An axiomatic foundation for multidimensional spatial models of elections with a valence dimension

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AN AXIOMATIC FOUNDATION FOR MULTIDIMENSIONAL SPATIAL MODELS OF ELECTIONS WITH A VALENCE DIMENSION[†]

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ABSTRACT. Recent works on political competition incorporate a valence dimension into the standard spatial model. The analysis of the game between candidates in these models is typically based on two assumptions about voters' preferences. One is that valence scores enter the utility function of a voter in an 'additively separable' way, so that the total utility can be decomposed into the 'ideological utility' from the implemented policy (based on the Euclidean distance) plus the valence of the winner. The second is that all the voters identically perceive the platforms of the candidates and agree about their score on the valence dimension.

The goal of this paper is to axiomatize collections of preferences that satisfy these assumptions. Specifically, we consider the case where only the ideal point in the policy space and the ranking over candidates are known for each voter. We characterize the case where there are policies $x_1, \dots, x_m$ for the m candidates and numbers $v_1, \dots, v_m$ representing valence scores, such that a voter with an ideal policy y ranks the candidates according to $v_i - ||x_i - y||^2$.

Keywords: Elections, Spatial models, Valence, Euclidean preferences.

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1. Introduction

Since the seminal works of Hotelling [21] and Downs [14], spatial models of elections are widely used in the political economy literature. Typically, these models identify the policy space with a finite dimensional Euclidean space. Each potential voter in the electorate is assumed to have an ideal point in the policy space, and his utility if a certain policy is implemented is decreasing in the Euclidean distance between this policy and his ideal point. Candidates then choose their platforms and each voter votes for the candidate with the closest platform to her ideal policy. Usually, the emphasis is on equilibrium analysis of the resulting game between candidates.

More recently, researchers incorporated a "valence" dimension to the standard model. This additional dimension influence voters' preferences and was shown to have an important effect on the outcome of the political game, both in theory and in empirical studies. This additional dimension may represent any non–policy issue on which candidates differ in the "score" they get from voters. Examples include charisma, experience, past success, communication skills, etc. The difference between the valence dimension and other dimensions (which are part of the policy space) is that all voters prefer high valence scores to low. References to works that incorporate valence issues can be found in the related literature section below.

Let C denote the set of candidates competing in some elections, and let the d dimensional Euclidean space $\mathbb{R}^d$ represent the policy space. When valence issues are present, the preferences of voters are defined over the set $\mathbb{R}^d \times C$. Indeed, the utility of a potential voter depends both on the implemented policy and on the valence of the winning candidate. Notice that we deal with a collection of preference orders, one for each voter. In almost all the works that we are aware of, the analysis is based on two fundamental assumptions about this collection of preferences, which we now discuss.

The first assumption concerns the preferences of individual voters. The utility function of a voter is assumed to be 'additively separable' in the policy and valence dimensions. That is, each voter has an 'ideological' utility function over policies and a valence index for candidates. The utility of a voter from a pair $(x, i) \in \mathbb{R}^d \times C$ can be decomposed into the utility from the implemented policy x plus the valence index of candidate i. More specifically, each voter is characterized by his ideal point $y \in \mathbb{R}^d$ and the valence scores

$\{v_i\}_{i \in C}$ that he gives to the various candidates. The utility he obtains from the pair (x, i) is given by $v_i - ||x - y||^2$.[1]

The second key assumption usually made is that all the voters perceive in the same way the alternatives they face. First, the voters agree on the location of the candidates in the policy space. That is, the beliefs of all voters regarding the policy that a certain candidate is going to implement if elected coincide. Although this seems like a rather strong assumption, it can be justified by the claim that candidates commit to a certain policy prior to the elections, and so this is the policy that voters anticipate will be implemented if the candidate is elected. But voters are also supposed to agree about the valence of each candidate. This is harder to justify, in particular since it seems reasonable that voters with different ideological views will also have different views of the valence of candidates. Notice that, if one allows to each potential voter to perceive the platforms and/or valences of the candidates differently, then the model may become completely untractable.

Obviously, it is very hard (not to say impossible) to extract the entire preferences of each voter over pairs of a winning candidate and an implemented policy. Therefore, it is not easy to check whether the aforementioned assumptions make sense in any particular political campaign. Thus, it seems an important matter to identify conditions on more easily observable data that guarantee consistency with the spatial model assumptions. Introducing such necessary and sufficient conditions is the main result of this paper.

Specifically, we assume that, for each potential voter in the electorate, only his ideal policy and his ranking of the candidates can be observed. While this may also seem quite demanding, it is much more reasonable than observing the entire utility function of the voter. We characterize the case where this data is consistent with voters having utility functions as above. That is, we characterize the case where there are platforms $\{x_i\}_{i \in C} \subseteq \mathbb{R}^d$ and numbers $\{v_i\}_{i \in C}$ representing valence scores, such that a voter with an ideal policy y ranks the candidates according to $v_i - ||x_i - y||^2$. We emphasize that the representation is for the collection of preference orders of all voters jointly, and not for the preferences of a single voter.

We use four conditions for the characterization. The first is that each voter preferences over candidates are rational (complete and transitive). The second is a continuity condition. The third and perhaps most important condition is convexity of the set of voters

[1]Notice that we take the square of the Euclidean norm (and not just the norm) as the 'ideological' utility function. We discuss this point in subsection 3.3.

preferring one candidate over another. There is a close connection between convexity and the Euclidean metric, as other metrics would typically induce non–convex sets. The last condition requires sufficient heterogeneity in the preferences. This is a more technical condition which is not necessary for the representation but is required for the sufficiency part of the proof.[2]

We think of our result as "good news" since it shows that if voters' preferences satisfy a set of rather natural axioms then they are consistent with the standard spatial model with a valence dimension.[3] From a theoretical viewpoint, the result provides a possible justification for the assumptions (discussed above) that allow to study the game between candidates. From an empirical perspective, the axioms may help to check whether the spatial model makes sense in any particular campaign.

