Translation invariance when utility streams are infinite and unbounded

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Abstract: The axiom *translation invariance* consists in asserting the invariance of the ranking of two utility streams if one applies the same translation to both. This axiom is significant in the characterization of utilitarian criteria in finite dimension. This characterization is achieved thanks to the "weak weighted utilitarianism theorem".The objective here is to propose a generalization of this theorem in a space of infinite and unbounded utility streams. A consequence of the suggested generalization is that, in the context of intergenerational choice, every maximal point with respect to a paretian utilitarian order granting comparable considerations to the present and the future, is also a maximal point with respect to some future-oriented criterion.

Keywords: Translation invariance – Infinite utility streams – Utilitarianism – Intergenerational equity

JEL Classification Numbers: C61, D63, D71, D99.

1 Introduction

The axiom *translation invariance* (following the terminology of Weibull 1985) consists in asserting the invariance of the ranking of two utility streams if one applies the same translation to both (formal definition in section 2). In the literature, it is also referred to as the *translation scale invariance* axiom (Basu-Mitra 2007a, Banerjee 2006), or the *invariance with respect to individual change of origin* axiom (d'Aspremont-Gevers 2002). It belongs to the set of axioms "concerned with separating formally superfluous details from potentially paramount information" (d'Aspremont-Gevers 2002, page 19). More precisely, it characterizes situations where individual utility is cardinal and where profits or losses of utility are comparable from an individual to another. It is thus checked for example if utility constitutes not only a representation of preferences, but also an objective measurement of satisfaction [2]. In addition, it does not require that a given value of utility represents the same satisfaction for all the individuals. For example, satisfaction 0 can correspond to two different baskets of goods for two different individuals. For this reason, it also corresponds to what is called in the literature *interpersonal unit comparability and non level comparability*, or *zero-independence* (for example in Lauwers-Vallentyne 2004).

This axiom is particularly significant in the characterization of utilitarian criteria (i.e. criteria based on a sum of utilities). Indeed, the characterizations

[2] Such a measurement presupposes the ability to give an objective meaning to the concept of satisfaction, what is subject to debate in the literature. On this topic, see for example d'Aspremont-Gevers (2002), section: Domain interpretation.

of certain versions of utilitarianism have recourse to versions of this axiom. For example: d'Aspremont-Gevers (1977), d'Aspremont-Gevers (2002) in the case of finite utility streams, Basu-Mitra (2007a), Lauwers-Vallentyne (2004), or Banerjee (2006) in the case of infinite streams.

The characterization given by d'Aspremont-Gevers (2002) is based on the weak weighted utilitarianism theorem (d'Aspremont-Gevers 2002, theorem 17, page 57). This theorem affirms that any order satisfying the axiom *weak translation invariance* (which is a weakened version of *translation invariance*) and also satisfying *weak Pareto*, is a subrelation to a weighted utilitarianism. This theorem applies in finite dimension, i.e. for a finite number of individuals.

The objective here is to propose a generalization of the weak weighted utilitarianism theorem to a space of infinite and unbounded utility streams. That will apply for example to intergenerational choice (i.e. intertemporal choice with infinite horizon) and unbounded utility streams.

Weibull (1985) also proposed a theorem (theorem A) exploring the consequences of the axiom *translation invariance* for an order defined on a general normed real vector space. However, the assumptions of Weibull theorem entail representability, that is, the existence of a real-valued order-preserving function. In the context of intergenerational choice, representability is too restrictive as it entails the impossibility to have simultaneously *anonymity* and *weak Pareto* (Basu-Mitra 2007b), which are usually considered as basic principles. Moreover, the theorem proposed here (theorem 5) requires *weak translation invariance* whereas Weibull theorem requires full *translation invariance.* The reason of these limitations is that Weibull theorem applies to general spaces, what does not make it possible to exploit properties specific to infinite utility streams, namely *weak Pareto.*

The generalization proposed here shows that, compared with the situation in finite dimension, it is added a term which I proposed to call: linear limits (definition 4). This result makes it possible, in particular, to highlight the relation between equitable utilitarianism for infinite and unbounded streams and linear limits. For example, a consequence is that every maximal point with respect to equitable utilitarianism is also a maximal point with respect to some positive linear limit. In the context of intergenerational choice, this means that equitable utilitarianism must comply entirely with long-term optimality. This result holds if we only impose that the order grants comparable considerations to the present and the future.

The exploitation of the suggested generalization is based on a decomposition of the dual of a space of infinite and unbounded real sequences to which the streams are supposed to belong: l_∞^r (section 3). The decomposition theorem used here (theorem 3) is a generalization to the unbounded case, of the decomposition theorem used in Lauwers (1998): the Yosida-Hewitt theorem. Chilchinisky (1996) also used the Yosida-Hewitt theorem to study long-term-oriented intertemporal criteria. Le Van-Saglam (2004) applied a similar decomposition for the determination of the Lagrange multipliers associated with the calculus of infinite horizon optimal growth. We will return to Chilchinisky (1996) and Lauwers (1998) in section 4.

