Uncertainty aversion and equilibrium existence in games with incomplete information

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UNCERTAINTY AVERSION AND EQUILIBRIUM EXISTENCE IN GAMES WITH INCOMPLETE INFORMATION*

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ABSTRACT. We consider games with incomplete information *à la* Harsanyi, where the payoff of a player depends on an unknown state of nature as well as on the profile of chosen actions. As opposed to the standard model, players' preferences over state–contingent utility vectors are represented by arbitrary functionals. The definitions of Nash and Bayes equilibria naturally extend to this generalized setting. We characterize equilibrium existence in terms of the preferences of the participating players. It turns out that, given continuity and monotonicity of the preferences, equilibrium exists in every game if and only if all players are averse to uncertainty (i.e., all the functionals are quasi–concave). We further show that if the functionals are either homogeneous or translation invariant then equilibrium existence is equivalent to concavity of the functionals.

Keywords: Games with incomplete information, equilibrium existence, uncertainty aversion, convex preferences.

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1. Introduction

Since Harsanyi [16], games with incomplete information have proved to be a powerful tool in the analysis of strategic situations where agents are uncertain regarding the specifics of the environment. The vast majority of applications assumes that the players in the game share a common prior probability distribution over the state space, and that each player is a Bayesian expected–utility maximizer with respect to (w.r.t.) this prior (and given the strategies of his opponents). With these assumptions on players' preferences, the appropriate solution concept for such games is the Bayes–Nash equilibrium, either in its ex–ante or interim forms. Existence of Bayes–Nash equilibrium in every game with incomplete information is guaranteed by a fixed–point argument.[1]

Starting with Ellsberg [9], a rich literature has developed showing consistent violations of the expected–utility maximization theory when decision makers are uncertain regarding the probabilities of relevant events. In particular, agents' preferences tend to exhibit *uncertainty aversion*, which cannot be explained within the subjective expected–utility framework. The experimental findings of Ellsberg [9] and his successors inspired economists to develop alternative theories of decision making under uncertainty (e.g., Schmeidler [27] and Gilboa and Schmeidler [14]).

Roughly speaking, we interpret uncertainty as the situation where the probabilities of some relevant events are ambiguous, and cannot be determined by the decision maker.[2] Under this interpretation, uncertainty is present in many real–life game–like situations. For instance, a firm in a Cournot oligopoly may be too uncertain regarding the demand function to assign a probability to the event that the intercept of this function is between the numbers $\underline{a}$ and $\overline{a}$; an oil company bidding for the rights to drill in a new site may not have enough information to assess the probability that the site has a capacity of ten million barrels. Notice that the uncertainty in both these examples concerns the probabilities of payoff relevant states of nature, and not the strategies of the opponents. If the firms compete in the same market for a long time, each firm probably knows the strategy of its competitors as a function of their information.

If, as in the above examples, the uncertainty is only regarding the state of nature then the definitions of Bayes and Nash equilibria can be naturally generalized to allow for arbitrary preferences over state–contingent utility vectors. We use the standard model of

[1]See Milgrom and Weber [23] for a general equilibrium existence result in games with incomplete information.

[2]As opposed to *risk* where the probabilities of outcomes are known. See Epstein [10].

a game with incomplete information, where each one of a finite set of players is endowed with a partition of the state space[3] that represents his information. Players' payoffs depend on the chosen action profile as well as on the realized state of nature. A strategy of a player is a function from states to (possibly mixed) actions that is measurable w.r.t. his partition. In any given state of nature, every strategy profile induces a probability distribution over pure action profiles. The utility of a player in this state of nature is his expected payoff according to this distribution. Thus, any strategy profile induces a real–valued function on the state space for each of the players. We refer to such a function as the *induced utility–vector*.

To define ex–ante equilibrium, assume that each player i is characterized by a functional J_i over the space of real–valued functions over the state space. If f is such a function then $J_i(f)$ represents the total utility that player i derives from f.[4] An ex–ante equilibrium is then simply defined as a strategy profile such that no player i can derive a higher utility (as measured by $J_i(f)$, where f is the induced vector for player i) by altering his strategy. In order to define interim equilibrium, assume that each player i is characterized by a family of functionals $\{J_i^F\}$, one for every non–empty event F. A strategy profile constitutes an interim equilibrium if, for every player i and for every element F in i's information partition, the restriction to F of the induced vector for i maximizes player i's utility (as measured by J_i^F) given the strategies of i's opponents.[5]

Since the domain of preferences over which the equilibrium concept is defined has been extended, it is natural to study the relation between existence of equilibrium and properties of players' preferences. On the one hand, we would like to know what kind of preferences guarantee equilibrium existence. On the other hand, we can take a 'revealed preferences' viewpoint and ask what can be learned about preferences from the observation that players have reached an equilibrium.

