 0$. That

[4] *Notice that local indeterminacy is compatible with a negative interest rate, which we will show in the next section.*

is to say, in this case, (T, D) lies on the line (AB). Finally, the two roots are complex conjugate of modulus 1, whenever (T, D) belongs to the segment [BC] which is defined by $D = 1$, $|T| \leq 2$. Since both roots are zero when both T and D are 0, then, by continuity, they have both a modulus less than one iff (T, D) lies in the interior of the triangle ABC, which is defined by $|T| < |1+D|$, $|D| < 1$. The steady state is then locally indeterminate given that there is a unique predeterminate variable K_t. If $|T| > |1 + D|$, the stationary state is a saddle-point. Finally, in the complementary region $|T| < |1 + D|, |D| > 1$, the steady state is a source.

The diagram can also be used to study local bifurcations. When the point (T, D) crosses the interior of the segment [BC], a *Hopf bifurcation* is expected to occur. If, instead, the point crosses the line (AB), one root goes through -1. In that case, a *flip bifurcation* is expected to occur. Finally, when the point crosses the line (AC), one root goes through $+1$, one expects an exchange of stability between the NSS and another steady state through a *transcritical bifurcation.*

As in Lloyd-Braga et al. (2007), we focus on two parameters, the elasticity of capital–labor substitution (σ, an independent parameter) and the elasticity of labor supply (ε_l, a bifurcation parameter), which varies from zero to $+\infty$. From the expressions of D and T given in Proposition 3, we find that $(T(\varepsilon_l),\ D(\varepsilon_l))$ describes a half-line Δ, which equation is

$$D = \mathcal{S}T - \mathcal{S}\frac{1-\sigma-\alpha\gamma\left(1-s\right)}{\left(1-\tau^{NSS}\right)\left(1-s\right)\left(1-\alpha\right)},$$

where the slope $\mathcal{S}$ is

$$\mathcal{S} = \frac{s}{\left(1-\tau^{NSS}\right)\left[\sigma-\alpha\left(1-\gamma\right)\left(1-s\right)\right]}. \tag{18}$$

As $\varepsilon_l \in (0, +\infty)$, the starting and end points of the half line Δ are:

$$\begin{aligned}
\lim_{\varepsilon_l \to +\infty} D(\varepsilon_l) &= D_1 = \frac{s}{(1-s)(1-\alpha)\left(1-\tau^{NSS}\right)}, \\
\lim_{\varepsilon_l \to +\infty} T(\varepsilon_l) &= T_1 = \frac{\frac{1-\sigma-\alpha\gamma(1-s)}{1-\tau^{NSS}} + \sigma - \alpha(1-\gamma)(1-s)}{(1-s)(1-\alpha)}.
\end{aligned} \tag{19}$$

Since $D(\varepsilon_l)$ decreases with ε_l and $\lim_{\varepsilon_l \to 0} D(\varepsilon_l) = +\infty$, the relevant part of the half line Δ thus starts in (T_1, D_1) for $\varepsilon_l = +\infty$ and points upwards to the right (or to the left) as ε_l decreases.

Here we assume gross substitutability, i.e. $\gamma \geq 1$. Then the half line Δ points upwards to the right as $\mathcal{S} > 0$. Thus a necessary condition for the existence of local indeterminacy is $D_1 < 1$. Notice that, for fixed values of τ^{NSS}, s and α, D_1 is independent of σ.

Assumption 4. $\gamma \geq 1$, $s \leq \frac{1-\tau^{NSS}}{2-\tau^{NSS}}$, and $\alpha < \frac{\left(1-\tau^{NSS}\right)(1-s)-s}{\left(1-\tau^{NSS}\right)(1-s)} \equiv \alpha_1$.

When $\alpha < \alpha_1$, $D_1 < 1$ holds for any $\sigma \in [0, +\infty)$ and local indeterminacy can occur. In this case, the stationary interest rate is negative. Let us analyze how the starting point $(T_1(\sigma),\, D_1(\sigma))$, given in (19), and the slope $\mathcal{S}(\sigma)$ change with σ. We define a flat half-line Δ_1 linking the points T_1 and D_1 for different values of $\sigma \in [0, +\infty)$.

$$\begin{aligned}
\lim_{\sigma \to 0} T_1 &= T_1^0 = \frac{1-\alpha(1-s)\left[1-\tau^{NSS}(1-\gamma)\right]}{(1-s)(1-\alpha)\left(1-\tau^{NSS}\right)}, \\
\lim_{\sigma \to +\infty} T_1 &= T_1^\infty = -\infty, \\
D_1 &= D_1^0 = \frac{s}{(1-s)(1-\alpha)\left(1-\tau^{NSS}\right)}.
\end{aligned}$$

In graphical terms, local indeterminacy can occur in the following three cases (see Figure 1):

- the point $\left(T_1^0, D_1^0\right)$ lies on the right side of the line AC and $D_1^0 < 1$.
- the point $\left(T_1^0, D_1^0\right)$ lies inside the triangle ABC and $D_1^0 < 1$.
- the point $\left(T_1^0, D_1^0\right)$ lies on the left side of the line AB and $D_1^0 < 1$.

Insert Figure 1 here.

Lemma 1. *$\mathcal{S}$ is decreasing with σ, and $\mathcal{S}_0 \equiv \mathcal{S}(\sigma = 0) = \frac{s}{\alpha(1-\tau^{NSS})(\gamma-1)(1-s)} < \frac{1}{\alpha(\gamma-1)}$.*

Proof. $\mathcal{S}_0 \equiv \mathcal{S}(\sigma = 0) = \frac{s}{\alpha(1-\tau^{NSS})(\gamma-1)(1-s)}$. $\tau^{NSS} < 1-\frac{s}{1-s}$ implies both $\mathcal{S} = \frac{1}{1-\tau^{NSS}} \frac{s}{[\sigma-\alpha(1-\gamma)(1-s)]} < \frac{1-s}{\sigma-\alpha(1-\gamma)(1-s)}$ and $\mathcal{S}_0 < \frac{1}{\alpha(\gamma-1)}$. ■

If $\alpha(\gamma - 1) > 1$, we have $\mathcal{S}_0 < 1$. In addition, $\mathcal{S}$ is less than 1. This inequality can be met for a sufficiently large γ.

It is easy for us to have the following properties. Case (1) arises if $\alpha\gamma < 1$ and $\alpha < \alpha_1$ hold; case (3) arises if $\alpha\gamma > \Lambda \equiv (1-2\alpha)\left(\frac{1}{\tau^{NSS}} - 1\right) + \frac{\frac{1+s}{1-s}}{\tau^{NSS}}$ and $\alpha < \alpha_1$ hold; and case (2) arises if $1 < \alpha\gamma < \Lambda$ and $\alpha < \alpha_1$ hold.[5] Moreover, we know that $\mathcal{S} < \mathcal{S}_0$. Notice that (1) $\mathcal{S}_0 > (<)1$ iff $\alpha\gamma < (>)\Psi \equiv \alpha + \frac{s}{(1-\tau^{NSS})(1-s)}$; and (2) $\Psi < 1$ holds when $\alpha < \alpha_1$. Therefore, we can summarize these results as follows.

