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Abstract

The Boruvka's algorithm, which computes the minimum cost spanning tree, is used to define a rule to share the cost among the nodes (agents). We show that this rule coincides with the folk solution, a very well-known rule of this literature.

Keywords: minimum cost spanning tree, Boruvka's algorithm, folk solution.

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OR/MS classification words: Networks/graphs (Tree algorithms); Games/group decisions (cooperative)

1 Introduction

In this paper we study minimum cost spanning tree problems (*mcstp*). Consider that a group of agents, located at different geographical places, wants some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source.

There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence they may share the cost of the distribution network. This example appears in Dutta and Kar (2004) [14]. Bergantiños and Lorenzo (2004, 2005, 2008) [1] [2] [3] study a real situation where villagers should pay the cost of constructing pipes from their respective houses to a water supplier. Other examples are communication networks, such as telephone, Internet, wireless telecommunication, or cable television.

The literature on *mcstp* starts by defining algorithms for constructing minimum cost spanning trees (*mt*). The first algorithm for finding an *mt* was developed in Boruvka (1926) [12]. Its purpose was an efficient electrical coverage of Bohemia. There are now two algorithms commonly used, Kruskal's algorithm developed in Kruskal (1956) [18] and Prim's algorithm developed in Prim (1957) [20]. All three are greedy algorithms that run in polynomial time. But constructing an *mt* is only a part of the problem. Another important issue is how to allocate the cost associated with the *mt* among the agents. Several authors have introduced rules in *mcstp* through some algorithms for constructing *mt*. The idea is to propose rules to divide the cost among the agents in a fair way[1].

Bird (1976) [9], Dutta and Kar (2004) [14], and Bergantiños and Vidal-Puga (2007a) [6] introduce three rules based on Prim's algorithm. Feltkamp *et al* (1994a) [16] introduce a rule based on Kruskal's algorithm. The rules introduced by Bergantiños and Vidal-Puga (2007a) [6] and Feltkamp *et al* (1994a) [16] coincide. A proof of this statement can be found, for instance, in Bergantiños and Lorenzo-Freire (2008) [5]. We call this rule the folk solution, which can be obtained in other ways. Let us mention some of them.

A simple *mcstp* is an *mcstp* where the cost of each arc is either 0 or 1. Norde *et al* (2004) [19] prove that each *mcstp* can be obtained as a linear combination of simple *mcstp* where all the coefficients are non-negative. Thus, we can generate a solution from the set of simple *mcstp* to the set of all *mcstp* by using the linear combination. Branzei *et al* (2004) [13] and Bergantiños and Vidal-Puga (2009) [8] prove that the folk solution can be obtained in this way. Bogomolnaia and Moulin (2008) [11] also apply this approach to *mcstp* for generating several solutions.

Another way of obtaining rules in *mcstp* is through cooperative games with transferable utility (*TU* games). Given an *mcstp* we associate a *TU* game. Later, we compute a cooperative solution in the *TU* game. The solution to the initial *mcstp* is the solution of the *TU* game. Bergantiños and Vidal-Puga (2007a, 2007b) [6] [7] prove that the folk solution can be obtained in this way by applying the Shapley value to several *TU* games.

Recently, Bogomolnaia *et al* (2008) [10] and Dutta and Mishra (2008) [15] apply the ideas behind the folk solution to some related problems.

Nevertheless, as far as we know, no rule has been introduced through Boruvka's

[1]In this paper we refer ro fairness as general principle to achieve, and not as a well-defined mathematical object.

algorithm. We do it. The idea behind this algorithm is the following. Initially the network is empty and each agent is a single component. We sequentially add to the network, for each connected component, the cheapest arc joining this connected component with some agent outside it and without introducing cycles. We divide the cost of any arc selected by Boruvka's algorithm following three principles. First, each agent is assigned to the arc selected by the component he belongs to. Each agent pays, partially, the cost of the assigned arc. Second, all agents pay the same proportion of the arc assigned. Namely, each agent i pays $pc_{a(i)}$ where $c_{a(i)}$ is the cost of the arc $a(i)$ assigned to agent i. Third, the proportion paid, p, should be as large as possible.