The main contribution of this paper (Theorem 1) demonstrates that, given that preferences are continuous and monotonic, *equilibrium exists in every game if and only if the preferences of all the players are represented by quasi–concave functionals*. Thus, the

[3]For simplicity, we restrict attention to finite state spaces. Our results can be extended to infinite spaces at the cost of adding standard technical assumptions on the various mathematical objects.

[4]Note that in the original Harsanyi model, $J_i(f)$ is the expected value of f w.r.t. the common prior.

[5]While it is common to consider interim–equilibria, we choose to focus on the notion of ex–ante equilibrium. The formulation of results in the ex–ante version is significantly more tractable and reduces notation. All the results given for the ex–ante version go through to the interim version. See Subsection 3.3 for a discussion of this point.

above questions have simple and clear answers. Furthermore, it is well known that quasi–concavity is the functional property that corresponds to uncertainty aversion (Schmeidler [27]). Our result therefore establishes a strong link between a behavioral property that is often observed and equilibrium existence.

We further study the relation between equilibrium existence and concavity. Concavity is perhaps a more intuitive definition of uncertainty aversion. Obviously, concavity implies equilibrium existence since it implies quasi–concavity. We show that the converse is also true given that the functionals that represent players' preferences are either translation invariant (Theorem 2) or homogeneous (Theorem 3). These results sharpen the connection between equilibrium existence and uncertainty aversion established in Theorem 1.

1.1. **An example.** To illustrate the definition of equilibrium and to motivate the results of the paper, consider the following example. There are two players $i = 1, 2$ and two states $\{s_1, s_2\}$. The action set of player 1 is $\{T, B\}$ and that of player 2 is $\{L, R\}$. The information partitions of both players are trivial. The payoffs are described in the diagram below, where 1 chooses a row and 2 a column.

	L	R
T	$1, 0$	$0, 0$
B	$0, 1$	$0, 0$

s_1

	L	R
T	$0, 0$	$0, 1$
B	$0, 0$	$1, 0$

s_2

Consider the case where both players are 'optimistic' in the sense that they evaluate any utility vector according to its maximal element. That is, if $f = (f(s_1), f(s_2))$ is the induced state–contingent utility for player i when a certain strategy profile is played, i's total utility is $J_i(f) = \max\{f(s_1), f(s_2)\}$ $(i = 1, 2)$.

We claim that with these preferences of the players there is no equilibrium in this game (since both players have trivial information partitions the ex–ante and interim versions coincide). Indeed, assume that player 1 plays $\beta T + (1-\beta)B$, and 2 plays $\alpha L + (1-\alpha)R$ for some $0 \leq \alpha, \beta \leq 1$. Then the induced vector for player 1 is $(\alpha\beta, (1-\alpha)(1-\beta))$ and for player 2 is $((1-\beta)\alpha, \beta(1-\alpha))$. Therefore, the images of the best response correspondences of both players consist of only pure strategies. But it is easy to see that there is no pure equilibrium in this game and, therefore, no equilibrium exists.

1.2. **Related literature.** The seminal works on non expected utility preferences are Schmeidler [27] and Gilboa and Schmeidler [14]. These papers initiated an extensive study of the axiomatic foundations of individual decision making under uncertainty. The list of papers emerging from this research is too long to be written here. Typically in this literature, players' preferences are independent of the state of nature. In our model this is not the case. Thus, given the strategies of their opponents, the players in our model face a state–dependent utility decision problem (see Karni [18] for a survey).

The ideas from individual decision making were later incorporated in interactive models of non–cooperative games with complete information. The main difficulty with defining equilibrium in this case is that strategies of players and beliefs about strategies of players are two different mathematical entities. Thus, the consistency of beliefs with actual strategies requirement that characterizes Nash equilibrium is usually hard to define for general preferences. Mukerji and Tallon [24] provides an extensive survey of this literature. Notice that in our model players' beliefs about their opponents' strategies coincide with the truth.

An especially relevant reference of the above literature is Crawford [7] who defined the notion of "equilibrium in beliefs". According to this solution concept every player uses just pure strategies, but players may believe that other players are mixing. To motivate this definition, Crawford constructs a simple example of equilibrium non–existence whenever preferences of players over distributions over pure action profiles are strictly quasi–convex. His example is similar to the one in the previous subsection. However, he did not attempt to provide a characterization of equilibrium existence, and the given example can not be easily generalized to prove such a characterization.

There are a few references that study ambiguity and, in particular, ambiguity aversion in games with incomplete information. Epstein and Wang [11] generalize the construction of a universal type space to a class of preferences that can accommodate uncertainty aversion. Kajii and Ui [17] study two different notions of equilibrium in games with incomplete information where players have maxmin preferences. One of these equilibrium concepts is a generalization of Crawford's "equilibrium in beliefs" for games with incomplete information. The other, called "mixed equilibrium", is a special case of the interim version of our equilibrium. Finally, Bade [3] considers incomplete information extensions *à la* Aumann [2] of normal–form games. However, her results are confined to the case where payoffs are state–independent.