Case 1. *There are two subcases in case 1. Subcase (1.1): $\alpha < \alpha_1$ and $\alpha\gamma < \Psi$ (< 1). In this subcase, $\mathcal{S}_0 > 1$ and $D_1^0 < 1$ hold, and the point $\left(T_1^0, D_1^0\right)$ lies on the right side of the line AC. Subcase (1.2): $\Psi < \alpha\gamma < 1$ and $\alpha < \alpha_1$. In this subcase, $\mathcal{S}_0 < 1$ and $D_1^0 < 1$ hold, and the point $\left(T_1^0, D_1^0\right)$ lies on the right side of the line AC.*

Case 2. *$1 < \alpha\gamma < \Lambda$ and $\alpha < \alpha_1$. In this subcase, $\mathcal{S}_0 < 1$ and $D_1^0 < 1$ hold, and the point $\left(T_1^0, D_1^0\right)$ lies inside the triangle ABC.*

Case 3. *$\alpha\gamma > \Lambda$ and $\alpha < \alpha_1$. In this subcase, $\mathcal{S}_0 < 1$ and $D_1^0 < 1$ hold, and the point $\left(T_1^0, D_1^0\right)$ lies on the left side of the line AB.*

[5] We find that when $D_1^0 < 1$ holds (or, $\alpha < \alpha_1$), $\Lambda > 1$ holds. For the first case, the point $\left(T_1^0, D_1^0\right)$ lies on the right side of the line AC, which means that $D_1^0 < T_1^0 - 1$. For the third case, the point $\left(T_1^0, D_1^0\right)$ lies on the left side of the line AB, which means that $D_1^0 < -T_1^0 - 1$. For the second case, the point $\left(T_1^0, D_1^0\right)$ lies inside the triangle ABC, which means that $D_1^0 > T_1^0 - 1$ and $D_1^0 > -T_1^0 - 1$ hold.

Since our purpose is to give conditions for local indeterminacy of equilibria under small labor income tax rates, an important issue is to study the intersections of the half line Δ with the lines AC, AB and BC. First, as Δ crosses the line AC, the vertical coordinate of the intersection point is[6]

$$\widehat{D}_{AC} = \frac{s\left[\frac{\sigma}{(1-\tau^{NSS})(1-s)(1-\alpha)} + \frac{\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)+\alpha\gamma(1-s)-1}{(1-\tau^{NSS})(1-s)(1-\alpha)}\right]}{\left(1-\tau^{NSS}\right)\sigma + \alpha\left(\gamma-1\right)\left(1-s\right)\left(1-\tau^{NSS}\right) - s}.$$

And we have the following results:

$$\begin{aligned}
\lim_{\sigma\to+\infty} \widehat{D}_{AC} &= \frac{s}{\left(1-\tau^{NSS}\right)^2\left(1-s\right)\left(1-\alpha\right)}, \\
\widehat{D}_{AC}\left(\sigma=0\right) &= \frac{s}{\left(1-\tau^{NSS}\right)\left(1-s\right)\left(1-\alpha\right)} \frac{\alpha\gamma\left(1-s\right) + \left(1-\tau^{NSS}\right)\left(1-s\right)\left(1-\alpha\right) - 1}{\alpha\gamma\left(1-s\right)\left(1-\tau^{NSS}\right) - s - \alpha\left(1-s\right)\left(1-\tau^{NSS}\right)}.
\end{aligned}$$

Second, as Δ crosses the line AB, the vertical coordinate of the intersection point is[7]

$$\widehat{D}_{AB} = \frac{s}{\left(1-\tau^{NSS}\right)^2\left(1-s\right)\left(1-\alpha\right)} \frac{\sigma + \alpha\gamma\left(1-s\right) - \left(1-\tau^{NSS}\right)\left(1-s\right)\left(1-\alpha\right) - 1}{\sigma + \alpha\left(\gamma-1\right)\left(1-s\right) + s/\left(1-\tau^{NSS}\right)}.$$

And we have the following results:

$$\begin{aligned}
\lim_{\sigma\to+\infty} \widehat{D}_{AB} &= \frac{s}{\left(1-\tau^{NSS}\right)^2\left(1-s\right)\left(1-\alpha\right)}, \\
\widehat{D}_{AB}\left(\sigma=0\right) &= \frac{s}{\left(1-\tau^{NSS}\right)^2\left(1-s\right)\left(1-\alpha\right)} \frac{\alpha\gamma\left(1-s\right) - \left(1-\tau^{NSS}\right)\left(1-s\right)\left(1-\alpha\right) - 1}{\alpha\left(\gamma-1\right)\left(1-s\right) + s/\left(1-\tau^{NSS}\right)}.
\end{aligned}$$

[6] When the half line Δ crosses the line AC, both $D = \mathcal{S}T - \mathcal{S}\frac{1-\sigma-\alpha\gamma(1-s)}{\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)}$ and $D = T-1$ hold.

[7] When the half line Δ crosses the line AB, both $D = \mathcal{S}T - \mathcal{S}\frac{1-\sigma-\alpha\gamma(1-s)}{\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)}$ and $D = -T-1$ hold.

Third, as Δ crosses the line BC, the horizontal coordinate of the intersection point is[8]

$$\widehat{T}_{BC} = \sigma[\frac{1-\tau^{NSS}}{s} - \frac{1}{\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)}] + \frac{\alpha(\gamma-1)(1-s)\left(1-\tau^{NSS}\right)}{s} + \frac{1-\alpha\gamma(1-s)}{\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)}.$$

And we have the following result:

$$\begin{aligned}\widehat{T}_{BC}(\sigma=0) &= \alpha\gamma\left[\frac{(1-s)\left(1-\tau^{NSS}\right)}{s} - \frac{1}{\left(1-\tau^{NSS}\right)(1-\alpha)}\right] \\ &\quad + \frac{1}{\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)} - \frac{\alpha(1-s)\left(1-\tau^{NSS}\right)}{s}.\end{aligned}$$

Moreover, when $\frac{1-\tau^{NSS}}{s} - \frac{1}{(1-\tau^{NSS})(1-s)(1-\alpha)} > 0$ (or, $1 > \frac{s}{(1-\tau^{NSS})^2(1-s)(1-\alpha)}$), $\widehat{T}_{BC}$ is increasing with σ, and tends to be $+\infty$ as σ goes to $+\infty$. When $\frac{1-\tau^{NSS}}{s} - \frac{1}{(1-\tau^{NSS})(1-s)(1-\alpha)} < 0$ (or, $\frac{s}{(1-\tau^{NSS})^2(1-s)(1-\alpha)} > 1$), $\widehat{T}_{BC}$ is decreasing with σ, and tends to be $-\infty$, as σ goes to $+\infty$.