1.1. **Related literature.** A few recent papers study questions related to the implications of assuming Euclidean preferences in spatial models. Degan and Merlo [12] ask under what conditions the assumption that voters vote ideologically (i.e., according to Euclidean preferences) is falsifiable, when data about the voting choices in several elections is available. Their answer is based on a relation between the dimension of the policy space and the number of elections.[4] Bogomolnaia and Laslier [8] find the exact number of dimensions required in order to be able to represent any preference profile of I voters over A alternatives. Knoblauch [24] provides a polynomial time algorithm to check whether a given finite preference profile has a one–dimensional Euclidean representation.

There are also several works that study similar questions for a more general class of preferences that include Euclidean preferences as a special case. Eguia [15] axiomatizes preference relations over lotteries over multi–attribute objects that admit a representation by some l_p norm. He also studies the case of multiple voters and characterize the case where their preferences can be jointly represented by such a norm. Kalandrakis [22] considers the case where a finite number of binary choices is observed, and characterizes the case where these choices can be rationalized by a concave utility function. He further studies the case where the rationalizing function has a bliss point.

[2] Nevertheless, our main result (Theorem 1) is an equivalence theorem. See Section 2 for details.

[3] Our conditions are not sufficient if one doesn't allow for a valence dimension, and we do not know how to characterize data consistent with spatial models without this additive term. See subsection 3.3.

[4] Some of their results generalize to the case where candidates get different valence scores from voters. See Section 3.2 in [12].

An important difference between our paper and all of the above is that we assume that, for any point in the policy space, the preferences of a voter with this ideal point over candidates are observed. All of the above papers deal with either a single preference relation or with a finite number of relations. While observing the preferences of a continuum of voters is impossible, we note that many previous works assume a continuous distribution of voters' ideal points. We further discuss this point in subsection 3.1.

Papers using spatial models of elections with valence issues similar to the one studied here are numerous in recent years. Examples include Ansolabehere and Snyder [1], Aragones and Palfrey [2], Degan [11], Dix and Santore [13], Enelow and Hinich [16], Gersbach [17], Groseclose [20], Kim [23] and Schofield [27] among others. These papers study different aspects of the political competition and provide various interpretations for the additive constant in the utility functions of the voters.

From a technical point of view, our main result is closely related to Theorem 1 in Azrieli and Lehrer [7], who characterize categorization systems that are generated by proximity to a set of prototypical cases. Furthermore, there is a surprisingly close connection between the result of this paper and the characterization of a collection of preference orders that can be represented by linear functionals.[5] Such characterizations appear in works on scoring rules (Myerson [26], Smith [28], Young [29]), case–based decision theory (Gilboa and Schmeidler [18]), expected utility in the context of games (Gilboa and Schmeidler [19]), relative utility (Ashkenazi and Lehrer [5]) and individual welfare functionals (Chambers and Hayashi [10]).

Finally, the mathematical object we deal with here is known in the geometry literature as (generalized) Voronoi diagram or (generalized) Dirichlet tessellation.[6] The most relevant papers in this literature are Ash and Bolker [3], [4] and Aurenhammer [6]. The book by Boots et al. [9] surveys applications of Voronoi diagrams in many different fields.

1.2. **Organization.** The next section contains the model and the main result of the paper, as well as a result regarding the uniqueness of the representation. In Section 3 we discuss several issues related to the model. In particular, we study the case of a finite set of voters, discuss the importance of the valence dimension for the result, and consider

[5] We thank Itzhak Gilboa for pointing out this connection.

[6] The word 'generalized' is added to indicate that there is an additive constant associated with each candidate. These objects are also called power diagrams in some places in the geometry literature.

the special cases of three candidates and of one–dimensional policy space. All the proofs are in Section 4.

2. Model and main result

2.1. **Setup.** Let $C = \{1, 2, \ldots, m\}$ be the set of candidates where $m \geq 2$. The policy space is taken to be $\mathbb{R}^d$ with[7] $d \geq 2$. Each potential voter is identified with her ideal point in the policy space and we assume that, for every $y \in \mathbb{R}^d$, there is a voter with y as her ideal policy. Thus, the set of voters is also $\mathbb{R}^d$. We will use the letters i, j, k, l to denote candidates (elements of C) and x, y, z to denote voters or policies (points in $\mathbb{R}^d$).

Our primitive is a collection of binary relations $\{\succeq_y\}_{y \in \mathbb{R}^d}$ over C, one for every voter $y \in \mathbb{R}^d$. The interpretation of $i \succeq_y j$ is that a voter with an ideal point y (weakly) prefers candidate i to candidate j. As usual, for any $i, j \in C$, we let $i \succ_y j$ if and only if both $i \succeq_y j$ and $j \not\succeq_y i$, and $i \sim_y j$ if and only if both $i \succeq_y j$ and $j \succeq_y i$.

2.2. **Axioms.** The following properties will be used for the characterization.

(A1) Weak order: For every $y \in \mathbb{R}^d$, $\succeq_y$ is complete and transitive.

(A2) Continuity: For every $i, j \in C$, the set $\{y \in \mathbb{R}^d \ : \ i \succ_y j\}$ is open.