Section 2 gives the weak weighted utilitarianism theorem (theorem 1), as well as some comments on the axioms used in this theorem. Section 3 specifies the working space, the norm and gives the decomposition theorem (theorem 3) with a corollary calculating a particular partial derivative of a real valued function interpreted as the sensitivity of the function to long-term changes. Section 4 generalizes theorem 1 (theorem 5). In the context of intergenerational choice, section 5 uses theorem 5 to establish the consequence pointed out above: the necessity to comply with long-term optimality. For the issues tackled in section 5, whether the infinite utility streams are bounded or not does not change the analysis. Therefore, section 5 will consider the more usual case where the infinite utility streams are bounded.

2 The weak weighted utilitarianism theorem

Denote $\mathbb{R}$ the real line and $\mathbb{N}^*$ the set of positive integers.

For an order R (i.e. a transitive and complete binary relation on a set of alternatives) and two alternatives x and y, "x is preferred or indifferent to y" is denoted $x \succsim y$, "x is preferred to y" is denoted $x \succ y$ and "x is indifferent to y" is denoted $x \sim y$. In this section, the set of alternatives is $\mathbb{R}^n$, where n is a positive integer representing the number of individuals.

Following the notation of d'Aspremont-Gevers (2002), the axioms used in this section are:

weak Pareto: $\forall x, y$ in $\mathbb{R}^n$, $x \succ y$ if $\forall i$ in $\{1, .., n\}$, $x_i > y_i$.

inv$(a_i + x_i)$: x, y in $\mathbb{R}^n, x \succsim_R y \Longrightarrow \forall a$ in $E, x + a \succsim_R y + a$

weak inv$(a_i + x_i)$: $\forall x, y$ in $\mathbb{R}^n$, $x \succ y \Longrightarrow \forall a$ in $\mathbb{R}^n$, $x + a \succsim y + a$.

minimal individual symmetry: $\forall i, j$ in $\{1, ..., n\}$, there exist x, y in $\mathbb{R}^n$ such that $x_i > y_i$, $x_j < y_j$, $x_k = y_k$ for all k in $\{1, ..., n\} / \{i, j\}$ and $x \sim y$.

anonymity: For all permutation π on $\{1, ..., n\}$ and all x in $\mathbb{R}^n$, $x \sim \pi x$, where $\pi x = \left(x_{\pi(i)}\right)_{i=1}^n$

The axiom *translation invariance* corresponds to $inv(a_i + x_i)$.

The axiom *weak Pareto* expresses a requirement of a minimal sensitivity of the order with respect to the components. The axiom $inv(a_i + x_i)$ was presented in section 1. The weakened form *weak* $inv(a_i+x_i)$ does not make it possible to have interpersonal unit comparability because a translation may transform a strict preference between two alternatives in indifference. The axiom *minimal individual symmetry* is an equity axiom that can accommodate to the incomparability of utilities. "It sets a limit on the influence any individual can exert on the social ranking when he (she) has a single opponent" (d'Aspremont-Gevers 2002, page 54). Finally the axiom *anonymity*, well known and used in the literature, expresses the interchangeability of the individuals to the eyes of the social order. It supposes the level-comparability of utilities.

Here is the weak weighted utilitarianism theorem.

Theorem 1 *(theorem 17, d'Aspremont-Gevers 2002) If an order R on $\mathbb{R}^n$ satisfies weak Pareto and weak $inv(a_i+x_i)$, there exists $\lambda \in \mathbb{R}^n_+/\left\{0\right\}$ such that $\forall x, y$ in $\mathbb{R}^n$*

$$\sum_1^n \lambda_i x_i > \sum_1^n \lambda_i y_i \Longrightarrow x \succ y$$

Moreover, if we add minimal individual symmetry (resp. anonymity), we must have every component of λ strictly positive (resp. strictly positive and equal).

3 Properties of the working spaces

3.1 Spaces of bounded growth-rate sequences

Let r be a nonnegative real. Utility streams are supposed to take value in the space

$$l_\infty^r = \left\{ x = (x_1, x_2, ...)/x_i \in \mathbb{R} \text{ and} \sup_{i\geq 1} |x_i| \, e^{-i.r} < +\infty \right\}$$

l_∞^r allows for infinite and unbounded utility streams but it requires bounded growth-rates of utility. This condition is justified since it is standard to consider on the one hand that the set of feasible consumption growth-rates is up-bounded, on the other hand that utility is a concave function of consumption.

Equipped with the norm:

$$\|x\| = \sup_{i \in \mathbb{N}^*} |x_i| \, e^{-i.r}$$

l_∞^r is a Banach vector space.

Theorem 2 have recourse to the extension form of the Hahn-Banach theorem asserting the existence of a continuous and linear extension to the whole space, for any continuous linear functional defined on a subspace of a Banach space. I refer to Luenberger (1968) for an expose of the extension form (page 111) and the geometric form (page 133) of the Hahn-Banach theorem. The geometric form asserts the existence of a continuous linear functional supporting a convex subset of a Banach space. It is invoked to prove theorem 5.