On the more applicative side, Salo and Weber [26], Lo [20], Bose et al. [5] and Levin and Ozdenoren [19] study auctions with uncertainty averse bidders. Bade [4] shows existence of equilibrium in a political game where parties are uncertain regarding the distribution of voters.

1.3. **Organization.** We proceed as follows. Section 2 formally defines the class of games with incomplete information that we consider and the notion of ex–ante equilibrium. In Section 3 we state and prove the main results, which relate uncertainty aversion to equilibrium existence. Interim equilibrium is defined and discussed in this section as well. In Section 4 we deal with some additional aspects of equilibrium existence. Namely, Subsection 4.1 discusses the implications of our results to familiar functional forms often appearing in the literature, and Subsection 4.2 hints to the relation between players' common preferences and symmetric equilibria of symmetric games. We conclude in Section 5.

2. Ex–ante equilibrium

An *environment* is a tuple $(S, N, \mathfrak{J} = \{J_i\}_{i \in N})$. The first component $S = \{s_1, s_2, \ldots, s_m\}$ is a non–empty finite set of *states of nature* (the state space). We assume $m \geq 2$ throughout. A *utility–vector* (vector, for short) is any function that maps S to $\mathbb{R}$. We will usually use the letters f, g to denote vectors. Addition of vectors and multiplication of vectors by scalars are performed pointwise. We can therefore identify the space of all vectors with the linear space $\mathbb{R}^m$. The constant vector f in which $f(s) = c$ for every $s \in S$ will be denoted by $\underline{c}$. $N = \{1, 2, \ldots, n\}$, where $n \geq 2$, is the set of players. For each $i \in N$, the functional $J_i : \mathbb{R}^m \to \mathbb{R}$ represents player i's preferences over vectors. The environment is fixed throughout the analysis.

Throughout the paper we maintain two mild assumptions on players preferences:

Continuity (C): For every player i, J_i is continuous over $\mathbb{R}^m$.

Monotonicity (M): For every player i and for every two vectors f, g, if $f(s) \geq g(s)$ $(f(s) > g(s))$ for every $s \in S$ then $J_i(f) \geq J_i(g)$ $(J_i(f) > J_i(g))$.

A *normal–form game with incomplete information* (game, for short) is defined by $G = (\{F_i\}_{i \in N}, \{A_i\}_{i \in N}, \{u_i\}_{i \in N})$. For each $i \in N$, F_i is player i's *information partition* – a partition of S, and A_i is the finite non–empty set of actions[6] of i. Denote by $A = \times_{i \in N} A_i$

[6]Here we assume that each player has the same set of actions in all states of nature. Our results can easily be extended to the case where the action set of a player vary across S as long as it is constant at each element of that player's information partition.

the set of all action profiles with typical element $a = (a_1, ..., a_n)$. The utility function of player $i \in N$ is $u_i : S \times A \to \mathbb{R}$. The set of all games (in a fixed environment) is denoted by Γ.

Let $G \in \Gamma$ and fix $i \in N$. A *strategy* for player i is an F_i–measurable function[7] $\sigma_i : S \to \Delta(A_i)$. The set of all strategies for player i is denoted Σ_i, and the set of all *strategy profiles* is $\Sigma = \times_{i \in N} \Sigma_i$ with typical element $\sigma = (\sigma_1, \ldots, \sigma_n)$. The probability with which player i plays the action $a_i \in A_i$ in state $s \in S$ according to σ_i is denoted $\sigma_i(s, a_i)$. As usual, σ_{-i} denotes the strategy profile of players other than i in which each player $j \neq i$ plays as in σ.

Every strategy profile σ in a game G induces a vector for each one of the players. Formally, the *induced vector of player i* is $f_i^{(\sigma)}$ where $f_i^{(\sigma)}(s) = \sum_{a \in A} \left(\prod_{j \in N} \sigma_j(s, a_j) \right) u_i(s, a)$ for every $s \in S$. We can now define our notion of ex–ante equilibrium.

Definition 1. *Let $G \in \Gamma$. A strategy profile $\sigma \in \Sigma$ is an* ex–ante $\mathfrak{J}$–equilibrium *of G if, for every $i \in N$ and for every $\sigma'_i \in \Sigma_i$, $J_i(f_i^{(\sigma)}) \geq J_i(f_i^{(\sigma'_i, \sigma_{-i})})$.*

3. Main results

3.1. **Equilibrium existence and uncertainty aversion.**

Uncertainty aversion has been a focus of the decision theory literature in the last two decades. Schmeidler's [27] seminal definition of uncertainty aversion[8] states that if the acts f, g satisfy $f \succeq g$, then for any $\alpha \in (0, 1)$, $\alpha f + (1 - \alpha) g \succeq g$. This axiom translates into *quasi–concavity* of the preferences representing functional (see Cerreia et al. [6] and Hanany and Klibanoff [15]).