Assumption 5. $s = \frac{1}{3}$ and $\tau^{NSS} = 0.285$.[9]

For the United States, estimates of the labor income tax rates range from 0.23 to 0.285 (Schmitt-Grohe and Uribe 1997, p. 983). First, we use the upper bound of the estimates to show our main results. Second, at the end of this section, we show that local indeterminacy occurs for income tax rates that are empirically plausible for the U.S. economy and for a large set of shares of first period consumption over the wage income.

In order to make the model analytically tractable, we let α be 0.3, which is less than $\alpha_1 = 0.3007$. For this numerical example, we set α (the share of first period consumption over the wage income) to be as large as possible in order to make the ratio of consumption expenditures over GDP as close as possible to 67.3% (see Lloyd-Braga et al.). Under this assumption, for the subcases (1.1),

[8] When the half line Δ crosses the line BC, both $D = \mathcal{S}T - \mathcal{S}\frac{1-\sigma-\alpha\gamma(1-s)}{\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)}$ and $D = 1$ hold.

[9] Since the model dynamics depend on the sign of $[\frac{s}{\left(1-\tau^{NSS}\right)^2(1-s)(1-\alpha)} - 1]$, we have to use numerical examples to show our main results.

(1.2) and case (2), there exist some values of σ, which make $\widehat{T}_{BC}(\sigma_c) = 2$ and $\widehat{T}_{BC}(\sigma_B) = -2$ hold respectively. Since $\widehat{T}_{BC}$ is decreasing with σ in this numerical case, $\sigma_B > \sigma_c$ holds. Straightforward computations show that

$$\begin{aligned}
\sigma_B &= (-2 - F)/E, \\
\sigma_c &= (2 - F)/E = 1 - \alpha\gamma(1-s) + \frac{2 - \frac{1-\tau^{NSS}}{s}[1 - \alpha(1-s)]}{E},
\end{aligned}$$

where $E = \frac{1-\tau^{NSS}}{s} - \frac{1}{(1-\tau^{NSS})(1-s)(1-\alpha)} < 0$ and

$$F = \alpha\gamma(1-s)\left[\frac{1-\tau^{NSS}}{s} - \frac{1}{(1-\tau^{NSS})(1-s)(1-\alpha)}\right] + \frac{1}{(1-\tau^{NSS})(1-s)(1-\alpha)} - \frac{\alpha(1-s)\left(1-\tau^{NSS}\right)}{s}.$$

Moreover, we analyze the intersection point of the lines Δ_1 and AC (or AB). $\widehat{D}_{AC}(\sigma_1) = \widehat{D}_{AB}(\sigma_2) = D_1 = \frac{s}{(1-s)(1-\alpha)(1-\tau^{NSS})}$ implies that

$$\sigma_2 = \frac{1}{\tau^{NSS}}\left[\left(1-\tau^{NSS}\right)(1-s)(1-2\alpha) + 1 + s\right] - \alpha\gamma(1-s) \text{ and } \sigma_1 = (1-s)(1-\alpha\gamma).$$

Straightforward computations show that

$$\sigma_2 - \sigma_c = \frac{1}{\tau^{NSS}}\left[\left(1-\tau^{NSS}\right)(1-s)(1-2\alpha) + 1 + s\right] - 1 - \frac{2 - \frac{1-\tau^{NSS}}{s}[1 - \alpha(1-s)]}{E}$$

does not depend on the value of γ. It is easy for us to find that $\sigma_1 < \sigma_c$ and $\sigma_2 < \sigma_B$.

For the subcase (1.1), we derive some other critical value of σ, which makes $\mathcal{S}(\sigma_S) = 1$ hold. It is easy to have $\sigma_s = -\alpha\gamma(1-s) + \frac{s}{1-\tau^{NSS}} + \alpha(1-s)$. Comparing σ_1 with σ_s, we find that when $D_1^0 < 1$, $\sigma_1 > \sigma_s$ holds.

Proposition 4. *Under Assumptions 1-5, let α be 0.3 ($\sigma_2 > \sigma_c$). When $\gamma \in (1, \frac{\Psi}{\alpha})$, subcase (1.1)* *occurs since $\sigma_s > 0$. We have the following results:*

(1) when $\sigma \in (0, \sigma_s)$, the slope of the half-line Δ is larger than 1 and the half-line Δ crosses the line AC at $\varepsilon_l = \varepsilon_l^T$. The NSS (1,1) is a saddle for $\varepsilon_l \in (\varepsilon_l^T, +\infty)$, undergoes a transcritical bifurcation *at $\varepsilon_l = \varepsilon_l^T$ and becomes a source for $\varepsilon_l \in (0, \varepsilon_l^T)$.*

(2) when $\sigma \in (\sigma_s, \sigma_1)$, the line Δ lies on the right side of the line AC and the NSS (1,1) is a saddle for $\varepsilon_l \in (0, +\infty)$.

(3) when $\sigma \in (\sigma_1, \sigma_c)$, the line Δ intersects the line AC and $\widehat{D}_{AC} < 1$. The NSS (1,1) is locally indeterminate for $\varepsilon_l \in (\varepsilon_l^T, +\infty)$, undergoes a transcritical bifurcation *at $\varepsilon_l = \varepsilon_l^T$ and becomes a saddle for $\varepsilon_l \in (0, \varepsilon_l^T)$.*

(4) when $\sigma \in (\sigma_c, \sigma_2)$, the line Δ can intersect both the segment BC and the line AC. The NSS (1,1) is locally indeterminate for $\varepsilon_l \in (\varepsilon_l^H, +\infty)$, undergoes a Hopf bifurcation *at $\varepsilon_l = \varepsilon_l^H$, becomes a source for $\varepsilon_l \in (\varepsilon_l^T, \varepsilon_l^H)$, undergoes a* transcritical bifurcation *at $\varepsilon_l = \varepsilon_l^T$ and becomes a saddle for $\varepsilon_l \in (0, \varepsilon_l^T)$.*

(5) when $\sigma \in (\sigma_2, \sigma_B)$, the line Δ can intersect the line AB, the segment BC and the line AC. The NSS (1,1) is a saddle point for $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$, undergoes a flip bifurcation *at $\varepsilon_l = \varepsilon_l^f$, becomes locally indeterminate for $\varepsilon_l \in \left(\varepsilon_l^H, \varepsilon_l^f\right)$, undergoes a* Hopf bifurcation *at $\varepsilon_l = \varepsilon_l^H$, becomes a source for $\varepsilon_l \in (\varepsilon_l^T, \varepsilon_l^H)$, undergoes a* transcritical bifurcation *at $\varepsilon_l = \varepsilon_l^T$ and becomes a saddle-point for $\varepsilon_l \in (0, \varepsilon_l^T)$.*

(6) when $\sigma \in (\sigma_B, +\infty)$, the line Δ can intersect the line AB and the line AC. The NSS (1,1) is a saddle point for $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$, undergoes a flip bifurcation *at $\varepsilon_l = \varepsilon_l^f$, becomes a source for $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^f\right)$, undergoes a* transcritical bifurcation *at $\varepsilon_l = \varepsilon_l^T$ and becomes a saddle point for $\varepsilon_l \in (0, \varepsilon_l^T)$.*

When $\gamma \in (\frac{\Psi}{\alpha}, \frac{1}{\alpha})$, subcase (1.2) occurs since $\sigma_s < 0$ and $\sigma_1 > 0$. Thus we have the following results.