The validity of the Hahn-Banach theorem in non separable Banach spaces relies on the axiom of choice (Luenberger 1968, page 111). Since l_∞^r is not separable, theorem 3 and theorem 5 both rely on the axiom of choice. This could raise objections because of the nonconstructiveness of the mathematical objects which existence is proved in theorem 3 and theorem 5. However, it is that complete constructiveness is not needed to draw some interesting and exploitable conclusions.

Denote l_∞^{r*} the set of continuous linear functionals on l_∞^r, i.e. the dual of l_∞^r. For $y \in l_\infty^{r*}$ and $x \in l_\infty^r$, the image of x by y is denoted $y\,(x)$. It is known that the dual of a Banach space is a Banach space, equipped with the norm

$$\|y\| = \sup_x \frac{|y(x)|}{\|x\|}, \quad y \in l_\infty^{r*}$$

Denote

$l_1^r = \left\{ x = (x_1, x_2, ...) / \; x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{+\infty} |x_i| \, e^{i.r} < +\infty \right\},$

$c_r = \{ x = (x_1, x_2, ...) / \; x_i \in \mathbb{R} \text{ and } x_i e^{-i.r} \text{ converges} \}$and

$c_0^r = \{ x = (x_1, x_2, ...) / \; x_i \in \mathbb{R} \text{ and } |x_i| \, e^{-i.r} \text{ converges to } 0 \}.$

Let δ_∞^r be the functional defined on c_r by: $\delta_\infty^r(x) = \lim_{i \longrightarrow +\infty} x_i . e^{-i.r}$.

Spaces corresponding to $r = 0$ are denoted respectively l_∞, l_∞^*, l_1, c, and c_0.

δ_∞^0 is denoted δ_∞.

Denote

$l_{\infty+} = \{ x = (x_1, x_2, ...) / \; x_i \in \mathbb{R} \text{ and } x_i \geq 0 \} .$

$$l_{\infty++} = \{x = (x_1, x_2, ...)/ \ x_i \in \mathbb{R} \text{ and } x_i > 0\}.$$

3.2 Decomposition of l_∞^{r*}

The Yosida-Hewitt theorem (Lauwers 1998, theorem 1) can be stated as follows:

Theorem 2 *(Yosida-Hewitt 1952) Let $y \in l_\infty^*$. Then we can write in a unique manner:*

$$y = y_1 + y_2$$

where y_1 is in l_1 and y_2 is such that its restriction to c is proportional to δ_∞.

Theorem 3 *Let $y \in l_\infty^{r*}$. Then we can write in a unique manner:*

$$y = y_1 + y_2$$

where y_1 verifies:

$$\sum_{i=1}^{+\infty} |y_{1i}| \, e^{i.r} < +\infty$$

and y_2 is such that its restriction to c_r is proportional to δ_∞^r.

Proof. [3] Consider the mapping I_r from l_∞ to l_∞^r, defined by

$$I_r (x_1, x_2, ...x_i, ...) = \left(x_1 e^r, x_2 e^{2r}, ...x_i e^{i.r}, ...\right)$$

I_r is obviously bijective and linear. Thus, it is an isomorphism. Moreover, $\|I_r (x)\| = \|x\|$ for all x in l_∞. Notice that $\|I_r (x)\|$ is evaluated in l_∞^r according to the formula $\|x\| = \sup_{i \in \mathbb{N}^*} |x_i| \, e^{-i.r}$ and $\|x\|$ is evaluated in l_∞ according to the formula $\|x\| = \sup_{i \in \mathbb{N}^*} |x_i|$. As a result, I_r is isometric.

We also have $I_r (l_{\infty++}) = l_{\infty++}$, $I_r (l_{\infty+}) = l_{\infty+}$, $I_r (c) = c_r$, $I_r (c_0) = c_0^r$ and $I_r (l_1) = l_1^r$.

We can associate to each y in l_∞^{r*}, a functional $I_r^* (y)$ as follows

$$\text{for all } x \text{ in } l_\infty, \ I_r^* (y) (x) = y(I_r(x))$$

[3] I owe to an anonymous referee the idea to use an isometric isomorphism in the proofs of theorem 3 and theorem 5. In this manner, these proofs are simpler than they were in the first version of paper.

It is easily checked that $I_r^*(y)$ is linear. I_r being isometric, the continuity of $I_r^*(y)$ results from the continuity of y. Thus, $I_r^*(y)$ is in l_∞^*. In addition, it is easily checked that the mapping I_r^* is linear and bijective from l_∞^{r*} to l_∞^*.

If y in l_∞^{r*} is such that the restriction of $I_r^*(y)$ to c is proportional to δ_∞ then the restriction of y to c_r is proportional to δ_∞^r.