Uncertainty Aversion (UA): J_i is quasi–concave for every player i. That is, $J_i(\alpha f + (1 - \alpha) g) \geq \min\{J_i(f), J_i(g)\}$ for every two vectors f, g and $\alpha \in (0, 1)$.

While the (UA) is relevant to the preferences of each decision maker separately, the next property reflects the interactive flavor of our model.

Equilibrium Existence (EE): There exists an ex–ante $\mathfrak{J}$–equilibrium in every game $G \in \Gamma$.

[7] For a finite set X, $\Delta(X)$ is the set of all probability measures over X.

[8] While this is the most commonly used definition of uncertainty aversion, there are alternative definitions in the literature such as those of Epstein [10] and Ghirardato and Marinacci [13].

We can now state our main result.

Theorem 1. *(UA) is equivalent to (EE).*

Proof. We start with the following lemma.

Lemma 1. *If J_i is not quasi–concave then there are vectors f, g such that $J_i(f) = J_i(g)$ and $J_i(\alpha f + (1-\alpha)g) < J_i(f)$ for every $\alpha \in (0, 1)$.*

Proof. If J_i is not quasi–concave then there are vectors f', g' and $\alpha_0 \in (0, 1)$ such that $J_i(f') = J_i(g')$ and $J_i(\alpha_0 f' + (1-\alpha_0)g') < J_i(f')$. By (C), the set $B := \{\beta \in [0, 1] : J_i(\beta f' + (1-\beta)g') \leq J_i(f')\}$ is compact and contains an interval around α_0. Let β_1 be the minimal element of B which is larger than α_0 and satisfies $J_i(\beta f' + (1-\beta)g') = J_i(f')$. Similarly, let β_2 be the maximal element of B which is smaller than α_0 and satisfies $J_i(\beta f' + (1-\beta)g') = J_i(f')$. Define $f = \beta_1 f' + (1-\beta_1)g'$ and $g = \beta_2 f' + (1-\beta_2)g'$. Then $J_i(f) = J_i(g)$ and $J_i(\alpha f + (1-\alpha)g) < J_i(f)$ for every $\alpha \in (0, 1)$. $\square$

(EE) $\Longrightarrow$ **(UA)**

Assume to the contrary that (EE) is satisfied, and that there is $i \in N$ such that J_i is not quasi–concave. Consider the following game $G \in \Gamma$. The information partitions of all the players are trivial. The action set of player i is $A_i = \{T, B\}$ and the action set of some arbitrary player $j \neq i$ is $A_j = \{L, R\}$. Each one of the other players (if there are any) has only one action and, therefore, these players have no influence on the outcome of the game.

The payoff function for player j is given by

$$u_j(s, a_i, a_j) = \begin{cases} 1; & a_i = T, a_j = L, \\ 1; & a_i = B, a_j = R, \\ 0; & a_i = T, a_j = R, \\ 0; & a_i = B, a_j = L. \end{cases}$$

Let f, g be as in Lemma 1. For every $0 < \delta < \frac{1}{2}$ define the intervals

$$\begin{aligned} X(\delta) &= \{\alpha f + (1-\alpha)g \ : \ \alpha \in [0.5 - \delta, 0.5 + \delta]\} \\ Y(\delta) &= \{\alpha f + (1-\alpha)g \ : \ \alpha \in [0, \delta]\} \\ Z(\delta) &= \{\alpha f + (1-\alpha)g \ : \ \alpha \in [1 - \delta, 1]\}. \end{aligned}$$

Let $s(\delta) = \max\{J_i(h) : h \in X(\delta)\}$, $t(\delta) = \min\{J_i(h) : h \in Y(\delta)\}$ and $r(\delta) = \min\{J_i(h) : h \in Z(\delta)\}$. By (C), there is $\delta^* > 0$ small enough such that both $s(\delta^*) < t(\delta^*)$ and

$s(\delta^*) < r(\delta^*)$. Define $f' = \delta^* g + (1-\delta^*)f$ and $g' = \delta^* f + (1-\delta^*)g$. Let the payoff function to player i be given by

$$u_i(s, a_i, a_j) = \begin{cases} f'(s); & a_i = T, a_j = L, \\ g'(s); & a_i = B, a_j = R, \\ f(s); & a_i = T, a_j = R, \\ g(s); & a_i = B, a_j = L. \end{cases}$$

Thus, the resulting bimatrix game in state $s \in S$ is given by the following diagram, where i is the rows player and j the columns player:

	L	R
T	$f'(s), 1$	$f(s), 0$
B	$g(s), 0$	$g'(s), 1$

The following sequence of claims proves that G has no ex–ante $\mathfrak{J}$–equilibrium.