(A3) Convexity: For every $i, j \in C$ and $y, z \in \mathbb{R}^d$, if $i \succeq_y j$ ($i \succ_y j$) and $i \succeq_z j$ then $i \succeq_{\alpha y + (1-\alpha) z} j$ ($i \succ_{\alpha y + (1-\alpha) z} j$) for every $\alpha \in (0, 1)$.

(A4) Heterogeneity: For every three distinct candidates $\{i, j, k\} \subseteq C$ there is $y \in \mathbb{R}^d$ such that $i \succ_y j \succ_y k$, and for every four distinct candidates $\{i, j, k, l\} \subseteq C$ the sets $\{y \in \mathbb{R}^d \ : \ i \sim_y j \sim_y k\}$ and $\{y \in \mathbb{R}^d \ : \ i \sim_y j \sim_y l\}$ are not equal.

The first property is standard. The second implies that if a voter with ideal point y strictly prefers candidate i over j then any voter with ideal point sufficiently close to y also prefers i over j. *(A3)* states that the set of voters preferring candidate i over j is convex. Finally, *(A4)* requires the population of voters to be sufficiently diverse in its preferences. Namely, for any (strict) ranking of every three candidates there should be a voter who ranks these candidates according to that given order; and for every three candidates there should be a voter that is indifferent between them but is not indifferent between them and some given fourth candidate. Note that if $m = 2$ then *(A4)* is trivially satisfied, and if $m = 3$ then the second part of *(A4)* is trivially satisfied.

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[7] Our result does not hold in the case $d = 1$. We elaborate on this case in subsection 3.6.

2.3. **Main result.** Before stating our result we need one more definition.

Definition 1. *Let* $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ *and* $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$. *We say that the set* $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^{d+1}$ *is in a* general position *if the following two conditions hold:*
(i) For every distinct $1 \leq i, j, k \leq m$*, the vectors* x_i, x_j, x_k *are affinely independent in* $\mathbb{R}^d$ *(equivalently,* $x_j - x_i$ *and* $x_k - x_i$ *are linearly independent in* $\mathbb{R}^d$*).*
(ii) For every distinct $1 \leq i, j, k, l \leq m$*, the sets*

$$\{y \in \mathbb{R}^d : v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2 = v_k - ||x_k - y||^2\}$$

and

$$\{y \in \mathbb{R}^d : v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2 = v_l - ||x_l - y||^2\}$$

are not equal.

Informally speaking, if a set of points is *not* in a general position then it has a 'degenerate structure'. We remark that if the points $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ are independently drawn from some continuous distribution over $\mathbb{R}^{d+1}$ then the resulting set will be in a general position with probability 1. The precise meaning of the term general position varies with the context in which it is used. The reader is referred to Matoušek (2002, pp. 3-5), where this concept is discussed in greater detail.

Theorem 1. *The following two statements are equivalent:*
(i) The collection of binary relations $\{\succeq_y\}_{y \in \mathbb{R}^d}$ *satisfies properties (A1) through (A4).*
(ii) There are points $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ *and numbers* $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$ *such that* $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ *is in a general position and, for every* $i, j \in C$ *and every* $y \in \mathbb{R}^d$, $i \succeq_y j$ *if and only if* $v_i - ||x_i - y||^2 \geq v_j - ||x_j - y||^2$.

The point x_i is the policy to be implemented if candidate i wins the elections and v_i is the score of i on the valence dimension ($1 \leq i \leq m$). Note that voters have common beliefs/views regarding the sets $\{x_1, x_2, \ldots, x_m\}$ and $\{v_1, v_2, \ldots, v_m\}$.

2.4. **Uniqueness.** Examining the proof of Theorem 1, one can see that the platforms and valences derived from the properties *(A1)-(A4)* are not unique. However, we do have the following connection between any two representations of the voters' preferences.

Proposition 1. *Assume* $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^{d+1}$ *represent the preferences* $\{\succeq_y\}_{y \in \mathbb{R}^d}$ *as in Theorem 1. Then* $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\} \subseteq \mathbb{R}^{d+1}$ *also* *represent* $\{\succeq_y\}_{y\in\mathbb{R}^d}$ *if and only if there is a positive number* $\alpha > 0$ *and a vector* $\beta \in \mathbb{R}^d$ *such that* $x_i' = \alpha x_i + \beta$ *for every* $1 \le i \le m$*, and such that the equation*[8]

$$v_i' - \alpha v_i = v_j' - \alpha v_j + \alpha(1-\alpha)(||x_j||^2 - ||x_i||^2) + 2\alpha\beta\cdot(x_i - x_j) \tag{1}$$

holds for every $i, j \in C$*. In particular, if* $x_i = x_i'$ *for* $1 \le i \le m$ *(i.e.,* $\alpha = 1$ *and* $\beta = 0$*) then there is some* $\gamma \in \mathbb{R}$ *such that* $v_i' = v_i + \gamma$ *for* $1 \le i \le m$*.*

This result can be interpreted as follows. We may rescale and change the origin of the policy space to get different sets of platforms that induce the same preferences. But once the unit of measurement and the origin are fixed the platforms are uniquely determined by the preferences. Moreover, once platforms are fixed, the relative valences of the various candidates (the differences $v_i - v_j$) are also unique.

3. Discussion and further results

3.1. **Finite set of voters.** From a practical point of view, it would be more interesting to find the testable implications of the spatial model assumptions for the preferences of a finite number of voters. The axiom *(A3)* implies that a finite sample of observations of voters' ideal points and rankings must have the property that the convex hulls of the ideal points of voters who prefer candidate i over j and those preferring j over i are disjoint, in order for it to be consistent with the spatial model. *(A1)* also gives an obvious necessary condition. The axioms *(A2)* and *(A4)* are not relevant for a finite sample.