We prove that the rule we introduce coincides with the folk solution. Our result gives more support to the folk solution as it can be obtained in several ways.

In Prim the cost of any arc is paid only by one agent. Fairness is recovered by taking the average over the set of allocations induced by the possible orders of the agents. In Kruskal and Boruvka the cost of any arc is divided between several agents. Fairness is obtained by dividing the cost of any arc in an equitable way.

Let us compare the definitions of the folk solution through Kruskal and Boruvka more carefully. In Kruskal's algorithm the mt is constructed by sequentially adding arcs with the lowest cost and without introducing cycles. Assume that we add arc (i, j), which links connected components S_i and S_j. We divide the cost of arc (i, j) among the agents according to the following principles. First, agents in a component S_i already connected to the source ($0 \in S_i$) pay nothing. Second, only agents who benefit directly when adding an arc, $S_i \cup S_j$, could pay something. Third, all agents in the same connected component pay the same. Fourth, the total amount paid by a group is proportional to the new agents to whom this group is connected (agents in S_i pay proportionally to $|S_j|$).

An important difference is that the order in which we add the arcs could be different. Moreover, in Kruskal at each step we add an arc, which is paid completely. In Boruvka, at each step we can add several arcs. At least one of them is paid completely but others can be paid only partially. Above we have mentioned the four principles for dividing the cost of an arc (i, j) following Kruskal. Principles one and three have also been applied with Boruvka. Principle two is similar in the sense that agents outside $S_i \cup S_j$ pay nothing. Whereas in Kruskal all agents in $S_i \cup S_j$ when $0 \notin S_i \cup S_j$ pay something, in Boruvka it is possible that agents in S_i or S_j pay nothing. Principle four is different. In Boruvka all agents in $S_i \cup S_j$ pay the same.

The paper is organized as follows. In Section 2 we define *mcstp*. In Section 3 we present our results. The proof of the main result is in Appendix.

2 The minimum cost spanning tree problem

In this section we introduce minimum cost spanning tree problems and revise some results of the literature that are relevant for this paper.

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given $N \subset \mathcal{N}$ finite, $|N|$ denotes the number of elements in N.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., |N|\}$.

A *cost matrix* $C = (c_{ij})_{i,j\in N_0}$ over N represents the cost of a direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we will work with undirected arcs, *i.e* $(i, j) = (j, i)$.

We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$, we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$. We denote the set of all cost matrices over N with all the costs different as $\mathcal{D}^N$, *i.e.* $C \in \mathcal{D}^N$ if $c_{ii'} \neq c_{jj'}$ when $(i, i') \neq (j, j')$.

A *minimum cost spanning tree problem*, briefly *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given an *mcstp* (N_0, C), we denote the *mcstp* induced by C in $S \subset N$ as (S_0, C).

A *graph* g over N_0 is a subset of $\{(i, j) : i, j \in N_0, i \neq j\}$. The elements of g are called *arcs*. Given $S \subset N_0$ we denote by g_S the restriction of g to the elements of S, *i.e.* $g_S = \{(i, j) \in g : i, j \in S\}$.

Given a graph g and a pair of nodes i and j, a *path* from i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$ and $j = i_l$.

A *tree* over N is a graph t satisfying that for all $i, j \in N_0$ there exists a unique path from i to j in g. Usually we write $t = \{(i^0, i)\}_{i\in N}$ where i^0 represents the first agent in the unique path in t from i to 0. We denote the set of trees over N as $\mathcal{T}_0^N$.