Claim 1. *The strategy $\frac{1}{2}T + \frac{1}{2}B$ is never a best response for player i.*

Proof. Fix any strategy $\alpha L + (1-\alpha)R$ $(0 \le \alpha \le 1)$ for player j. If player i plays $\frac{1}{2}T + \frac{1}{2}B$ then the induced vector for him is $\frac{1}{2}\alpha f' + \frac{1}{2}(1-\alpha)f + \frac{1}{2}\alpha g + \frac{1}{2}(1-\alpha)g' = \left(\frac{1}{2} + \frac{\delta^*}{2} - \alpha\delta^*\right) f + \left(\frac{1}{2} - \frac{\delta^*}{2} + \alpha\delta^*\right) g \in X(\delta^*)$. On the other hand, if i plays B his induced vector is in $Y(\delta^*)$. By construction, the latter gives i a higher payoff than the former. □

Claim 2. *There is no equilibrium in which player j plays a mixed strategy.*

Proof. By the previous claim, there cannot be an equilibrium where player i mixes with equal probabilities between his two strategies. For any other strategy of player i, (M) implies that the best response for player j is (only) in pure strategies. □

Claim 3. *If j plays a pure strategy then player i has a unique best response which is a pure strategy.*

Proof. This is a straightforward consequence of the construction. □

Claim 4. *There is no equilibrium in pure strategies.*

Proof. Easy to check. □

The Combination of these claims proves that no equilibrium exists in G.

(UA) $\Longrightarrow$ **(EE)**

Existence of ex–ante equilibrium is guaranteed by the following argument. Since this is a standard argument, we only provide the outline of the proof and omit the details.

For every $i \in N$, define $BR_i : \Sigma_{-i} \to \Sigma_i$ to be player i's best response correspondence. Quasi–concavity of J_i guarantees that BR_i is convex valued. Continuity of J_i implies that BR_i is upper semi–continuous and that it has compact values. (EE) is, therefore, a consequence of Brouwer's fixed point theorem. $\square$

3.2. **Equilibrium existence and concavity.** There are many instances in the literature where uncertainty aversion is represented by concavity rather than quasi–concavity of the functional. This is no coincidence since, to quote Schmeidler [27], "Intuitively, uncertainty aversion means that "smoothing" or averaging utility distributions makes the decision maker better off . . . Concavity captures best the heuristic meaning of uncertainty aversion". We thus define

Concavity (CON): J_i is concave for every player i.

The purpose of this subsection is to show that (EE) and (CON) are equivalent given that the functionals have some additional natural properties. Obviously, if the functionals of all the players are concave then (by Theorem 1) equilibrium exists in every game. Thus, the difficulty is to prove the converse, that (EE) implies (CON).

The following standard properties will be used for the results. In each of them we mean that the relevant property holds for every $J_i \in \mathfrak{J}$.

Translation Invariance (TI): For every vector f and a constant vector $\underline{c}$, $J_i(f + \underline{c}) = J_i(f) + c$.

Homogeneity (H): For every vector f and for every $\alpha \geq 0$, $J_i(\alpha f) = \alpha J_i(f)$.

The first result of this section uses the (TI) property.

Theorem 2. *If (TI) is satisfied then (CON) is equivalent to (EE).*

Proof. We only prove that (EE) implies (CON), since the other direction follows from Theorem 1. By repeating the argument of Lemma 1, if J_i is not concave then it is possible to find vectors f, g such that $J_i(\alpha f + (1 - \alpha)g) < \alpha J_i(f) + (1 - \alpha)J_i(g)$ for every $\alpha \in (0, 1)$. Fix such f, g and choose a number $M > 0$ large enough such that both $J_i(f + \underline{M}) > J_i(g)$ and $J_i(g + \underline{M}) > J_i(f)$. Existence of such a number is guaranteed by (TI).

Consider the game $G \in \Gamma$ with payoffs as in the diagram below (like in the proof of Theorem 1, all the information partitions are trivial and there are only two players with more than one action. Player i chooses T or B and j chooses L or R).