According to theorem 2, for any y in l_∞^{r*}, we can write

$$I_r^*(y) = z_1 + z_2$$

with $z_1$in $l_1$and z_2 such that its restriction to c is proportional to δ_∞. We then have $y = I_r^{*-1}(z_1) + I_r^{*-1}(z_2)$, with $I_r^{*-1}(z_1)$ in l_1^r and the restriction of $I_r^{*-1}(z_2)$ to c_r is proportional to δ_∞^r. ■

3.3 Sensitivity to long-term interest

Corollary 4 *Let f be a function from l_∞^r to $\mathbb{R}$, Frechet-differentiable at $x_0 \in l_\infty^r$. Denote $\delta f(x_0)$ the Frechet-differential of f at x_0. By definition, $\delta f(x_0) \in l_\infty^{r*}$. Let $\delta f_1(x_0)$ and $\delta f_2(x_0)$ be the components of $\delta f(x_0)$ as defined in theorem 3. Denote the restriction of $\delta f_2(x_0)$ to c_r by $\delta f_2(x_0)_{\lfloor c_r}$. Then, there is a real which is denoted $\frac{\partial f}{\partial \infty}(x_0)$ such that*

$$\delta f_2(x_0)_{\lfloor c_r} = \frac{\partial f}{\partial \infty}(x_0)\, \delta_\infty^r$$

Moreover, let $r_n(h)$ be the sequence of c_r obtained by setting to 0 the n first terms of h, then

$$\frac{\partial f}{\partial \infty}(x_0) = \lim_{\|h\| \longrightarrow 0, h \in c_r, \delta_\infty^r(h) \neq 0} \frac{\limsup_n f(x_0 + r_n(h)) - f(x_0)}{\delta_\infty^r(h)}$$

The same formula holds with $\liminf$.

Proof. Existence of $\frac{\partial f}{\partial \infty}(x_0)$ results from theorem 3. Let $h \in c_r$. Since f is a function from l_∞^r to $\mathbb{R}$, Frechet-differentiable at $x_0 \in l_\infty^r$, for all $\varepsilon > 0$ there is $\alpha > 0$ such that:

$$\|h\| < \alpha \Longrightarrow \frac{|f(x_0 + h) - f(x_0) - \delta f(x_0).h|}{\|h\|} < \varepsilon$$

But $\|h\| < \alpha \Longrightarrow \|r_n(h)\| < \alpha$ for all $n \geq 1$, then

$$|f(x_0 + r_n(h)) - f(x_0) - \delta f(x_0).r_n(h)| < \varepsilon \|r_n(h)\|$$

Thus

$$\left| f(x_0 + r_n(h)) - f(x_0) - \sum_{i=n+1}^{+\infty} \frac{\partial f}{\partial x_i}(x_0).h_i - \frac{\partial f}{\partial \infty}(x_0).\delta_\infty^r(h) \right| < \varepsilon \|r_n(h)\|$$

Moreover, $\|r_n(h)\| = \sup_{i>n} |h_i| e^{-ri}$. It is a positive and decreasing sequence converging to $|\delta_\infty^r(h)|$. We have also $\sum_{i=n+1}^{+\infty} \frac{\partial f}{\partial x_i}(x_0).h_i \longrightarrow 0$ when $n \longrightarrow +\infty$. Then

$$\left| \limsup_n f(x_0 + r_n(h)) - f(x_0) - \frac{\partial f}{\partial \infty}(x_0).\delta_\infty^r(h) \right| \leq \varepsilon |\delta_\infty^r(h)|$$

which gives

$$\left| \frac{\limsup_n f(x_0 + r_n(h)) - f(x_0)}{\delta_\infty^r(h)} - \frac{\partial f}{\partial \infty}(x_0) \right| \leq \varepsilon$$

This proves that

$$\frac{\partial f}{\partial \infty}(x_0) = \lim_{\|h\| \longrightarrow 0, h \in c_r, \delta_\infty^r(h) \neq 0} \frac{\limsup_n f(x_0 + r_n(h)) - f(x_0)}{\delta_\infty^r(h)}$$

The same proof applies for $\liminf$. ■

$\frac{\partial f}{\partial x_i}(x_0)$ measures the sensitivity of f to changes in x_i whereas $\frac{\partial f}{\partial \infty}(x_0)$ measures the sensitivity of f to changes in x_n when n tends to infinity. If f represents an intertemporal criterion, the sequence $\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), ...$ represents the criterion's sensitivity to short-term interest and $\frac{\partial f}{\partial \infty}(x_0)$ represents the criterion's sensitivity to long-term interest.

4 Generalization of the weak weighted utilitarianism theorem

The proof of the weak weighted utilitarianism theorem is based on the geometric version of Hahn-Banach theorem. As said in section 3, the Hahn-Banach theorem holds in l_∞^r. This allows to generalize theorem 1. This generalization is theorem 5. By clarifying the structure of l_∞^{r*}, theorem 3 will then make it possible to exploit theorem 5, as in corollary 6 and corollary 7.