Proposition 5. *Under Assumptions 1-5, let* α *be* 0.3 ($\sigma_2 > \sigma_c$)*. When* $\gamma \in (\frac{\Psi}{\alpha}, \frac{1}{\alpha})$*, subcase (1.2) occurs.*

(1) when $\sigma \in (0, \sigma_1)$*, the line* Δ *lies on the right side of the line AC, and the NSS (1,1) is always a saddle point for* $\varepsilon_l \in (0, +\infty)$*.*

(2) when $\sigma \in (\sigma_1, \sigma_c)$*, the line* Δ *intersects the line AC and* $\widehat{D}_{AC} < 1$*. The NSS (1,1) is locally indeterminate for* $\varepsilon_l \in \left(\varepsilon_l^T, +\infty\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

(3) when $\sigma \in (\sigma_c, \sigma_2)$*, the line* Δ *can intersect both the segment BC and the line AC. The NSS (1,1) is locally indeterminate for* $\varepsilon_l \in \left(\varepsilon_l^H, +\infty\right)$*, undergoes a Hopf bifurcation at* $\varepsilon_l = \varepsilon_l^H$*, becomes a source for* $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^H\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

(4) when $\sigma \in (\sigma_2, \sigma_B)$*, the line* Δ *can intersect the line AB, the segment BC and the line AC. The NSS (1,1) is a saddle for* $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$*, undergoes a flip bifurcation at* $\varepsilon_l = \varepsilon_l^f$*, becomes locally indeterminate for* $\varepsilon_l \in \left(\varepsilon_l^H, \varepsilon_l^f\right)$*, undergoes a Hopf bifurcation at* $\varepsilon_l = \varepsilon_l^H$ *, becomes a source for* $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^H\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

(5) when $\sigma \in (\sigma_B, +\infty)$*, the line* Δ *can intersect both the line AB and the line AC. The NSS (1,1) is a saddle for* $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$*, undergoes a flip bifurcation at* $\varepsilon_l = \varepsilon_l^f$*, becomes a source for* $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^f\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

There are two subcases in case 2. In subcase (2.1), when $\gamma \in (\frac{1}{\alpha}, \frac{1}{\alpha}\{\frac{1}{1-s} + \frac{1}{E(1-s)}[2 - \frac{(1-\alpha(1-s))(1-\tau^{NSS})}{s}]\})$, we have $\sigma_1 < 0$ and $\sigma_c > 0$. The local dynamics can be summarized as follows.

Proposition 6. *When subcase (2.1) occurs, we have the following results:*

(1) when $\sigma \in (0, \sigma_c)$*, the line* Δ *intersects the line AC and* $\widehat{D}_{AC} < 1$*. The NSS (1,1) is locally indeterminate for* $\varepsilon_l \in \left(\varepsilon_l^T, +\infty\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle point for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

(2) when $\sigma \in (\sigma_c, \sigma_2)$, the line Δ can intersect both the segment BC and the line AC. The NSS (1,1) is locally indeterminate for $\varepsilon_l \in \left(\varepsilon_l^H, +\infty\right)$, undergoes a Hopf bifurcation at $\varepsilon_l = \varepsilon_l^H$, becomes a source for $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^H\right)$, undergoes a transcritical bifurcation at $\varepsilon_l = \varepsilon_l^T$ and becomes a saddle point for $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$.

(3) when $\sigma \in (\sigma_2, \sigma_B)$, the line Δ can intersect the line AB, the segment BC and the line AC. The NSS (1,1) is a saddle point for $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$, undergoes a flip bifurcation at $\varepsilon_l = \varepsilon_l^f$, becomes locally indeterminate for $\varepsilon_l \in \left(\varepsilon_l^H, \varepsilon_l^f\right)$, undergoes a Hopf bifurcation at $\varepsilon_l = \varepsilon_l^H$, becomes a source for $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^H\right)$, undergoes a transcritical bifurcation at $\varepsilon_l = \varepsilon_l^T$ and becomes a saddle point for $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$.

(4) when $\sigma \in (\sigma_B, +\infty)$, the line Δ can intersect both the line AB and the line AC. The NSS (1,1) is a saddle point for $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$, undergoes a flip bifurcation at $\varepsilon_l = \varepsilon_l^f$, becomes a source for $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^f\right)$, undergoes a transcritical bifurcation at $\varepsilon_l = \varepsilon_l^T$ and becomes a saddle point for $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$.

In subcase (2.2), when $\gamma \in (\frac{1}{\alpha}\{\frac{1}{1-s} + \frac{1}{E(1-s)}[2 - \frac{(1-\alpha(1-s))(1-\tau^{NSS})}{s}]\}, \frac{\Lambda}{\alpha})$, we have $\sigma_c < 0$, $\sigma_2 > 0$ and $\widehat{T}_{BC} < 2$ for all $\sigma \in (0, +\infty)$. The local dynamics can be summarized as follows.

Proposition 7. *When subcase (2.2) occurs, we have the following results:*

(1) when $\sigma \in (0, \sigma_2)$, the line Δ can intersect both the segment BC and the line AC. The NSS (1,1) is locally indeterminate for $\varepsilon_l \in \left(\varepsilon_l^H, +\infty\right)$, undergoes a Hopf bifurcation at $\varepsilon_l = \varepsilon_l^H$, becomes a source for $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^H\right)$, undergoes a transcritical bifurcation at $\varepsilon_l = \varepsilon_l^T$ and becomes a saddle point for $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$.

(2) when $\sigma \in (\sigma_2, \sigma_B)$, the line Δ can intersect the line AB, the segment BC and the line AC. The NSS (1,1) is a saddle point for $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$, undergoes a flip bifurcation at $\varepsilon_l = \varepsilon_l^f$, becomes locally indeterminate for $\varepsilon_l \in \left(\varepsilon_l^H, \varepsilon_l^f\right)$, undergoes a Hopf bifurcation at $\varepsilon_l = \varepsilon_l^H$, becomes a source

for $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^H\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle point for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

(3) when $\sigma \in (\sigma_B, +\infty)$*, the line* Δ *can intersect the line AB and the line AC. The NSS (1,1) is a saddle point for* $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$*, undergoes a flip bifurcation at* $\varepsilon_l = \varepsilon_l^f$*, becomes a source for* $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^f\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle point for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

There are two subcases in case 3. In subcase (3.1), when $\gamma \in (\frac{\Lambda}{\alpha}, \frac{1}{\alpha}\{\frac{1}{1-s} - \frac{1}{E(1-s)}[2 + \frac{(1-\alpha(1-s))(1-\tau^{NSS})}{s}]\})$, we have $\sigma_B > 0$ and $\sigma_2 < 0$. The local dynamics can be summarized as follows.