As for sufficiency, it is tempting to try to prove a similar representation result to that of Theorem 1 for the case of a finite sample of voters. Of course, *(A3)* should be modified to require disjointness of convex hulls of sets of ideal points of voters with opposite preferences.[9]

For the case of two candidates, it is easy to see that (the modified) *(A3)* is sufficient for a representation. However, if there are at least three candidates this is no longer true. We demonstrate the problem with the following example. Let $d = 2$, $C = \{1, 2, 3\}$ and fix some $\epsilon > 0$. The set of voters, denoted Y, consists of six voters with the ideal points

$$Y = \{y_1 = (\epsilon,\epsilon),\ y_2 = (-\epsilon,-\epsilon),\ y_3 = (-\epsilon,-4),\ y_4 = (\epsilon,-4),\ y_5 = (4,\epsilon),\ y_6 = (4,-\epsilon)\}.$$

[8]For two vectors $z, w \in \mathbb{R}^d$ we denote by $z \cdot w = \sum_{i=1}^d z_i w_i$ the standard inner product in $\mathbb{R}^d$.

[9]Assume for simplicity that only strict preferences are allowed.

The preferences of these six voters are as follows. Voters $\{y_2, y_3\}$ prefer candidate 1 over candidate 2 (the rest of the voters prefer candidate 2 over candidate 1). Voters $\{y_1, y_2, y_3, y_5\}$ prefer candidate 1 over candidate 3, and voters $\{y_1, y_5\}$ prefer candidate 2 over candidate 3. Figure 1 illustrates the location of the voters' ideal points in the policy space and their rankings.

It is easy to check that the above condition of disjointness of the convex hulls is satisfied. However, we claim that these preferences are not consistent with a spatial model. Indeed, assume to the contrary that there are $\{(x_1, v_1), (x_2, v_2), (x_3, v_3)\}$ that represent these preferences as in Theorem 1. The locations of the points y_1, y_2, y_3, y_4 and the preferences of these voters imply that the line $\{y \in \mathbb{R}^2 \ : \ v_1 - ||y - x_1||^2 = v_2 - ||y - x_2||^2\}$ should be close to both points $(0, 0)$ and $(0, -4)$. Similarly, the line $\{y \in \mathbb{R}^2 \ : \ v_1 - ||y - x_1||^2 = v_3 - ||y - x_3||^2\}$ should be close to both points $(4, 0)$ and $(0, -4)$, and the line $\{y \in \mathbb{R}^2 \ : \ v_2 - ||y - x_2||^2 = v_3 - ||y - x_3||^2\}$ should be close to both points $(0, 0)$ and $(4, 0)$.

Now, for sufficiently small ϵ, it must be the case that the point $\bar{y} = (1, -1)$ is in the triangle generated by these three lines. It means that at this point we must have

$$v_1 - ||\bar{y} - x_1||^2 < v_2 - ||\bar{y} - x_2||^2 < v_3 - ||\bar{y} - x_3||^2 < v_1 - ||\bar{y} - x_1||^2,$$

a contradiction. If there was a voter with ideal point $\bar{y}$ and transitive preferences over candidates this could not have been happening. The characterization in the case of a finite voter's set remains unresolved.

3.2. **Euclidean preferences.** Our model does not presume any specific kind of preferences of the voters over the policy space. The primitive only consists of a collection of preferences over candidates indexed by points in $\mathbb{R}^d$. The Euclidean preferences are derived from the axioms.

Another approach would be to assume from the start that voters' preferences over policies are given by the Euclidean distance from their ideal point, and that valence scores are additively separable. In other words, one could test only the second assumption of the spatial model, that the subjective views of voters regarding the implemented policies and valences of the candidates are identical. In this case the model would consist of sets $\{x_i(y)\}_{i \in C} \subseteq \mathbb{R}^d$ and $\{v_i(y)\}_{i \in C} \subseteq \mathbb{R}$ for every $y \in \mathbb{R}^d$. It is easy to see that one can obtain a similar result to that of Theorem 1 in this case.

The Euclidean norm is intimately related to the convexity axiom *(A3)*. Other norms (such as the 'sup–norm' or the 'city–block metric') typically induce non–convex sets. A thorough study of the relation between convexity and the Euclidean norm, as well as of the kind of preferences induced by other norms is beyond the scope of this paper.

3.3. **The valence dimension.** The utility function of a voter with an ideal point y that we derive in Theorem 1 is of the form $v_i - ||x_i - y||^2$. Thus, we use the square of the Euclidean norm (and not just the norm) as the 'ideological' utility function. If instead voters' preferences are represented by the utility function $v_i - ||x_i - y||$ then the induced sets of supporters of candidates may not be convex. For instance, let $x_1 = (0,0)$, $x_2 = (1,0)$, $v_1 = 0$, and $v_2 = 1$. Then voters with ideal points $y = (0,1)$ and $y' = (0,-1)$ strictly prefer candidate 2 over candidate 1. However, a voter with an ideal point $y'' = \frac{y+y'}{2} = (0,0)$ is indifferent between the candidates. Thus, using the square of the norm is a consequence of the convexity axiom *(A3)*.

Using the square of the norm is natural also due to the following reason. We would like to think of the valence dimension as equally important to the policy dimensions. Recall that we think of the valence dimension as a dimension on which all voters agree that more is better. An alternative way to put this is to say that the ideal point of every voter is $+\infty$ along this dimension. For the sake of the argument, assume that we replace $+\infty$ by a large enough constant M. Then the utility of a voter if candidate i wins should be measured according to the distance between his ideal point (y, M) and the the point (x_i, v_i). This implies that we should add the valence score to the square of the norm of the difference in the policy space and not to the norm.