Given an *mcstp* (N_0, C) and a graph g, we define the *cost* associated with g as $c(N_0, C, g) = \sum\limits_{(i,j)\in g} c_{ij}$. When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

Any graph g over N_0 induces a partition of N_0 as follows: We say that $S \subset N_0$ is a *connected component* induced by g if two conditions hold. First, for any $i, j \in S$, there exists a path in g connecting nodes i and j. Second, for each $i \in S$ and $j \in N_0 \backslash S$, there exist no path in g connecting nodes i and j. The set of connected components is a partition of N_0, which we denote as $\mathcal{P}(N_0, g)$. Clearly, if t is a tree $\mathcal{P}(N_0, t) = \{N_0\}$.

A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree $t \in \mathcal{T}_0^N$ such that $c(t) = \min_{g\in\mathcal{T}_0^N} c(g)$. It is well-known in the literature on *mcstp* that there exists an *mt*, even though it does not need to be unique. Given an *mcstp* (N_0, C) we denote by $m(N_0, C)$ the cost associated with any *mt* t in (N_0, C).

Probably, the most famous algorithms for computing the *mt* associated with an *mcstp* are the ones introduced in Boruvka (1926) [12], Kruskal (1956) [18], and Prim (1957) [20].

A (*cost allocation*) *rule* is a function f such that for each *mcstp* (N_0, C), we have $f(N_0, C) \in \mathbb{R}^N$ and $\sum_{i \in N} f_i(N_0, C) = m(N_0, C)$. As usual, $f_i(N_0, C)$ represents the cost assigned to agent i.

Feltkamp *et al* (1994a) [16] introduce a rule based on Kruskal's algorithm. Bergantiños and Vidal-Puga (2007a) [6] introduce a rule based on Prim's algorithm. Bergantiños and Lorenzo-Freire (2008) [5] prove that both rules coincide. We call this rule the folk solution and we denote it by φ.

We briefly discuss the definition of the folk solution through the algorithms of Prim and Kruskal.

- Prim's algorithm. Idea: starting from the source we construct an *mt* by sequentially connecting agents with the lowest cost and without introducing cycles.

  Bird (1976) [9] defines a rule when the *mcstp* has a unique *mt*: each agent pays his connection cost.

  When several *mt* exist, Dutta and Kar (2004) [14] connect the agent with the lowest index according to a predetermined order μ of the set of agents. The allocation induced by each order μ could be unfair. Fairness is recovered by computing the average over the set of all possible orders μ.

  Bergantiños and Vidal-Puga (2007a) [6] prove that the folk solution can be obtained by applying the previous procedure based on Prim's algorithm to the irreducible problem (N_0, C^*). C^* is obtained from C by reducing the cost of any arc as much as possible without changing the total cost of connecting all agents to the source.

- Kruskal's algorithm. Idea: the *mt* is constructed by sequentially adding arcs with the lowest cost without introducing cycles.

  Feltkamp *et al* (1994a) [16] define the folk solution through Kruskal's algorithm. We now define it using the formulation given in Bergantiños *et al* (2008) [4].

  Assume that we add arc (i, j), which links connected components S_i and S_j. We divide the cost of arc (i, j) among agents in N according with the following principles:

1. Agents already connected to the source pay nothing (if $0 \in S_i$, each agent in S_i pays nothing).
2. Only agents who benefit directly when adding an arc could pay something (agents in $N \backslash (S_i \cup S_j)$ pay nothing).
3. All agents in the same connected component pay the same.
4. The total amount paid by a group is proportional to the new agents to whom this group is connected (agents in S_i pay proportionally to $|S_j|$).

Then, agent i pays

$$\begin{cases} \frac{|S_j|}{|S_i \cup S_j||S_i|} & \text{if } 0 \notin S_i \cup S_j \\ \frac{1}{|S_i|} & \text{if } 0 \in S_j \\ 0 & \text{if } 0 \in S_i. \end{cases}$$

3 A rule based on Boruvka's algorithm

Boruvka (1926) [12] provides an algorithm for computing an mt. We provide a way of sharing the cost of any arc selected by Boruvka's algorithm. We first describe Boruvka's algorithm in a formal way.