	L	R
T	$f(s), 1$	$f(s)+M, 0$
B	$g(s)+M, 0$	$g(s), 1$

Assume that player j plays the strategy $\alpha L + (1-\alpha)R$ for some $\alpha \in [0,1]$. By (TI) and the construction of f, g, it cannot be that $f-g$ is constant since in that case J_i is linear on the interval $[f, g]$. It follows that $\alpha f + (1-\alpha)(f+\underline{M}) \neq \alpha(g+\underline{M}) + (1-\alpha)g$, so every two different strategies of player i induce different vectors for him. If player i plays $\beta T + (1-\beta)B$ (where $0 < \beta < 1$) then the induced vector for i is given by $\beta f + (1-\beta)g + [\alpha(1-\beta) + \beta(1-\alpha)]\underline{M}$. Using (TI) and the construction of f, g we obtain

$$\begin{aligned}
&J_i\left(\beta f + (1-\beta)g + [\alpha(1-\beta) + \beta(1-\alpha)]\underline{M}\right) = \\
&J_i\left(\beta f + (1-\beta)g\right) + [\alpha(1-\beta) + \beta(1-\alpha)]M < \\
&\beta J_i(f) + (1-\beta)J_i(g) + [\alpha(1-\beta) + \beta(1-\alpha)]M = \\
&\beta J_i(\alpha f + (1-\alpha)(f+\underline{M})) + (1-\beta)J_i(\alpha(g+\underline{M}) + (1-\alpha)g \leq \\
&\max\{J_i(\alpha f + (1-\alpha)(f+\underline{M})), J_i(\alpha(g+\underline{M}) + (1-\alpha)g)\}.
\end{aligned}$$

It follows that, no matter what the strategy of player j is, a (strictly) best response for player i is a pure strategy. But if i plays a pure strategy then the best response for player j is also a pure strategy. Since it is easy to see that there is no pure strategy equilibrium in G the proof is complete.

□

Our next aim is to obtain a similar result to that of Theorem 2 when (TI) is replaced by (H). However, to achieve this we must restrict attention to the class of games $G \in \Gamma$ with non–negative payoffs and, correspondingly, to functionals with[9] $\mathbb{R}^m_+$ as a domain.

Theorem 3. *Assume that each J_i is defined only over $\mathbb{R}^m_+$ and consider the class of all games in Γ with non–negative payoffs. If (H) is satisfied then (CON) is equivalent to (EE).*

[9] $\mathbb{R}^m_+$ is the set of all non–negative utility vectors.

Proof. We only prove that (EE) implies (CON). Assume that J_i is not concave. Then it is possible to find f, g like in the proof of Theorem 2. Moreover, by C, we can find such f, g in the interior of $\mathbb{R}^m_+$.

We now construct a game G with non–negative payoffs and with no equilibrium. The information partitions and the action sets are the same as in the proof of Theorem 2. The payoffs are given in the diagram below.

	L	R
T	$0,1$	$f(s),0$
B	$g(s),0$	$0,1$

Notice that (H) and (M) when combined imply that each J_i is strictly positive on the interior of $\mathbb{R}^m_+$. Since $f(s), g(s) > 0$ for every s, $J_i(f), J_i(g) > 0$. This implies that there is no pure equilibrium. (H) and the construction of f, g imply that f, g are linearly independent. Thus, if player j plays the strategy $\alpha L + (1-\alpha)R$ for some $\alpha \in [0,1]$, player i's induced vector when he plays T is different from the induced vector when he plays B.

If i plays $\beta T + (1-\beta)B$ (where $0 < \beta < 1$) then the induced vector for i is given by $\alpha(1-\beta)g + \beta(1-\alpha)f$. Defining $\gamma = \alpha(1-\beta) + \beta(1-\alpha) > 0$ and using (H) and the construction of f, g we get

$$J_i\left(\alpha(1-\beta)g + \beta(1-\alpha)f\right) = \gamma J_i\left(\frac{\alpha(1-\beta)}{\gamma}g + \frac{\beta(1-\alpha)}{\gamma}f\right) <$$
$$\alpha(1-\beta)J_i(g) + \beta(1-\alpha)J_i(f) = (1-\beta)J_i(\alpha g) + \beta J_i((1-\alpha)f) \leq$$
$$\max\left\{J_i(\alpha g), J_i((1-\alpha)f)\right\}.$$

It follows that, no matter what is the strategy of player j, a (strictly) best response for player i is a pure strategy. But if i plays a pure strategy then the best response for player j is also a pure strategy. This implies that G has no ex–ante $\mathfrak{J}$–equilibrium.

□

Remark 1. *Theorem 3 is not true without the restriction of the domain to the non–negative orthant. Indeed, let $m = 2$ and $J_i(f) = \max\{f(s_1) + f(s_2), 2(f(s_1) + f(s_2))\}$ for every $i \in N$. Then each J_i is homogenous, (strictly) monotone, continuous, quasi–concave but not concave. Thus,* (EE) *is satisfied while* (CON) *is not.*

Remark 2. *Theorems 2 and 3 show that, under the additional assumptions of (TI) or (H), concavity is equivalent to the seemingly weaker condition of quasi–concavity. For the* *case of homogeneity, a direct proof of this fact is already well known (see Rader* [25]*, page* 98 *Theorem* 6*, and Shephard* [28]*, page* 31 *Proposition* 7*). For the case of translation invariance, we are not aware of a similar result in the literature.*