Proposition 8. *When subcase (3.1) occurs, we have the following results:*

(1) when $\sigma \in (0, \sigma_B)$*, the line* Δ *can intersect the line AB, the segment BC and the line AC. The NSS (1,1) is a saddle point for* $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$*, undergoes a flip bifurcation at* $\varepsilon_l = \varepsilon_l^f$*, becomes locally indeterminate for* $\varepsilon_l \in \left(\varepsilon_l^H, \varepsilon_l^f\right)$*, undergoes a Hopf bifurcation at* $\varepsilon_l = \varepsilon_l^H$ *, becomes a source for* $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^H\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle point for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

(2) when $\sigma \in (\sigma_B, +\infty)$*, the line* Δ *can intersect the line AB and the line AC. The NSS (1,1) is a saddle for* $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$*, undergoes a flip bifurcation at* $\varepsilon_l = \varepsilon_l^f$*, becomes a source for* $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^f\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$*.*

In subcase (3.2), when $\gamma \in (\frac{1}{\alpha}\{\frac{1}{1-s} - \frac{1}{E(1-s)}[2 + \frac{(1-\alpha(1-s))(1-\tau^{NSS})}{s}]\}, +\infty)$, we have $\sigma_B < 0$ and $\widehat{T}_{BC} < -2$ for all $\sigma \in (0, +\infty)$. The local dynamics can be summarized as follows.

Proposition 9. *When subcase (3.2) occurs, we have the following results:*

when $\sigma \in (0, +\infty)$*, the line* Δ *can intersect both the line AB and the line AC. The NSS (1,1) is a saddle point for* $\varepsilon_l \in \left(\varepsilon_l^f, +\infty\right)$*, undergoes a flip bifurcation at* $\varepsilon_l = \varepsilon_l^f$*, becomes a source* *for* $\varepsilon_l \in \left(\varepsilon_l^T, \varepsilon_l^f\right)$*, undergoes a transcritical bifurcation at* $\varepsilon_l = \varepsilon_l^T$ *and becomes a saddle point for* $\varepsilon_l \in \left(0, \varepsilon_l^T\right)$.

Insert Figures 2 through 7 here.

In order to provide economic intuitions, we compute several derivatives, using (3) and (s-3) in Appendix.

$$\frac{dl_t}{dw_t}\frac{w_t}{l_t} = \varepsilon_l > 0, \frac{dl_t}{dR_{t+1}}\frac{R_{t+1}}{l_t} = (1-\alpha)\,\varepsilon_l > 0 \text{ and } \frac{dl_t}{d\tau_t}\frac{\tau_t}{l_t} = -\frac{\tau^{NSS}}{1-\tau^{NSS}}\varepsilon_l < 0 \tag{20}$$

It is easy for us to obtain the following derivatives from (13):

$$\frac{K_{t+1}}{R_{t+1}}\frac{dR_{t+1}}{dK_{t+1}} = -\frac{1-s}{\sigma} < 0, \frac{dR_{t+1}}{dL_{t+1}}\frac{L_{t+1}}{R_{t+1}} = \frac{1-s}{\sigma} > 0. \tag{21}$$

Now it is known that local indeterminacy (cyclical equilibrium path) can arise only if the elasticity of capital-labor substitution is less than the share of capital in total income and the share of first period consumption over the wage income is small enough. Let's first consider the case without endogneous labor income tax rates and use the economic interpretation provided by Lloyd-Braga et al. (2007, p. 527) to show how local indeterminacy can arise: we assume that an instantaneous increase in the capital stock K_t from the steady state occurs at time t. This generates two opposite effects: a contemporary effect consists in an increase in the wage rate w_t. It implies, from Eq. (20), an increase in the labor supply l_t. Because $K_{t+1} = (1-\alpha)w_t l_t(1-\tau_t)$ is satisfied each period, a higher capital stock in the next period is expected.[10] But at the same time, an expectation effect plays in the opposite direction: a higher K_{t+1} is followed by a decrease in the interest factor. And the latter

[10]The budget constraint $K_{t+1} = (1-\alpha)w_t l_t$ holds when $\tau_t = 0$, i.e. the case without endogenous income tax rates.

implies, from Eq. (20), a decrease in the current labor supply. A cyclical path can arise only if the expectation effect dominates the contemporary effect and generates a decrease in the wage income which would decrease savings at time t and capital at time $t+1$. This requires that the elasticity of capital-labor substitution be less than the share of capital in total income and the share of first period consumption over the wage income be small enough. Indeed, adding labor income taxes can dampen the contemporary effect since $\frac{dl_t}{d\tau_t}\frac{\tau_t}{l_t} = -\frac{\tau^{NSS}}{1-\tau^{NSS}}\varepsilon_l < 0$, thus making local indeterminacy more likely to occur. In other words, although from Eq. (20), an increase in the wage rate implies an increase in the labor supply, the magnitude of the increase in the labor supply is decreasing with labor income tax rates. As a result, a higher capital stock in the next period (K_{t+1}) is expected but the magnitude of the capital stock (in the next period) is negatively related to income tax rates.

Up to now, we have shown that in an aggregate OLG model with elastic labor supply and a reasonable share of first period consumption over the wage income, local indeterminacy can occur with empirical estimates (the upper bound) of labor income tax rates. Moreover, we will show that local indeterminacy can occur with an elasticity of intertemporal substitution in consumption slightly greater than unity, a large enough elasticity of capital–labor substitution and a large enough elasticity of the labor supply. All of these conditions can be found within infinite horizon models. As in Lloyd-Braga et al. (2007, p. 528), we let γ be 1.1 and we drive $\sigma_s = 0.4462$, $\sigma_1 = 0.446667$, $\sigma_c = 0.446668$, $\sigma_2 = 5.1274$ and $\sigma_B = 5.1415$. According to Proposition 4, we consider the case $\sigma \in (\sigma_c, \sigma_2) = (0.446668,\ 5.1274)$. In order to be compatible with the empirical estimates reported by Duffy and Papageorgiou (2000), we assume that $\sigma \in [1.14,\ 3.24]$. The NSS (1,1) is locally indeterminate for $\varepsilon_l \in \left(\varepsilon_l^H, +\infty\right)$ with ε_l^H a Hopf bifurcation value. Assuming that $\sigma \in [1.14,\ 3.24]$, we get $\varepsilon_l^H = 1000$.