In this section and the next one, the axioms *weak Pareto*, *weak* $inv(a_i + x_i)$ and *minimal individual symmetry* are the same than the correspondent axioms in the finite case, except that the space of alternatives is l_∞^r instead of $\mathbb{R}^n$. The axiom *anonymity* has several versions in the infinite case. Each version corresponds to a requirement of invariance of the ranking with respect to a given set of permutations. The bigger the set of permutation, the higher the level of anonymity. For example:

finite anonymity corresponds to invariance with respect to finite permutations.

fixed step anonymity corresponds to invariance with respect to fixed step permutations. A permutation σ on the set of positive integers $\mathbb{N}^*$ is said to be *fixed step* iff there exists a partition of $\mathbb{N}^*$: N_1, N_2...such that $\forall i, j, |N_i| = |N_j|$ and σ can be written as the composition of permutations $\sigma_1 \circ \sigma_2 \circ$...where for all i and j such that $j \neq i$, σ_i leaves invariant all the elements of N_j.

Since finite permutations constitute a subset of the set of fixed step permutations, *fixed step anonymity* is stronger than *finite anonymity.* I refer to Fleurbaey-Michel 2003 for the definitions of the different versions of anonymity.

An other axiom is needed:

super weak Pareto: $\forall x, y$ in l_∞^r, $x \succ y$ if $\inf(x_i - y_i)e^{-ri} > 0$.

Theorem 5 *If an order R on l_∞^r satisfies super weak Pareto and weak $inv(a_i+x_i)$, there exists a non-null, continuous and positive (in the sense that if $x_i \geq 0$ for all i then $\varphi(x) \geq 0$) linear functional φ on l_∞^r such that, for all x, y in l_∞^r :$\varphi(x) > \varphi(y) \Longrightarrow x \succ y$.*

Proof. With the help of some adaptations, the proof is the same one as that of theorem 1. This proof is exposed in detail in d'Aspremont-Gevers (2002, page 57). I repeat the stages where adaptations are necessary, in particular when it is referred to l_∞^r or to the interior of its positive cone, or to properties related to its norm.

Denote $\overset{\circ}{l}{}_{\infty++}^r$ the interior of $l_{\infty++}$ and $\overset{\circ}{l}{}_{\infty+}^r$ the interior of $l_{\infty+}$ in l_∞^r (i.e. with respect to the norm $\|x\| = \sup_{i\in\mathbb{N}^*} |x_i| e^{-i.r}$). Consider the isomorphism I_r (defined in the proof of theorem 3). The images of $l_{\infty++}$ and $l_{\infty+}$ by I_r are respectively $l_{\infty++}$ and $l_{\infty+}$. Moreover, I_r being isometric, $\overset{\circ}{l}{}_{\infty++}^r$ is the image of the interior of $l_{\infty++}$ by I_r and $\overset{\circ}{l}{}_{\infty+}^r$ is the image of the interior of $l_{\infty+}$ by I_r.

It is known that the interiors of each of the sets $l_{\infty++}$ and $l_{\infty+}$ in l_∞ are the same set $\{x \in l_\infty / \inf x_i > 0\}$. As a result, $l_{\infty++}$ and $l_{\infty+}$ have also the same interior in l_∞^r: the set $\{x \in l_\infty^r / \inf x_i e^{-i.r} > 0\}$. Thus

$$\overset{\circ}{l^r_{\infty+}} = \overset{\circ}{l^r_{\infty++}} = \left\{x \in l_\infty^r / \inf x_i e^{-i.r} > 0\right\} \tag{1}$$

In comparison with the proof of d'Aspremont-Gevers (2002), it is necessary to replace the positive cone of $\mathbb{R}^n$, $P = \{p \in \mathbb{R}^n / p_i > 0 \text{ for all } i\}$ by $\overset{\circ}{l^r_{\infty++}}$. The sets S and Q are the same as in d'Aspremont-Gevers (2002): $S = \{s \in l_\infty^r / s \succsim 0\}$ and $Q = \left\{q \in l_\infty^r / q = s + p, s \in S \text{ and } p \in \overset{\circ}{l^r_{\infty++}}\right\}$. We can write $Q = \cup_{s\in S}\left(s + \overset{\circ}{l^r_{\infty++}}\right)$. Thus, being an union of open subsets, Q is open.

Suppose we can show that Q is convex. Thanks to the Hahn-Banach theorem, there exist a non-null and continuous linear functional, say φ, supporting Q.This writes $\forall q \in Q, \varphi(q) > 0$. Let x, y in l_∞^r be such that $y \succsim x$. For all p in $\overset{\circ}{l^r_{\infty++}}$, according to (1), we have $\liminf p_i e^{-ir} > 0$. Thus, for any real θ in $]0, 1[$, $\inf \theta p_i e^{-ir} > 0$. Thanks to *super weak Pareto*, we have $y + \theta p \succ x$. Then, *weak inv*$(a_i + x_i)$ yields $y - x + \theta p \succsim 0$. Thus $y - x + \theta p + (1-\theta)p$ is in Q. Thus $\varphi(y - x + p) > 0$ for all p in $\overset{\circ}{l^r_{\infty++}}$. Since 0 is clearly in the adherence of $\overset{\circ}{l^r_{\infty++}}$ and φ is continuous, we have $\varphi(y - x) \geq 0$. We have shown that $y \succsim x$ implies $\varphi(y - x) \geq 0$. We deduce that $\varphi(x) > \varphi(y) \Longrightarrow x \succ y$. Moreover, let x be in $l_{\infty+}$, that is, $x_i \geq 0$ for all i. We now prove that $\varphi(x) \geq 0$, what establishes the positivity of φ. Let α be a positive real and p be in $\overset{\circ}{l^r_{\infty++}}$. According to (1), $\inf p_i e^{-ir} > 0$. Thus, since $x_i \geq 0$ and $\alpha > 0$, $\inf(x_i + \alpha p_i)e^{-ir} > 0$. Denote $y(\alpha) = x + \alpha p$. By *super weak Pareto,* it results that $y(\alpha) \succ 0$. Thus, since for all x, y in l_∞^r we have $\varphi(x) > \varphi(y) \Longrightarrow x \succ y$, we deduce $\varphi(y(\alpha)) \geq \varphi(0) = 0$. We can check that $\lim_{\alpha \longrightarrow 0} y(\alpha) = x$. By continuity of φ, we deduce that $\varphi(x) \geq 0$.