Let π be an order over the set of all possible arcs. Namely

$$\pi : \{(i,j) : i,j \in N_0, i \neq j\} \to \left\{1, 2, ..., \begin{pmatrix} |N| \\ 2 \end{pmatrix}\right\}.$$

Remember that we are taking $(i,j) = (j,i)$.

Boruvka's algorithm (associated with the order π).

Step 1: Let $g^{\pi,0} = \emptyset$. Notice that $\mathcal{P}(N_0, g^{\pi,0}) = \{\{0\}, \{1\}, ..., \{|N|\}\}$.

Assume we have reached Step s $(s = 1, 2, ...)$ and we have defined $g^{\pi,s-1}$.

Step s: For each $T \in \mathcal{P}(N_0, g^{\pi,s-1})$, $0 \notin T$, let $\left(i^{\pi,T}, j^{\pi,T}\right) \in T \times (N_0 \backslash T)$ be such that $c_{i^{\pi,T} j^{\pi,T}} = \min\{c_{ij} : i \in T, j \in N_0 \backslash T\}$. In case there is more than one possible arc, we select the one with the lowest position in the order π.

$$g^{\pi,s} = g^{\pi,s-1} \cup \left\{\left(i^{\pi,T}, j^{\pi,T}\right) : T \in \mathcal{P}\left(N_0, g^{\pi,s-1}\right)\right\}.$$

It is well known that if for each $T \in \mathcal{P}(N_0, g^{\pi,s-1})$ the arc $\left(i^{\pi,T}, j^{\pi,T}\right)$ is selected through π, then $g^{\pi,s}$ is a graph with no cycles.

If $\mathcal{P}(N_0, g^s) = \{N_0\}$, then $g^{\pi,s}$ is a tree and the process is over. If $\mathcal{P}(N_0, g^{\pi,s}) \neq \{N_0\}$, then we go to Step $s+1$.

The process finishes in a finite number of steps. The tree obtained by this procedure is denoted by t^π. It is well known that for each order π, t^π is an mt. Moreover, given an mt t, there exists an order π such that t^π coincides with t. It is possible that $t^\pi = t^{\pi'}$, even if π and π' are different orders. For instance, if all the costs are different, $t^\pi = t^{\pi'}$ for all π and π'.

When no confusion arises we write g^s, i^T, ... instead of $g^{\pi,s}$, $i^{\pi,T}$, ... respectively.

Remark 1. We have presented Boruvka's algorithm in a different way. Usually, the condition $0 \notin T$ does not appear. We have added it in order to adapt the algorithm to our objective: to divide the cost of the mt among the agents. If $0 \in T$, then agents in T do not need to be connected to more agents.

Let us apply Boruvka's algorithm to the following examples.

Example 1. $N = \{1, 2\}$ and C is given by $c_{01} = 10$, $c_{02} = 100$, and $c_{12} = 2$. This situation can be represented by Figure 1.

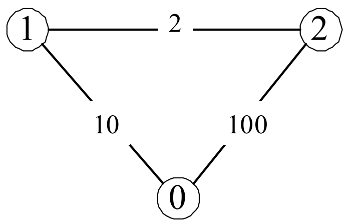

Figure 1: A two-node case.

Since all the costs are different, it is not necessary to specify the order π.

1. Step 1. $\left(i^{\{1\}}, j^{\{1\}}\right) = (1,2)$, and $\left(i^{\{2\}}, j^{\{2\}}\right) = (2,1)$. Then, $g^1 = \{(1,2)\}$.
2. Step 2. $\mathcal{P}(N_0, g^1) = \{\{0\}, N\}$ and $\left(i^N, j^N\right) = (1,0)$.

   Now $g^2 = \{(0,1), (1,2)\}$ is the mt.

Example 2. Let (N_0, C) be the $mcstp$ given by Figure 2:
Since all the costs are different, it is not necessary to specify the order π.