At the end of this section, we show that local indeterminacy occurs for income tax rates that are empirically plausible for the U.S. economy and for a large set of shares of first period consumption over the wage income. First, we require $\alpha > \alpha_2 \equiv 1 - \frac{s}{(1-\tau^{NSS})^2(1-s)}$ and $\alpha < \alpha_1$ to guarantee that $\widehat{T}_{BC}$ is decreasing with σ and $D_1^0 < 1$. Second, we require that $\sigma_2 - \sigma_c > 0$. The latter requirement holds if and only if $M > 0$ where

$$\begin{aligned} M &= s\left(1-\tau^{NSS}\right)(1-s)(1-2\alpha) - (1-\tau^{NSS})^3(1-s)^2(1-\alpha)(1-2\alpha) \\ &\quad + \left(1+s-\tau^{NSS}\right)\left[s-\left(1-\tau^{NSS}\right)^2(1-s)(1-\alpha)\right] + 2s\tau^{NSS}(1-\tau^{NSS})(1-s)(1-\alpha) \\ &\quad -\tau^{NSS}\left(1-\tau^{NSS}\right)^2(1-s)(1-\alpha) + \tau^{NSS}(1-\tau^{NSS})^2(1-s)^2(1-\alpha)\alpha. \end{aligned}$$

Numerical results show that when $s = \frac{1}{3}$, all of these propositions hold for a large set of α and τ^{NSS}, which satisfy $\alpha_2 < \alpha < \alpha_1$ and $\sigma_2 - \sigma_c > 0$. More precisely, for income tax rates that range from 0.23 to 0.285, all of these propositions hold for the set of α's that lie between the dotted line and the dashed line and are denoted by $(\times)$ (see Figure 8).

Insert Figure 8 here.

5. Concluding Remarks

This paper embeds a balanced budget rule in an OLG model with consumption in both periods of life, homothetic preferences and in which the share of first period consumption over the wage income is not large. We show that under gross substitutability, local indeterminacy can occur when the steady state labor income tax rates are not too large. In numerical examples, for empirical estimates of labor income tax rates, local indeterminacy requires the elasticity of capital–labor substitution and the elasticity of the labor supply to be sufficiently large. This is in contrast to the previous result that local indeterminacy can occur only if the elasticity of capital-labor substitution is less than the share of capital in total income. Moreover, we show that local indeterminacy can occur with an elasticity of intertemporal substitution in consumption slightly greater than unity, a large enough elasticity of capital–labor substitution and a large enough elasticity of the labor supply. All of these conditions can be found within infinite horizon models and from this point of view, Diamond meets Ramsey again.[11]

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Appendix:

A.1. Proof of Proposition 1

If $(K^{*}, l^{*}) = (1, 1)$ is a normalized steady state of the dynamic system (16), we have the following results:

$$1 = \left[1 - \alpha\left(Af'(1)\right)\right]\left[Aw(1) - g\right] \equiv G(A), \tag{s-1}$$

$$v'(1/B)/B = u_1\left[1, h\left(Af'(1)\right)\right]\left[Aw(1) - g\right]. \tag{s-2}$$

It is easy to show that $V(B) = v'(1/B)/B$ is invertible since $V'(B) < 0$. $Aw(1) - g > 0$ holds since $g = \tau^{NSS} A\omega(1)$, where $\tau^{NSS} \in (0, 1)$ is the steady state labor income tax rate. Moreover, we get $\frac{G'(A)A}{G(A)} = \frac{Aw(1)}{Aw(1)-g} - \alpha(R)(1 - \gamma(R)) > 0$ when we derive the elasticity of $G(A)$ using the elasticity of $\alpha(R)$. Then we know that $G(A)$ is a strictly increasing function for any $\gamma(R) > 0$. Since $\alpha(R) \in (0, 1)$, $\lim_{z \to 0}(1 - \alpha(z)) \le 1$ and thus $\lim_{A \to 0}[1 - \alpha(Af'(1))]A = 0$. Then a unique $A^{*} > 0$ can satisfy equation (s-1) iff $\lim_{A \to +\infty} G(A) > 1$. We can easily get B^{*} from (s-2) after we pin down the unique A^{*} from (s-1).

[11] To save space, we do not discuss the case where local indeterminacy exists in a Cobb-Douglas economy and explore the under- versus over- accumulation properties of the NSS.

A.2. Proof of Proposition 2.

In the NSS, $R^* = A^* f'(1)$ holds. From the proposition above, we have $A^* = \frac{1}{(1-\alpha)(1-\tau^{NSS})w(1)}$. It is easy to know that $R^* = \frac{s}{(1-\alpha)(1-\tau^{NSS})(1-s)}$.

A.3. Proof of Proposition 3.

Using the same method as in Lloyd-Braga et al. (2007), we obtain the following equation (all evaluated around the NSS):

$$\frac{du_1(1,h(R))}{dR}\frac{R}{u_1(1,h(R))} = u_{12}(1,h(R))\frac{d\widehat{c}/c}{dR}\frac{R}{u_1(1,h(R))} = 1-\alpha(R). \tag{s-3}$$

After tedious algebra, we have

$$\begin{bmatrix} dK_{t+1} \\ dl_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\alpha(1-\gamma)\frac{s-1}{\sigma} & \alpha(1-\gamma)\frac{1-s}{\sigma} \\ (1-\alpha)\frac{s-1}{\sigma} & (1-\alpha)\frac{1-s}{\sigma} \end{bmatrix}^{-1} * \begin{bmatrix} \frac{1}{1-\tau^{NSS}}\frac{s}{\sigma} & \frac{1}{1-\tau^{NSS}}\frac{\sigma-s}{\sigma} \\ -\frac{1}{1-\tau^{NSS}}\frac{s}{\sigma} & \frac{1}{\varepsilon_l}+1-\frac{1}{1-\tau^{NSS}}\frac{\sigma-s}{\sigma} \end{bmatrix} \begin{bmatrix} dK_t \\ dl_t \end{bmatrix}.$$

A.4. The values of ε_l^T, ε_l^F and ε_l^H.

$\widehat{D}_{AC} = \frac{\mathcal{S}}{1-\mathcal{S}}\left[1-\frac{1-\sigma-\alpha\gamma(1-s)}{(1-\tau^{NSS})(1-s)(1-\alpha)}\right] = \frac{s}{(1-s)(1-\alpha)}\frac{1}{1-\tau^{NSS}}\frac{1+\varepsilon_l^T}{\varepsilon_l^T}$ implies that

$$\varepsilon_l^T = \{\frac{\mathcal{S}}{(1-\mathcal{S})s}\left[\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)-1+\sigma+\alpha\gamma(1-s)\right]-1\}^{-1}.$$

$\widehat{D}_{AB} = \frac{-\mathcal{S}}{1+\mathcal{S}}\left[1+\frac{1-\sigma-\alpha\gamma(1-s)}{(1-\tau^{NSS})(1-s)(1-\alpha)}\right] = \frac{s}{(1-s)(1-\alpha)}\frac{1}{1-\tau^{NSS}}\frac{1+\varepsilon_l^F}{\varepsilon_l^F}$ implies that

$$\varepsilon_l^F = \{\frac{-\mathcal{S}}{(1+\mathcal{S})s}\left[\left(1-\tau^{NSS}\right)(1-s)(1-\alpha)+1-\sigma-\alpha\gamma(1-s)\right]-1\}^{-1}.$$