Theorem 1 is not true if we require all candidates to have the same score (zero, w.l.o.g.) on the valence dimension. The reader is referred to Azrieli and Lehrer (2007, Example 6.2) for an example. Thus, more restrictions must be imposed on preferences in order to allow a representation of the utility in the form $-||x_i - y||^2$. Finding natural additional axioms that distinguish this case from the more general one studied in this paper is an interesting direction for future research.

3.4. **The cases $m = 2$ and $m = 3$.** In contrast to the claim of the previous subsection, if there are only two or three candidates then it is possible to represent the voters' preferences without resorting to valences. The case $m = 2$ is trivial since one only needs to choose the platforms x_1 and x_2 in equal distance from the hyperplane separating the voters that prefer candidate 1 from those preferring candidate 2. In the case $m = 3$ we state this fact as a proposition.

Proposition 2. *Assume $m = 3$. The preferences $\{\succeq_y\}_{y\in\mathbb{R}^d}$ satisfy properties (A1) through (A4) if and only if there are $x_1, x_2, x_3 \in \mathbb{R}^d$ in a general position such that $i \succeq_y j$ if and only if $||x_i - y||^2 \leq ||x_j - y||^2$.*

3.5. **Observing just the first best.** Theorem 1 requires that we observe the entire ranking of each voter over C. It might be hard to extract this information from voters. A more plausible assumption is that only the most preferred candidate(s) is (are) observed for each voter. A possible way to formalize this is to assume that the primitive is a function $f : \mathbb{R}^d \to 2^C$, with the interpretation that $f(y) \subseteq C$ is the set of candidates which voter y prefers the most. We do not know how to get a similar result to that of Theorem 1 in this case when the dimension of the policy space is $d \geq 2$. However, it turns out that when $d = 1$ a simple characterization is possible (see the next subsection).

3.6. **The case $d = 1$.** If the policy space is one dimensional (as is the case in many papers) then Theorem 1 is no longer true, even if appropriately modified. The reason for this failure is that the set of voters who are indifferent between some three candidates is typically empty. This set plays a major role in the proof of the main result. Nevertheless, we can get a representation similar to that of Theorem 1 if we assume that only the most preferred candidates are observed for each voter (as in the previous subsection). We will use the following properties for the characterization.

(B1) For every $i \in C$, the set $\{y \in \mathbb{R} \ : \ f(y) = \{i\}\}$ is not empty and open.

(B2) For every $i \in C$ and $y, z \in \mathbb{R}$, if $i \in f(y)$ ($\{i\} = f(y)$) and $i \in f(z)$ then $i \in f(\alpha y + (1-\alpha)z)$ ($\{i\} = f(\alpha y + (1-\alpha)z)$) for every $\alpha \in (0, 1)$.

Before stating the result, we need a definition analogue to Definition 1.

Definition 2. *The set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^2$ is* well-ordered *if there is a permutation $\pi : C \to C$ such that the following two conditions hold:*
(i) $x_{\pi(1)} < x_{\pi(2)} < \ldots < x_{\pi(m)}$.
(ii) $a_{\pi(1)\pi(2)} < a_{\pi(2)\pi(3)} < \ldots < a_{\pi(m-1)\pi(m)}$ *where* $a_{\pi(i)\pi(i+1)} = \frac{x^2_{\pi(i)} - x^2_{\pi(i+1)} + v_{\pi(i+1)} - v_{\pi(i)}}{2(x_{\pi(i)} - x_{\pi(i+1)})}$ *for* $i = 1, 2, \ldots, m-1$.

Proposition 3. *The correspondence $f : \mathbb{R} \to 2^C$ satisfies properties (B1) and (B2) if and only if there is a well-ordered set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^2$ such that $f(y) = argmax\{v_i - (x_i - y)^2 \ : \ i \in C\}$.*

4. Proofs

4.1. **Proof of Theorem 1.** The proof of Theorem 1 is similar to the proof of the main result in Azrieli and Lehrer (2007). We therefore provide an outline of the proof and only detail those steps that did not appear in that paper.

A simple but important observation is that, for any $x_i \neq x_j \in \mathbb{R}^d$ and $v_i, v_j \in \mathbb{R}$, the set $\{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2\}$ is an affine subspace of dimension $d-1$ (a hyperplane), perpendicular to the direction $x_i - x_j$. Indeed, a simple computation shows that this set can be rewritten as $\{y \in \mathbb{R}^d \ : \ y \cdot (x_i - x_j) = \frac{1}{2}(v_j - v_i + ||x_i||^2 - ||x_j||^2)\}$. Similarly, the set $\{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 > v_j - ||x_j - y||^2\}$ is an open half space in $\mathbb{R}^d$ (given that $x_i \neq x_j$).

(ii)* implies *(i):

Fix the sets $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ and $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$. Property *(A1)* is obviously satisfied. Denote $A_{ij} = \{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2\}$ and $B_{ij} = \{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 > v_j - ||x_j - y||^2\}$. By property *(i)* of Definition 1, $x_i \neq x_j$ for every $i \neq j \in C$. Thus, each B_{ij} is open and convex and each A_{ij} is the boundary of the closed half space $B_{ij} \cup A_{ij}$. This shows that properties *(A2)* and *(A3)* are satisfied.