It remains now to show that Q is convex.

Let s, s' be in S. For all p in $\overset{\circ}{l^r_{\infty++}}$, $s \succsim 0$ implies, by *super weak Pareto,* $s + p \succ 0$. By *weak inv*$(a_i + x_i)$, this implies $s + p + s' \succsim s'$. So, by transitivity, since $s' \succsim 0$,we have $s + p + s' \succsim 0$. In other words, $s + p + s' \in S$. This implies that, for all p' in $\overset{\circ}{l^r_{\infty++}}$, $s + p + s' + p'$ is in Q. Thus Q is closed under addition. To show the convexity of Q, it is enough to show that $\mu q \in Q$ whenever $q \in Q$ and μ is a positive real.

d'Aspremont-Gevers (2002) show that for any $s \in S$ and $p \in P$, for all positive integers k, m, and for any real θ in $]0, 1[$, we have $\left(\frac{k}{m}\right)(s + \theta p) \in S$. This holds in the present setting (when P is replaced with $\overset{\circ}{l^r_{\infty++}}$) and the proof is literally the same. It is then omitted. Now let $q = s + p$ be in Q (with $s \in S$ and $p \in \overset{\circ}{l^r_{\infty++}}$), and let μ be a positive real. Let (m_n) and (k_n) be two sequences of positive integers such that $\lim \frac{k_n}{m_n} = \mu$. Denote $p'_n = \left(\mu - \frac{k_n}{m_n}\right)(s + \theta p) + \mu(1-\theta)p$. The sequence $\mu(1-\theta)p$ is obviously in $\overset{\circ}{l^r_{\infty++}}$. Moreover, we can also check that $\lim p'_n = \mu(1-\theta)p$. As a result, $\overset{\circ}{l^r_{\infty++}}$ being open, there exist a positive integer N such that $p'_N \in \overset{\circ}{l^r_{\infty++}}$. We have $\mu q = \mu(s+p) = \left(\frac{k_N}{m_N}\right)(s+\theta p) + p'_N$. Since $\left(\frac{k_N}{m_N}\right)(s+\theta p)$ is in S and p'_N in $\overset{\circ}{l^r_{\infty++}}$, it results that μq is in Q. ■

Let $\varphi = \varphi_1 + \varphi_2$ be the decomposition of φ given by theorem 3. In the context of intergenerational choice (or intertemporal choice with infinite horizon), the component φ_1 corresponds to *discounted utilitarianism.* According to the definition of l^r_1, the coefficients of φ_1 , denoted φ_{1n}, tend exponentially towards 0 at infinity. Consequently, φ_1 is only sensitive to short-term interest. Concerning the component φ_2, for all x in l^r_∞, $\varphi_2(x)$ depends only on limits of sequences obtained from subsequences of x. Consequently, φ_2 is only sensitive to long-term interest (the coefficient $\frac{\partial\varphi}{\partial\infty}$,measuring the sensitivity of φ to changes in long-term well-being, depends only on φ_2). We may say that φ_1 is the short-term component and φ_2 the long-term component of the order. As φ_2 is linear, I suggest to name functionals like φ_2 (i.e. which restriction to c_r is proportional to δ^r_∞) "linear limits".

Definition A linear limit on l^r_∞ is a functional on l^r_∞ which restriction to c_r is proportional to δ^r_∞.