And $\widehat{T}_{BC} = \frac{1}{\mathcal{S}} + \frac{1-\sigma-\alpha\gamma(1-s)}{(1-\tau^{NSS})(1-s)(1-\alpha)} = \frac{1}{(1-s)(1-\alpha)}\{\frac{1-\sigma-\alpha\gamma(1-s)}{1-\tau^{NSS}} + \frac{1+\varepsilon_l^H}{\varepsilon_l^H}\left[\sigma - \alpha\left(1-\gamma\right)\left(1-s\right)\right]\}$ implies that

$$\varepsilon_l^H = \{\frac{\left(1-s\right)\left(1-\alpha\right)}{\mathcal{S}\left[\sigma - \alpha\left(1-\gamma\right)\left(1-s\right)\right]} - 1\}^{-1}.$$

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Tables and Figures

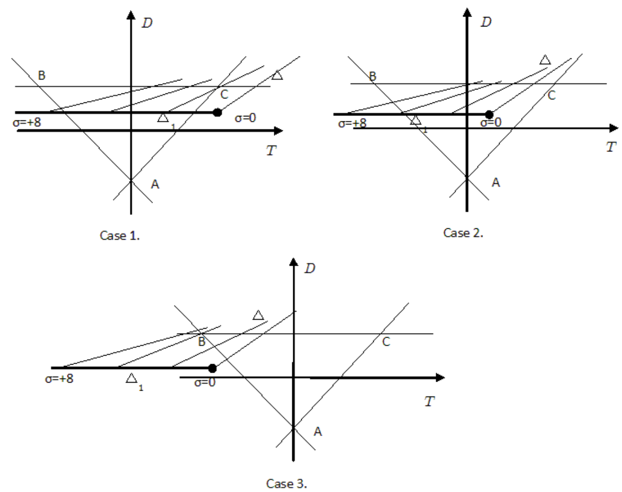

Figure 1. Local dynamics.

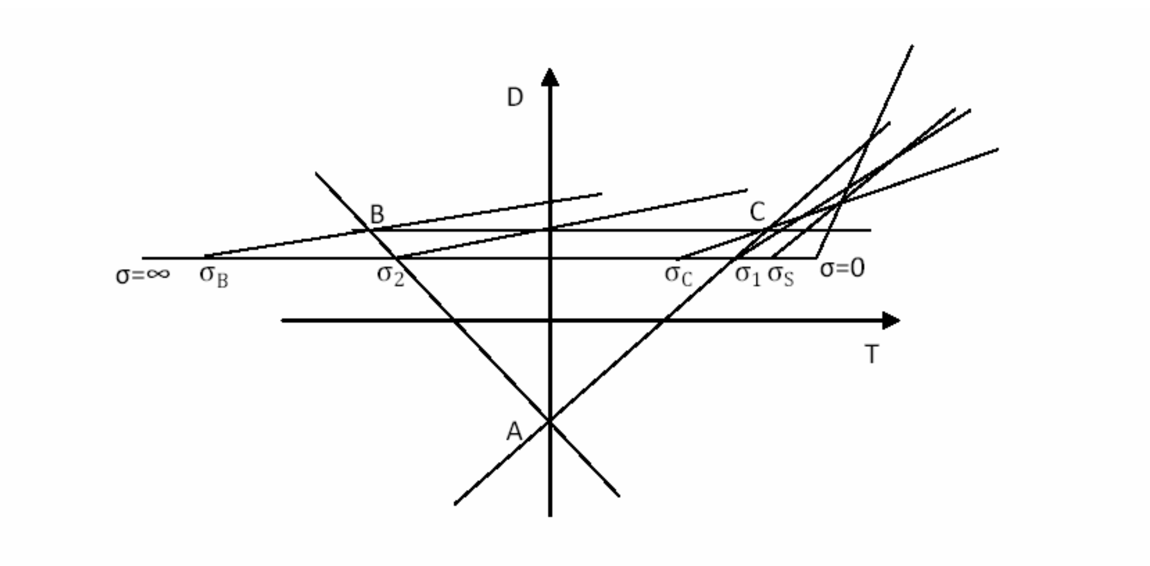

Figure 2. Subcase (1.1)

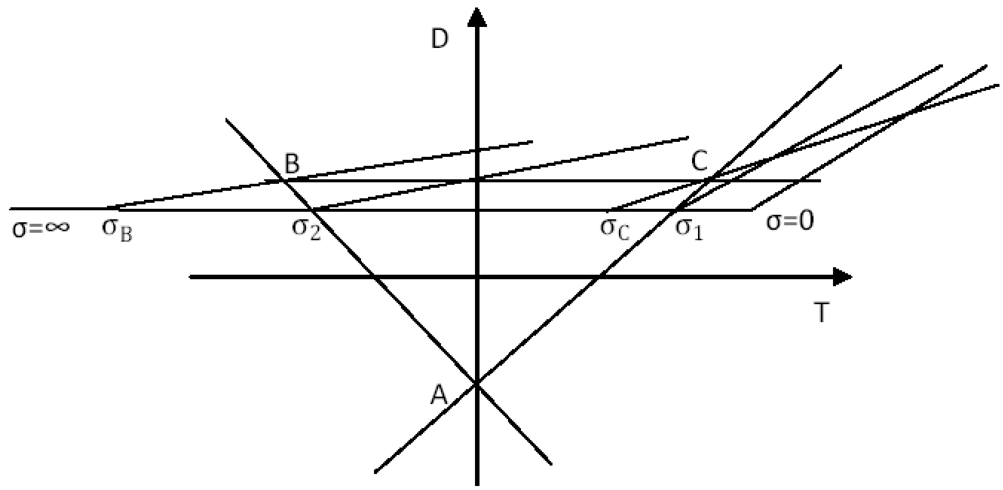

Figure 3. Subcase (1.2)

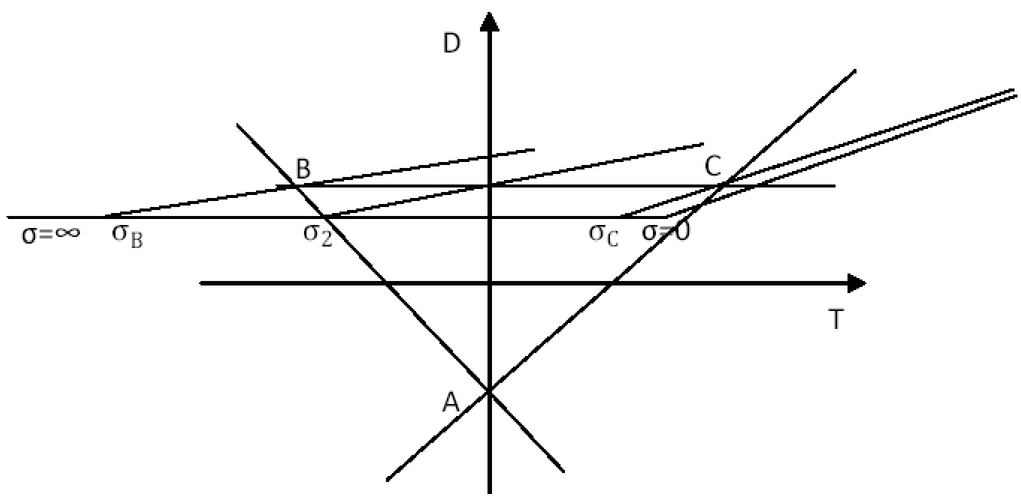

Figure 4. Subcase (2.1).