Property *(A4)* is satisfied because the set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is in a general position. Indeed, take any distinct $i, j, k \in C$. We need to show that there is some y with $i \succ_y j \succ_y k$. If this was not true then it must be that B_{ij} and B_{jk} do not intersect. But this can only happen if $x_i - x_j$ and $x_j - x_k$ are linearly dependent, a contradiction to the assumption of general position (property *(i)*). Finally, take any distinct $i, j, k, l \in C$. By the general position assumption (property *(ii)*) we have that $B_{ij} \cap B_{jk}$ and $B_{ij} \cap B_{jl}$ are not equal. This proves that *(A4)* is satisfied. $\square$

(i)* implies *(ii):

The proof is constructive. We first find the platforms $x_1, x_2, \ldots, x_m$ of the candidates, and then construct the valences $v_1, v_2, \ldots, v_m$. We need however to state some preliminary claims. The proofs of all these claims can be found in Azrieli and Lehrer (2007).

Claim 1. *For every ordered pair (i, j) of distinct candidates there is a non-zero vector $s_{ij} \in \mathbb{R}^d$ and a number $c_{ij} \in \mathbb{R}$ such that $\{y \in \mathbb{R}^d \ : \ i \succeq_y j\} = \{y \in \mathbb{R}^d \ : \ s_{ij} \cdot y \leq c_{ij}\}$.*

Moreover, these vectors and numbers can be chosen such that $s_{ji} = -s_{ij}$ *and* $c_{ji} = -c_{ij}$ *for every* (i, j)*.*

Fix a collection $\{s_{ij}, c_{ij}\}_{i,j \in C}$ as in Claim 1 until the end of the proof.

Claim 2. (A4) *implies that, for every* $i, j, k \in C$*, the vectors* s_{ij} *and* s_{ik} *are linearly independent.*

Claim 3. *For every* $i, j, k \in C$*, the vectors* s_{ij}, s_{ik} *and* s_{jk} *are* not *linearly independent.*

For $t, s \in \mathbb{R}^d$, denote by $R(t, s)$ the ray that starts at t and continues in the direction of s. That is $R(t, s) = \{t + \alpha s \ : \ \alpha \geq 0\}$.

Claim 4. *If* $x_1, x_2 \in \mathbb{R}^d$ *satisfy* $x_2 - x_1 = \alpha s_{12}$ *for some* $\alpha > 0$ *then, for every* $3 \leq i \leq m$*, the rays* $R(x_1, s_{1i})$ *and* $R(x_2, s_{2i})$ *intersect .*

We are now in the position to construct the sets $\{x_1, x_2, \ldots, x_m\}$ and $\{v_1, v_2, \ldots, v_m\}$. The point x_1 is chosen arbitrarily. Next, define $x_2 = x_1 + \alpha_{12} s_{12}$, where $\alpha_{12} > 0$ is arbitrary. For every $3 \leq i \leq m$, define x_i to be the unique point of intersection (by Claim 4) of the rays $R(x_1, s_{1i})$ and $R(x_2, s_{2i})$. A key point in the proof is that, when $\{x_1, x_2, \ldots, x_m\}$ are defined in this way, then, for every $1 \leq i, j \leq m$, $x_j - x_i = \alpha_{ij} s_{ij}$ for some $\alpha_{ij} > 0$. This fact follows from Proposition 1 (page 26) in Azrieli and Lehrer (2007). Finally, choose v_1 arbitrarily and define $v_i = v_1 - ||x_1||^2 + ||x_i||^2 - 2\alpha_{1i} c_{1i}$ for every $2 \leq i \leq m$.

It is useful to note that $\alpha_{ij} s_{ij} = \alpha_{1j} s_{1j} - \alpha_{1i} s_{1i}$ for every $3 \leq i, j \leq m$. Indeed, the left–hand side of the equality is $x_j - x_i$ while the right–hand side is $(x_j - x_1) - (x_i - x_1)$. This implies also that $\alpha_{ij} c_{ij} = \alpha_{1j} c_{1j} - \alpha_{1i} c_{1i}$. To see this, take $y \in \mathbb{R}^d$ such that $1 \sim_y i$ and $1 \sim_y j$ (the existence of such y is guaranteed by Claim 2). Transitivity implies that $i \sim_y j$. So $y \cdot s_{1i} = c_{1i}$, $y \cdot s_{1j} = c_{1j}$ and $y \cdot s_{ij} = c_{ij}$. Multiplying these equalities by α_{1i}, α_{1j} and α_{ij} correspondingly, and subtracting the first from the second we get the above equality.

To complete the proof we need to check that the set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is in a general position and that, for every $i, j \in C$ and $y \in \mathbb{R}^d$, $i \succeq_y j$ if and only if

$v_i - ||x_i - y||^2 \geq v_j - ||x_j - y||^2$. For the latter we have

$$\begin{aligned} & i \succeq_y j \iff s_{ij} \cdot y \leq c_{ij} \iff (x_j - x_i) \cdot y \leq \alpha_{ij} c_{ij} \iff (x_j - x_i) \cdot y \leq \alpha_{1j} c_{1j} - \alpha_{1i} c_{1i} \\ & \iff (x_j - x_i) \cdot y \leq \frac{1}{2} \left(v_1 - v_j + ||x_j||^2 - ||x_1||^2 \right) - \frac{1}{2} \left(v_1 - v_i + ||x_i||^2 - ||x_1||^2 \right) \\ & \iff (x_j - x_i) \cdot y \leq \frac{1}{2} \left(v_i - v_j + ||x_j||^2 - ||x_i||^2 \right) \iff v_i - ||x_i - y||^2 \geq v_j - ||x_j - y||^2. \end{aligned}$$

For the former, the vectors x_i, x_j, x_k are affinely independent since $x_j - x_i = \alpha_{ij} s_{ij}$ and $x_k - x_i = \alpha_{ik} s_{ik}$, and these are linearly independent vectors by Claim 2. Finally, the sets $\{y \in \mathbb{R}^d : i \sim_y j \sim_y k\}$ and $\{y \in \mathbb{R}^d : i \sim_y j \sim_y l\}$ are not equal by *(A4)*. This proves that $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is in a general position. $\square$

4.2. **Proof of Proposition 1.** First, it is easy to check that if there are $\alpha > 0$ and $\beta \in \mathbb{R}^d$ such that $x'_i = \alpha x_i + \beta$ for $1 \leq i \leq m$, and in addition equation (1) is satisfied then $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ and $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\}$ represent the same preferences.