3.3. **Interim equilibrium.** While our main results are stated for the ex–ante version of the equilibrium, it is possible to obtain similar results in the interim version as well. In order to define interim equilibrium we first need to revise the notion of an environment. An *interim environment* is a tuple $\left(S, N, \mathfrak{J} = \{J_i^F\}_{i\in N, F\subseteq S}\right)$. The first two components S and N are as described in Section 2. Given an *event* $F \subseteq S$, an F*–vector* is any function from F to $\mathbb{R}$. Once F is fixed we identify the space of all F–vectors with the linear space $\mathbb{R}^{|F|}$. For each $i \in N$ and $F \subseteq S$, the functional $J_i^F : \mathbb{R}^{|F|} \to \mathbb{R}$ represents player i's preferences over F–vectors.

Given such an environment, a game with incomplete information, a strategy, a strategy profile and an induced vector (over S) are defined as in Section 2. Given a vector f over S and an event $F \subseteq S$, we denote by $f|_F$ the F–vector which is the restriction of f to F.

Definition 2. *Let $G \in \Gamma$. A strategy profile $\sigma \in \Sigma$ is an* interim $\mathfrak{J}$–equilibrium *of G if, for every $i \in N$, every $F \in F_i$, and for every $\sigma'_i \in \Sigma_i$, $J_i^F(f_i^{(\sigma)}|_F) \geq J_i^F(f_i^{(\sigma'_i,\sigma_{-i})}|_F)$.*

Note that when considering the notion of interim environment, the analogous definition of an ex–ante $\mathfrak{J}$–equilibrium in a game G is a strategy profile σ, such that for every $i \in N$ and for every $\sigma'_i \in \Sigma_i$, $J_i^S(f_i^{(\sigma)}) \geq J_i^S(f_i^{(\sigma'_i,\sigma_{-i})})$. In this revised model, the properties presented in Section 3 can be reformulated as possible properties of J_i^S (instead of J_i), resulting with the analogs to Theorems 1, 2 and 3.

In order to characterize interim $\mathfrak{J}$–equilibrium existence, we need to adapt the properties presented in Section 3 to suit the interim environment. By "adapt" we mean that each relevant property holds for every J_i^F. Once this is done, one can prove analogue results to Theorems 1, 2 and 3. Since such an exercise provides no additional insights we omit the details.

4. Further aspects of equilibrium existence

4.1. **Special functional forms.** In this section we apply the results of Section 3 to a variety[10] of families of representing functionals, when considered in a strategic environment as discussed above.

Gilboa and Schmeidler [14] axiomatize preference orders that are determined by the minimal expected–utility w.r.t. some convex and compact set of priors. Given a set of probability measures $P_i \subseteq \Delta(S)$, we say that J_i *represents maxmin preferences w.r.t.* P_i if[11] $J_i(f) = \min_{p \in P_i} p \cdot f$ for every vector f. The following is a consequence of the Gilboa–Schmeidler axiomatization (see in particular Lemma 3.5 in [14]) when combined with Theorem 3 (or with Theorem 2). It can be seen as an alternative characterization of maxmin preferences in a multi–player environment.

Corollary 1. *The functionals $\{J_i\}_{i\in N}$ satisfy $(H), (M), (TI)$ and (EE) if and only if there is a family of sets $\{P_i\}_{i\in N}$, where $P_i \subseteq \Delta(S)$ is convex and compact for each i, such that J_i represents maxmin preferences w.r.t. P_i. Moreover, each set P_i is uniquely determined by J_i.*

Maccheroni et al. [21] axiomatize *variational preferences*, which generalize the model of maxmin preferences. The functional form of a variational preference is $J_i(f) = \min_{p\in\Delta(S)} \big(p \cdot f + c(p)\big)$, where $c : \Delta(S) \to [0, \infty]$ is a grounded[12] convex and lower–semicontinuous functional. A consequence of the axiomatization of Maccheroni et al. along with Theorem 2 is the following.

Corollary 2. *The functionals $\{J_i\}_{i\in N}$ satisfy $(M), (TI)$ and (EE) if and only if every J_i represents a variational preference.*

Remark 3. *Similar functional forms to the maxmin and variational preferences appear in the risk assessment literature (see Artzner et al.* [1]*, Delbaen* [8]*, and Föllmer and Schied* [12]*).*

Choquet integral preferences (Schmeidler [27]) are often used as an alternative to expected–utility maximization. It is well known that the Choquet integral w.r.t. any

[10] The representing functionals discussed below are merely a partial list of those appearing in the literature of axiomatic decision theory.

[11] If $p \in \Delta(S)$ and $f \in \mathbb{R}^m$ is a vector then $p \cdot f = \int_S f dp$ denotes the expected value of f according to the probability measure p.