1. Step 1. $\left(i^{\{1\}}, j^{\{1\}}\right) = (1,2)$, $\left(i^{\{2\}}, j^{\{2\}}\right) = (2,1)$, and $\left(i^{\{3\}}, j^{\{3\}}\right) = (3,1)$

   Then, $g^1 = \{(1,2), (1,3)\}$.

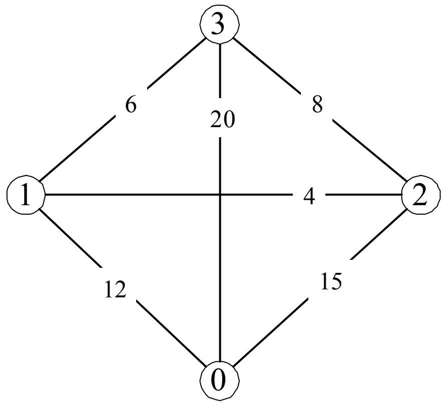

Figure 2: A three-node case with all costs different.

2. Step 2. $\mathcal{P}(N_0, g^1) = \{\{0\}, N\}$ and $(i^N, j^N) = (1, 0)$.

   Now $g^2 = \{(0,1), (1,2), (1,3)\}$ is the *mt*.

We now introduce a rule in *mcstp* based on Boruvka's algorithm. Our first idea is the following. At each step, each connected component select an arc. The cost of each selected arc is divided equally among all agents belonging to the components selecting this arc. Let us clarify it in Example 1.

Example 1.

1. Step 1. Connected components $\{1\}$ and $\{2\}$ select $(1,2)$. Thus, c_{12} is divided equally between agents 1 and 2.

2. Step 2. Connected component $\{1,2\}$ selects $(0,1)$. Thus, c_{01} is divided equally between agents 1 and 2.

Finally, each agent $i \in N$ pays, $\frac{1}{2}c_{12} + \frac{1}{2}c_{01} = 6$.

Example 2 shows that we must elaborate more our previous idea in order to get a rule.

Example 2.

1. Step 1. Connected components $\{1\}$ and $\{2\}$ select $(1,2)$ whereas $\{3\}$ selects $(1,3)$. Arc $(1,2)$ should be paid by agents 1 and 2. But, what happens with $(1,3)$?

   Assume that c_{13} is paid by agent 3.

2. Step 2. The component N selects $(0,1)$. We consider two ways of sharing the cost of $(0,1)$.

   (a) We divide the cost equally among all agents, which is the way to proceed following the general idea we are applying. Thus, agents 1 and 2 pay $\frac{1}{2}c_{12}+\frac{1}{3}c_{01}$ and agent 3 pays $c_{13}+\frac{1}{3}c_{01}$.
   Let (N_0, C^{ε}) be the *mcstp* where $c^{\varepsilon}_{01}=6+3\varepsilon, c^{\varepsilon}_{02}=6+4\varepsilon$, $c^{\varepsilon}_{03}=6+5\varepsilon$, $c^{\varepsilon}_{12}=6$, $c^{\varepsilon}_{13}=6+\varepsilon$, and $c^{\varepsilon}_{23}=6+2\varepsilon$. Applying the previous idea to (N_0, C^{ε}) we obtain that agents 1 and 2 pay around 5 whereas agent 3 pays around 8. Thus, the quasi-symmetric problem (N_0, C^{ε}) has an asymmetric solution. Since, we are trying to get a fair rule, this procedure does not seem to be a good idea.