Now, assume that $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ and $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\}$ represent the same preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$. It follows from the proof of Theorem 1 that for every $i, j \in C$ there is a positive number, say $t_{ij} > 0$, such that $x_j - x_i = t_{ij}(x'_j - x'_i)$ (with the convention $t_{ij} = -t_{ji}$). Fix some three candidates $i, j, k \in C$. Sum up the equalities $x_j - x_i = t_{ij}(x'_j - x'_i)$, $x_i - x_k = t_{ki}(x'_i - x'_k)$, $x_k - x_j = t_{jk}(x'_k - x'_j)$ and rearrange the terms to obtain $(x'_i - x'_j)(t_{ki} - t_{ij}) + (x'_k - x'_j)(t_{jk} - t_{ki}) = 0$. But the vectors x'_i, x'_j, x'_k are affinely independent so $t_{ki} - t_{ij} = t_{jk} - t_{ki} = 0$. It follows that $t_{ij} = t_{ki} = t_{jk}$, so there is a number $\alpha > 0$ such that $x_j - x_i = \alpha(x'_j - x'_i)$ for every $i, j \in C$. Now, define $\beta = x_1 - \alpha x'_1$. For every $2 \leq i \leq m$ we have $x_1 - x_i = \alpha(x'_1 - x'_i)$ or $x_i - \alpha x'_i = x_1 - \alpha x'_1 = \beta$. That is, $x'_i = \alpha x_i + \beta$ for every $1 \leq i \leq m$.

Finally, we must have $\frac{1}{2}\left(v_i - v_j + ||x_j||^2 - ||x_i||^2\right) = \frac{1}{2}\left(v'_i - v'_j + ||x'_j||^2 - ||x'_i||^2\right)$ for every $i, j \in C$. Substituting $\alpha x_i + \beta$ for x'_i and $\alpha x_j + \beta$ for x'_j and rearranging we obtain equation (1). In particular, if $x'_i = x_i$ and $x'_j = x_j$ then $v'_i - v_i = v'_j - v_j$. Define $\gamma = v'_1 - v_1$. It follows that $v'_i = v_i + \gamma$ for every $1 \leq i \leq m$. $\square$

4.3. **Proof of Proposition 2.** The *if* part follows from Theorem 1, so we only need to prove the *only if* part. By Theorem 1, there are $(x_1, v_1), (x_2, v_2), (x_3, v_3)$ in a general position that represent the preferences. It follows that the vectors $x_1 - x_2$ and $x_1 - x_3$ are linearly independent. Therefore, there is $\beta \in \mathbb{R}^d$ that solves the two equations

$\beta \cdot (x_1 - x_2) = \frac{v_2 - v_1}{2}$ and $\beta \cdot (x_1 - x_3) = \frac{v_3 - v_1}{2}$. Notice that the same vector β must satisfy also $\beta \cdot (x_2 - x_3) = \frac{v_3 - v_2}{2}$. Define $x'_i = x_i + \beta$ for $i = 1, 2, 3$.

By Proposition 1, the set $\{(x'_1, v'_1), (x'_2, v'_2), (x'_3, v'_3)\}$ represent the same preferences as $\{(x_1, v_1), (x_2, v_2), (x_3, v_3)\}$ if the equation $v'_i - v'_j = v_i - v_j + 2\beta \cdot (x_i - x_j)$ is satisfied for every $i, j \in C$. By construction, the vector β satisfies $\beta \cdot (x_i - x_j) = \frac{v_j - v_i}{2}$ for every i, j. It follows that $v'_1 = v'_2 = v'_3 = 0$ solve the above equations. That is, $\{(x'_1, 0), (x'_2, 0), (x'_3, 0)\}$ represent the preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$. □

4.4. **Proof of Proposition 3.** Assume first that the correspondence f can be represented as in the proposition. We can assume w.l.o.g. that π is the identity, so $x_1 < x_2 < \ldots < x_m$ and $a_{12} < a_{23} < \ldots < a_{(m-1)m}$. It is also convenient to denote $a_{01} = -\infty$ and $a_{m(m+1)} = +\infty$. Now, for every $1 \leq i \leq m-1$, a simple computation shows that $v_i - (x_i - y)^2 \geq v_{i+1} - (x_{i+1} - y)^2$ if and only if $y \leq a_{i(i+1)}$ (the same equivalence holds when the weak inequalities are replaced by strict ones). It follows that candidate i ($1 \leq i \leq m$) is the unique maximizer of $\{v_j - (x_j - y)^2 \ : \ j \in C\}$ if and only if $y \in (a_{(i-1)i}, a_{i(i+1)})$ and that i is a maximizer (not necessarily unique) of this expression if and only if $y \in [a_{(i-1)i}, a_{i(i+1)}]$. This shows that f satisfies properties *(B1)* and *(B2)*.