Lauwers (1998) gives examples of linear limits on l_∞ : medial limits and integrals against measures based on free ultrafilters. If the conditions were added that φ_1 and φ_2 are both non-null, the form $\varphi_1 + \varphi_2$ corresponds to what Chilchinisky (1996) called *sustainable preference.* This form respects at the same time short-term and long-term interests. Chilchinisky (1996) axiomatized that by introducing two axioms: *non dictatorship of the present* and *non dictatorship of the future.* If the condition of *stationarity* is imposed, Lauwers (1998) showed (lemma 2) that one of the two components φ_1 or φ_2 must be null. For the definitions of *non dictatorship of the future, non dictatorship of the present* and *stationarity,* I refer respectively to Chilchinisky (1996) and Lauwers (1998). Moreover, Lauwers (1998) showed that φ_2 may guarantee a level of anonymity higher than *finite anonymity.* Fleurbaey-Michel (2003) noticed that this level of anonymity, which may be referred to as *Lauwers anonymity,* is higher than *fixed step anonymity.* But the incompatibility of *Lauwers anonymity* with *weak Pareto* makes that they regard it as too high, opinion which seems to be followed in the literature. Likewise, most authors reject linear limits as social welfare functions because they fail to check *weak Pareto* which is seen as a minimal sensitivity axiom. To clarify more the boundary of the clash between anonymity and the Pareto axioms, Mitra-Basu (2007) characterize the class of permutations for which utility streams can be pronounced to be indifferent without conflicting with the strong Pareto axiom. The set of fixed-step permutations is included in that class.

5 Application

5.1 Equitable utilitarianism

Axiom *weak Pareto* obviously entails *super weak Pareto*. As axioms *weak Pareto* and *weak* $inv(a_i + x_i)$ are often used, theorem 5 should be useful in fields such as the study of links between axioms, or the axiomatization of social welfare relations for infinite and unbounded utility streams. For example, the following corollary shows that, to some extent, linear limits must nevertheless be satisfied in a certain way if one wishes to satisfy *super weak Pareto*, *weak* $inv(a_i + x_i)$ and *finite anonymity*. These three axioms may be considered as minimal axioms for equitable intergenerational utilitarianism. Linear limits are an example of orders satisfying *super weak Pareto, weak* $inv(a_i + x_i)$ and *finite anonymity*.

In growth theory, models generally suppose a positive growth rate, i.e. $r > 0$. But in the literature dealing with the evaluation of infinite utility streams, the case $r = 0$ is more usual. In the present section, I set $r = 0$. The following analysis can be easily extended to the case $r > 0$.

Corollary 6 *Let R be an order on l_∞ satisfying super weak Pareto, weak $inv(a_i + x_i)$ and minimal individual symmetry (resp. finite anonymity). Let φ be the linear functional given by theorem 5 and $\varphi = \varphi_1 + \varphi_2$ the decomposition of φ given by theorem 3. We must either have every component of φ_1 positive or $\varphi_1 = 0$ (resp. $\varphi_1 = 0$).*

Proof. From *minimal individual symmetry*, it is clear that if a component of φ_1 is positive, every other component of φ_1 must also be positive. Suppose now that R satisfies *finite anonymity*. Let e_n be the sequence of l_∞ such that $e_{ni} = 0$ if $i \neq n$ and $e_{nn} = 1$. We have $\varphi_2(e_n) = 0$. Thus, $\varphi(e_n) = \varphi_1(e_n) = \varphi_{1n}$. Suppose there is n, m such that $\varphi_{1n} > \varphi_{1m}$. Then we would have $\varphi(e_n) > \varphi(e_m)$, what would imply $e_n \succ e_m$. This contradicts that R is finite anonymous since e_m can be obtained from e_n by a finite permutation.

As a result, we have $\varphi_{1n} = \varphi_{1m}$ for all $n, m \geq 1$. Now make m tend to infinity. Then φ_{1m} tends to 0 because the sum $\sum_{i=1}^{+\infty} |\varphi_{1i}|$ converges. Consequently, $\varphi_{1i} = 0$ for all $i \geq 1$. So $\varphi_1 = 0$ and $\varphi = \varphi_2$. ■

Remark 1 *A linear limit is an example of an order satisfying super weak Pareto, weak inv*$(a_i + x_i)$ *and finite anonymity. Since linear limits are generally rejected as social welfare functions because they fail to check weak Pareto, one may wonder if there exists an order satisfying weak Pareto, weak inv*$(a_i + x_i)$ *and finite anonymity. I established the existence of such an order in a paper entitled: "On the extension of a preorder under translation invariance" (available at: http://ideas.repec.org/p/pra/mprapa/15407.html). However, the linear functional in corollary 6 does not inherit the property weak Pareto as in the finite-dimension case (theorem 1). This undoubtedly makes the ranking given by the linear functional less meaningful in this situation than in the finite-dimension one*[4] *.*

A consequence of corollary 6 is that every maximal point in a subset s of l_∞ with respect to an order R on l_∞ satisfying *super weak Pareto, weak inv*$(a_i + x_i)$ and *finite anonymity*, is also a maximal point in s with respect to some positive linear limit (the positivity of φ_2 results from the positivity of φ). It is in that sense that I said that linear limits must nevertheless be satisfied, despite their insensivity. Since in the context of intergenerational choice φ_2 determines the optimal long-term behavior, we can express this by saying that R must comply entirely with long-term optimality.

5.2 The intransigence of the future

Consider now an order R on l_∞ satisfying *super weak Pareto* and *weak inv*$(a_i + x_i)$. Let $\varphi_1 + \varphi_2$ be the decomposition of R given by theorem 3 and theorem 5. I show that if R only checks the weaker assumption $\varphi_2 \neq 0$ instead of *finite anonymity*, the consequence of corollary 6, pointed out above, nevertheless holds.