   (b) Any *mt* has $|N|$ arcs and $|N|$ agents. Since the cost of each arc should be divided among the $|N|$ agents, it seems reasonable to require that the sum of the proportions of the costs of the *mt* that each agent pays should be 1. In case (a) the sum of these proportions for agents 1 and 2 is $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ whereas for agent 3 is $1+\frac{1}{3}=\frac{8}{6}$.
   If we want to make these proportions equal among them, then c_{01} should be paid between agents 1 and 2 because agent 3 has already paid the cost of arc $(1,3)$ at Step 1. Thus, agents 1 and 2 pay $\frac{1}{2}c_{12}+\frac{1}{2}c_{01}$ and agent 3 pays c_{13}.
   Let (N_0, C^{ε}) be the *mcstp* where $c^{\varepsilon}_{01}=20, c^{\varepsilon}_{02}=20+\varepsilon$, $c^{\varepsilon}_{03}=20+2\varepsilon$, $c^{\varepsilon}_{12}=6$, $c^{\varepsilon}_{13}=6+\varepsilon$, and $c^{\varepsilon}_{23}=6+2\varepsilon$. Applying the previous idea to (N_0, C^{ε}) we obtain that agents 1 and 2 pay around 13 whereas agent 3 pays around 6. Again, the quasi-symmetric problem (N_0, C^{ε}) has an asymmetric solution. Since, we are trying to get a fair rule, this procedure does not seem to be a good idea.

Remark 2. Feltkamp, Tijs, and Muto (1994b) [17] introduce a rule called the Decentralized Rule in $\mathcal{D}^N$. This rule is defined using the ideas of 2 (b).

In both cases, (a) and (b), we do not find a fair rule. The problem is motivated because, in Step 1, c_{13} is only paid by agent 3. Thus, we have decided to change the way in which we divide c_{13}. We follow the "same proportion" approach. We require that all agents must pay the same proportion of the arc they are selecting. Thus, it could be possible that some arcs are only paid partially. In this case, we focus on "paid connected components", namely agents who are connected through arcs paid completely, instead of connected components, and we apply the ideas mentioned above. Let us clarify this procedure in Example 2.

1. Connected components $\{1\}$ and $\{2\}$ select $(1,2)$ whereas $\{3\}$ selects $(1,3)$.

   (a) Arc $(1,2)$ should be paid by agents 1 and 2 and arc $(1,3)$ by agent 3. Thus, the proportion paid is $p = \frac{1}{2}$. Notice that if $p > \frac{1}{2}$, then agents 1 and 2 pay more than the cost or arc $(1,2)$.

   Agents 1 and 2 pay $\frac{1}{2}c_{12}$ and agent 3 pays $\frac{1}{2}c_{13}$.

   (b) Now there are two paid connected components $\{1,2\}$ and $\{3\}$. Arc $(1,3)$ joins both components but only half of the cost was paid.

   $\{1,2\}$ selects $(1,3)$ and $\{3\}$ selects $(1,3)$, as before. Thus, agents 1, 2, and 3, pay the same proportion of the cost of arc $(1,3)$ not paid yet.

   Agents 1, 2, and 3 pay $\frac{1}{6}c_{13}$.

2. The component N selects $(0,1)$.

   (a) The cost of arc $(0,1)$ is divided equally among all agents in the component.

   Agents 1, 2, and 3 pay $\frac{1}{3}c_{01}$.

Finally, agents 1 and 2 pay $\frac{3}{6}c_{12}+\frac{1}{6}c_{13}+\frac{2}{6}c_{01}$ whereas agent 3 pays $\frac{3}{6}c_{13}+\frac{1}{6}c_{13}+\frac{2}{6}c_{01}$. Notice that at each step, each agent pays the same proportion of the cost of an arc. The arc each agent pays depends on his position on the matrix C.

We now explain, in an informal way, how to compute this rule (β^{π}), summarizing the ideas explained above. Initially all agents are isolated. At Step $s-1$ agents are partitioned into paid connected components. We describe Step s. Each one of these components select a non paid arc following Boruvka's algorithm. The cost of the arcs selected at Step s is divided according with the following principles:

- Each agent pays a proportion, p, of the cost of the arc selected by the component he belongs to.

- This proportion is equal for all players, not only inside each component, but across components..

- The proportion paid should be as large as possible. Namely, if each agent pays $p' > p$, then there exists an arc such that the amount paid by the agents assigned to this arc is larger than the cost of the arc.