Conversely, assume that f satisfies *(B1)* and *(B2)*. These properties imply that there is a permutation of the candidates, w.l.o.g. the identity, and a sequence of numbers $a_{12} < a_{23} < \ldots < a_{(m-1)m}$ such that $f(y) = \{i\}$ if and only if $y \in (a_{(i-1)i}, a_{i(i+1)})$ and $f(y) = \{i, i+1\}$ if and only if $y = a_{i(i+1)}$ for $1 \leq i \leq m$.

Take any set of points $x_1 < x_2 < \ldots < x_m$. Define $v_1 = 0$ and, for every $1 \leq i \leq m-1$, let $v_{i+1} = 2a_{i(i+1)}(x_i - x_{i+1}) - x_i^2 + x_{i+1}^2 + v_i$. Rearranging, this gives $a_{i(i+1)} = \frac{x_i^2 - x_{i+1}^2 + v_{i+1} - v_i}{2(x_i - x_{i+1})}$ for $i = 1, 2, \ldots, m-1$. Thus, the set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^2$ is well-ordered. Finally, we need to check that $f(y) = argmax\{v_i - (x_i - y)^2 \ : \ i \in C\}$. This is true since

$$i \in f(y) \iff y \in [a_{(i-1)i}, a_{i(i+1)}] \iff \frac{x_{i-1}^2 - x_i^2 + v_i - v_{i-1}}{2(x_i - x_{i+1})} \leq y \leq \frac{x_i^2 - x_{i+1}^2 + v_{i+1} - v_i}{2(x_{i-1} - x_i)}$$

$$\iff v_i - (y - x_i)^2 \geq v_{i-1} - (y - x_{i-1})^2 \text{ and } v_i - (y - x_i)^2 \geq v_{i+1} - (y - x_{i+1})^2$$

$$\iff v_i - (y - x_i)^2 \geq v_j - (y - x_j)^2 \text{ for all } j \neq i.$$

□

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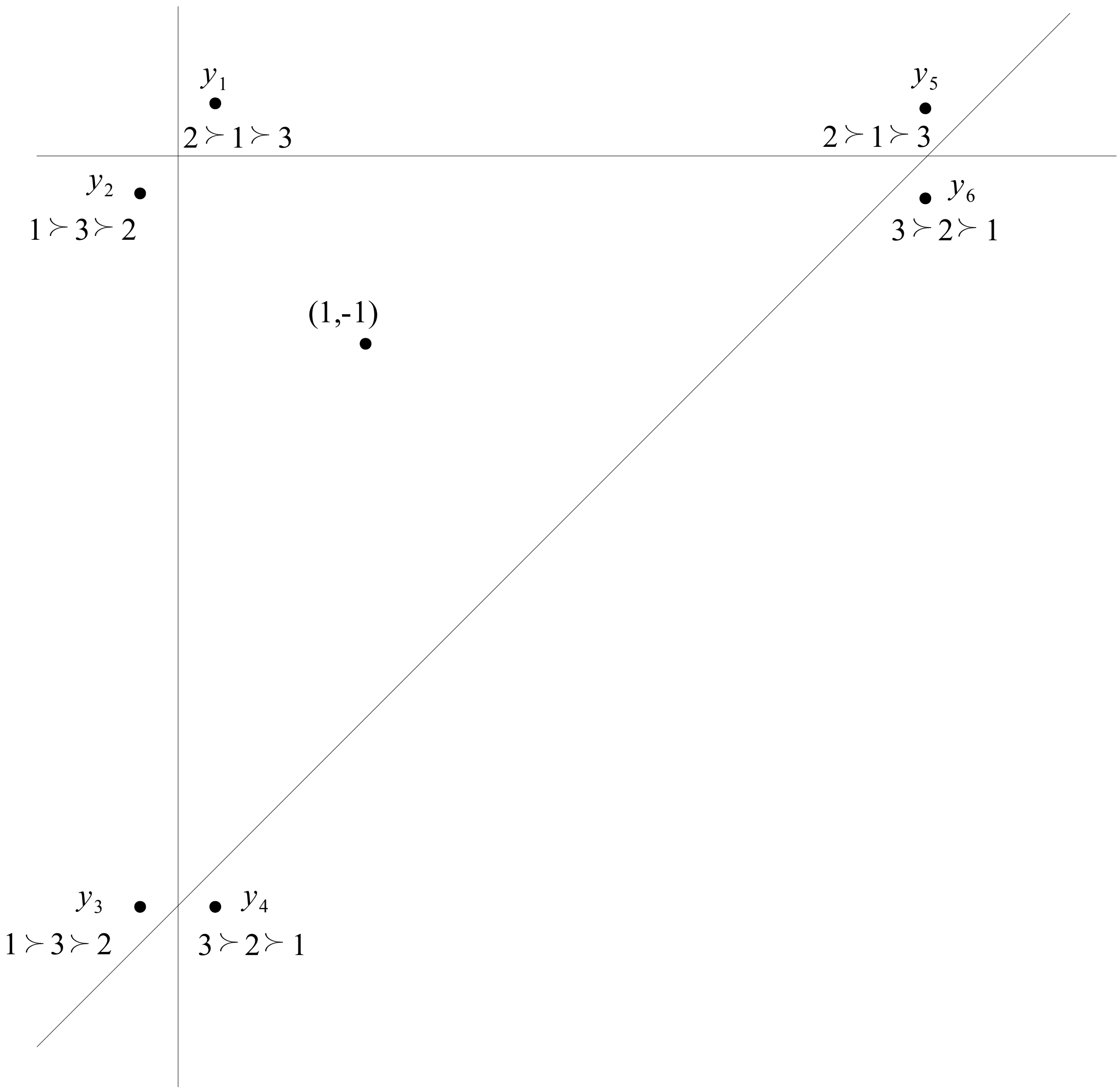

Figure 1