Let s be the set (included in l_∞) of feasible utility streams starting from some initial conditions. It is not unrealistic to suppose that s satisfies the two conditions:

[4] In the first version of this paper, I used *weak Pareto* in the statement of theorem 5. I owe to an anonymous referee the introduction of *super weak Pareto* and the observation that theorem 5 holds with *super weak Pareto*.

Condition A: For any x, y in s and any date n,there is an integer $m \geq n+1$ and a vector $(z_{n+1,...,}z_m)$ such that the stream

$$x_n z_m y = (x_1, ..., x_n, z_{n+1,...,}z_m, y_{m+1}, y_{m+2}, ...)$$

is in s.

Condition B: For any x in s, if y in l_∞ is such that $x_i \geq y_i$ for all i in $\mathbb{N}^*$, then y is in s.

Condition A says that it is always possible to jump from any stream x to any stream y, if necessary with the help of some transitional period of sacrifice: $z_{n+1,...,}z_m$.

Condition B says that it is always feasible to throw away utility.

Corollary 7 *Suppose that the set s of feasible utility streams satisfies conditions A and B. Let R be an order on l_∞ satisfying super weak Pareto, weak $inv(a_i + x_i)$ and such that its long-term component φ_2 is non null. Then every maximal point in s with respect to R is also a maximal point in s with respect to φ_2.*

Proof. Suppose that x in s is a maximal point for R. Suppose there exists y in s such that

$$\varphi_2(x) < \varphi_2(y)$$

Let n be a positive integer and $(z_{n+1,...,}z_m)$ the sequence given by condition A. Denote $x_n y$ the following stream:

$$(x_n y)_i = x_i \text{ for } i \text{ in } \{1, ..., n\}$$
$$(x_n y)_i = \inf(x_i, z_i) \text{ for } i \text{ in } \{n+1, ..., m\}$$
$$(x_n y)_i = y_i \text{ for } i \geq m+1$$

where $(x_n y)_i$ is the i^{th} component of $x_n y$.

Condition B imply $(x_n y) \in s$. Moreover, $\|x_n y\| \leq \sup(\|x\|, \|y\|)$ for all n. Denote φ_{1i} the i^{th} component of φ_1. We have

$$|\varphi_1(x_n y) - \varphi_1(x)| = \left|\sum_1^\infty ((x_n y)_i - x_i)\varphi_{1i}\right| \leq \sup(\|x\|, \|y\|) \sum_{m+1}^\infty |\varphi_{1i}|$$

Since $m > n$ and $\sum_1^\infty |\varphi_{1i}| < \infty$, we have $\lim_{n\to\infty} \sum_{m+1}^\infty |\varphi_{1i}| = 0$. Therefore

$$\lim_{n\to\infty} \varphi_1(x_n y) = \varphi_1(x)$$

For all n in $\mathbb{N}^*$, $\varphi_2(x_n y) = \varphi_2(y)$ and $(\varphi_1 + \varphi_2)(x_n y) = \varphi_1(x_n y) + \varphi_2(y)$. We then have $\lim_{n\to\infty}(\varphi_1 + \varphi_2)(x_n y) = \varphi_1(x) + \varphi_2(y) > (\varphi_1 + \varphi_2)(x)$. Thus, there would exist N in $\mathbb{N}^*$ such that for all $n \geq N$, $(\varphi_1 + \varphi_2)(x_n y) > (\varphi_1 + \varphi_2)(x)$. Therefore, x would not be maximal in s for $\varphi_1+\varphi_2$, what implies that x would not be maximal in s for R. A contradiction. ■

Corollary 7 shows that, under *super weak Pareto* and *weak* $inv(a_i + x_i)$, as soon as φ_2 is non null, it "imposes its views" in the sense that optimality according to R entails optimality according to φ_2. It is remarkable that R need not be equitable to be "under the orders" of φ_2.

In the context of intergenerational choice, if $\varphi_2 = 0$ and $\varphi_1 \neq 0$, R is present-oriented and if $\varphi_1 = 0$ and $\varphi_2 \neq 0$, R is future-oriented. If $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$, we may say that we grant to the present and the future comparable considerations. Hence, it is possible to restate corollary 7 as follows: under *super weak Pareto* and *weak* $inv(a_i + x_i)$, if an intergenerational order R grants to the present and the future comparable considerations, it must comply entirely with long-term optimality. In other words, showing some fairness between the present and the future results in satisfying the future fully. One could call this property: *the intransigence of the future.*

Notice that the assumption $\varphi_2 \neq 0$ is not formally needed in the proof of corollary 7. However, if φ_2 were null, long-term optimality would not correspond to optimality according to φ_2.

This consequence of corollary 7 might suggest that the future has too much power. But on the other hand future is majority and giving power to majority is generally seen as desirable. Moreover, complying with long-term optimality is compatible with Chichilnisky axiom *non dictatorship of the future.* It does not entail insensivity toward the present.

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