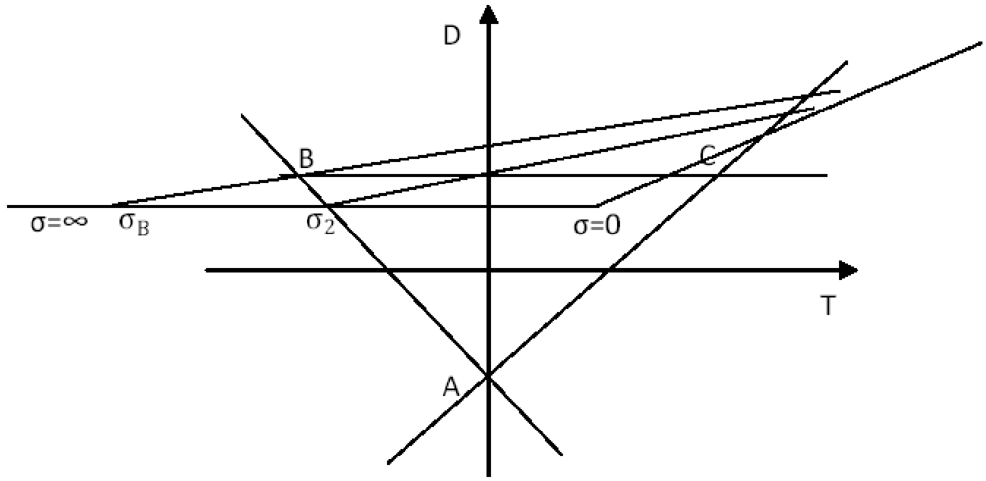

Figure 5. Subcase (2.2)

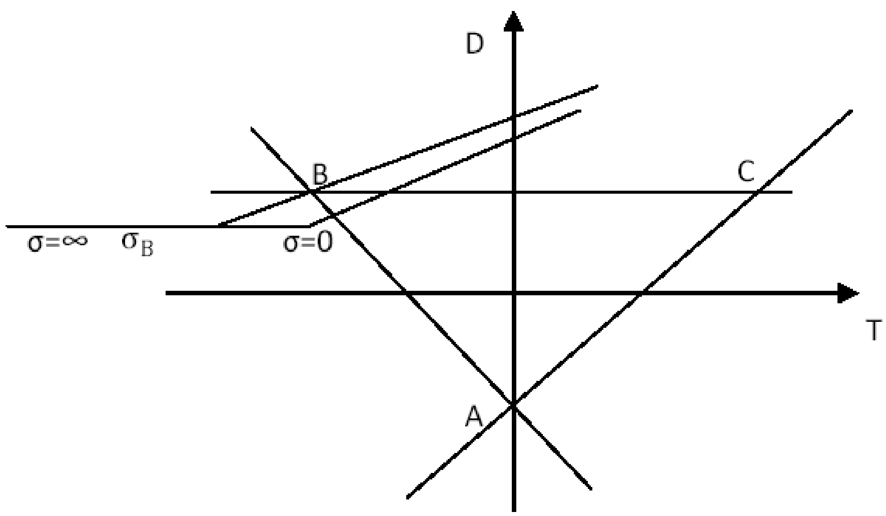

Figure 6. Subcase (3.1)

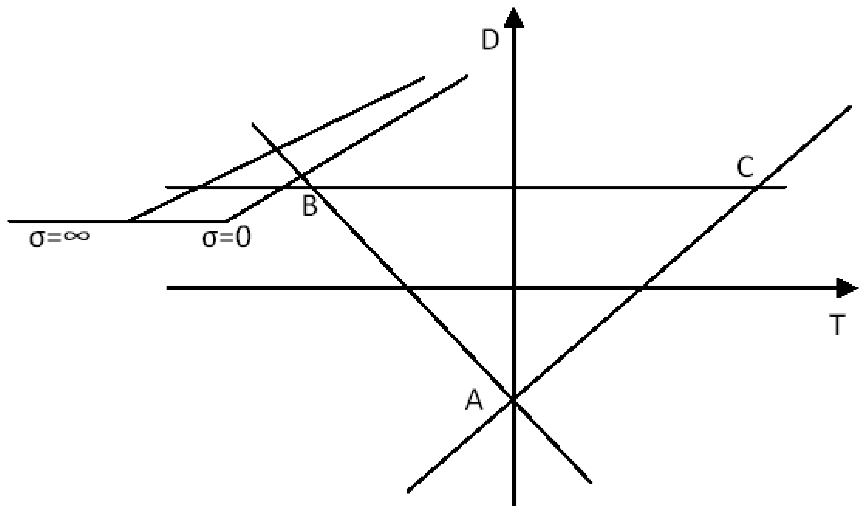

Figure 7. Subcase (3.2)

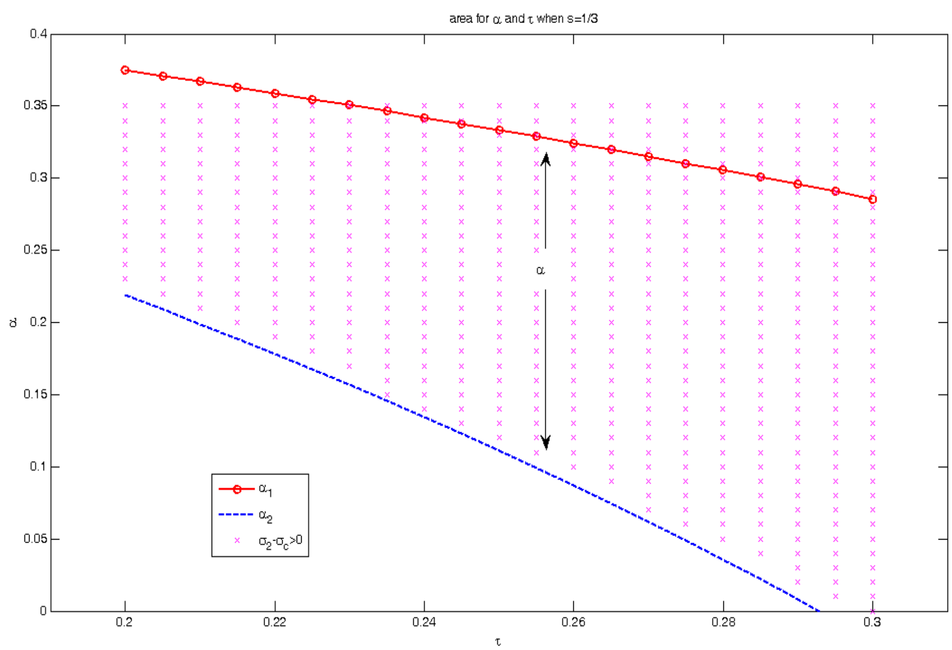

Figure 8. The area of α and τ^{NSS} $(s = \frac{1}{3})$.