[12] A function c is grounded if its infimum over the domain is zero.

capacity[13] v satisfies the $(C),(M),(H)$ and (TI) properties. If v is convex ($v(E)+v(F) \leq v(E \cup F) + v(E \cap F)$ for every two events $E, F \subseteq S$) then the Choquet integral w.r.t. v is a concave functional. Moreover, it coincides with the maxmin functional w.r.t. the core[14] of v.

However, if v is not convex then the Choquet integral w.r.t. v is not a concave functional. In this case, Theorem 2 implies that, if one of the players has Choquet preferences w.r.t. a non–convex capacity, then there is a game G with no equilibrium. Note that in the example presented in the introduction both players' preferences are represented by the Choquet integral w.r.t. the capacity that assigns 1 to every non–empty set of states. This capacity is non–convex, and indeed, the proposed game has no equilibrium. This example can be easily generalized to provide a constructive proof of ex–ante equilibrium non–existence whenever one of the players has Choquet preferences w.r.t. some non–convex capacity.

4.2. **Symmetry and common preferences.** A natural question in our multi–agent environment is what characterizes the case where different players have similar preferences? We say that agents i and j have *common preferences* if for every f, g, $J_i(f) \geq J_i(g)$ if and only if $J_j(f) \geq J_j(g)$. What seems to be a natural candidate for the characterization of common preferences is existence of symmetric equilibria in symmetric games.

Fix two players $i, j \in N$. For a given action profile a in some game $G \in \Gamma$ with $A_i = A_j$, we denote a^{ij} the action profile in which players i and j exchange their actions while any other player plays the same as in a. That is, $a_i^{ij} = a_j$, $a_j^{ij} = a_i$ and $a_k^{ij} = a_k$ for every $k \in N \setminus \{i, j\}$. A game $G \in \Gamma$ is *ij–symmetric* if $F_i = F_j$, $A_i = A_j$, and for every $s \in S$ and $a \in A$, $u_i(s, a) = u_j(s, a^{ij})$ and $u_k(s, a) = u_k(s, a^{ij})$ for every player $k \neq i, j$.

We now state an additional property.

***ij*–Symmetric Equilibrium (*ij*-SE)**: If G is ij–symmetric then there is an ex–ante $\mathfrak{J}$–equilibrium σ in G such that $\sigma_i = \sigma_j$.

When players i and j have common preferences it is a standard exercise to show that $(ij - SE)$ is satisfied. The question is therefore whether the converse holds as well.

[13]A capacity is a set function $v : 2^S \to \mathbb{R}$ satisfying $v(\emptyset) = 0$, $v(S) = 1$ and $v(E) \leq v(F)$ whenever $E \subseteq F$.

[14]The core of a capacity v is the set of probability measures p over S satisfying $p(E) \geq v(E)$ for every event E.

In consumer theory, Mas–Collel [22] studies the relation between common preferences over consumption bundles and common demand correspondences. He assumes standard continuity, monotonicity and quasi–concavity of preferences, and shows that under an additional mild lipschitzian condition, preferences are the same if and only if they induce the same demand correspondences.

Assuming Mas–Collel's lipschitz condition, if the preferences of players i and j are not common, then the induced demand functions are different. It means that we can find a budget set such that the players will choose differently from this set. Thus, when preferences satisfying the lipschitzian condition are distinct, one can construct a game where the pure actions of each of the two players give the extreme points of this budget set (independently of the actions of the other player). In this case there will be no symmetric equilibrium. Therefore, under the mild lipschitzian condition, $(ij - SE)$ implies common preferences.

Mas-Collel further constructs an example, where two distinct preferences, not having the lipschitzian property, induce the same demand correspondences. This example implies that there is no hope to deal with the general case using the previous method of proof, and one should take into account more general games. We leave this issue open.

5. Conclusion

The paper extends the definitions of Nash and Bayes equilibria in games with incomplete information to the case where players perceived ambiguity is not necessarily through a unique prior. Rather, player's preferences are represented by general functionals over state–contingent utility vectors. The main result of this paper shows that such equilibria exist in every game if and only if all players are averse to uncertainty; that is, all preferences representing functionals are quasi–concave. With the additional properties of homogeneity or translation invariance, equilibrium existence in every game is equivalent to concavity of the functionals. While these results are not surprising from a mathematical point of view, they provide an interesting link between the attitude of agents to uncertainty over the state of nature and the existence of self–enforcing agreements in interactive situations.

In our view, the more interesting implication is that equilibrium existence requires uncertainty aversion. While there are many works that deal with sufficient conditions (on preferences) for equilibrium existence, we are not aware of any work that asked the converse question: What can be learned on preferences from the existence of equilibrium? Our results show that equilibrium existence may have important and meaningful implications on preferences.

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