Let π be some order of the arcs N_0, C a cost matrix, and t^π (or simply t) the arc selected following Boruvka's algorithm associated with π. We now define β^π formally.

Step 0. We define:

- $a_i^{0,\pi} = \emptyset$ for all $i \in N$.

  In general, $a_i^{s,\pi}$, or simply a_i^s, denotes the arc in t that agent i pays partially in Step s.

- $p^{0,\pi} = 0$.

  In general, $p^{s,\pi}$, or simply p^s, denotes the proportion of the cost of the arc that each agent pays in Step s.

- $\varrho_{ij}^{0,\pi} = 0$ for all $(i, j) \in t$.

  In general, $\varrho_{ij}^{s,\pi}$, or simply ϱ_{ij}^s, denotes the proportion of the cost of arc (i, j) already paid in Step s. Namely, $\varrho_{ij}^s = \sum_{r=0}^{s} p^r$

- $A^0(\pi) = t$.

  In general, $A^{s,\pi}$, or simply A^s, denotes the set of non-completely paid arcs in Step s. Thus, $A^s = \left\{(i, j) \in t : \varrho_{ij}^s < 1\right\}$. We denote $\overline{A}^s = t \backslash A^s = \left\{(i, j) \in t : \varrho_{ij}^s = 1\right\}$.

- $f_i^{0,\pi} = 0$ for all $i \in N$.

  In general, $f_i^{s,\pi}$, or simply f_i^s, denotes the cost that agent i pays in Step s. Thus, $f_i^s = p^s c_{a_i^s}$.

Assume that we have defined Step r for all $r < s$. We now define Step s. For simplicity, we omit reference to the order π.

Given a connected component $T \in \mathcal{P}\left(N_0, \overline{A}^{s-1}\right)$, $0 \notin T$, we select the arc $\left(i^T, j^T\right)$ as in Boruvka's algorithm. Namely, $c_{i^T j^T} = \min\left\{c_{ij} : i \in T, j \in N_0 \backslash T\right\}$ and for all $i' \in T$, $j' \in N_0 \backslash T$ such that $c_{i'j'} = \min\left\{c_{ij} : i \in T, j \in N_0 \backslash T\right\}$, $\pi\left(i^T, j^T\right) \leq \pi\left(i', j'\right)$. It is obvious that $\left(i^T, j^T\right) \in t$. Moreover, if component T selected $\left(i^T, j^T\right)$ in Step $s-1$ and $\left(i^T, j^T\right)$ was not completely paid at the beginning of Step s ($\left(i^T, j^T\right) \in A^{s-1}$), component T also selects $\left(i^T, j^T\right)$ in Step s.

Given $k \in T \in \mathcal{P}\left(N_0, \overline{A}^{s-1}\right)$, we define $a_k^s = \left(i^T, j^T\right)$. That is, each agent will pay the cost of the arc selected by Boruvka's algorithm for the component he belongs to.

For each arc $(i, j) \in A^{s-1}$, let $N_{ij}^s = \left\{k \in N : a_k^s = (i, j)\right\}$ be the set of agents that will pay the cost of arc (i, j). We define

$$p^s = \min\left\{\frac{1 - \varrho_{ij}^{s-1}}{\left|N_{ij}^s\right|} : (i, j) \in A^{s-1}, N_{ij}^s \neq \emptyset\right\}.$$

Notice that, assuming that all agents must pay the same proportion of the cost the arc, p^s is the maximum proportion that agents can pay in Step s.

For each $(i,j) \in A^{s-1}$, we define $\varrho_{ij}^s = \varrho_{ij}^{s-1} + \left|N_{ij}^s\right| p^s$. Thus, $\varrho_{ij}^s \leq 1$ for each $(i,j) \in A^{s-1}$. Moreover, there exists at least one $(i,j) \in A^{s-1}$ such that $\varrho_{ij}^s = 1$. Thus, $A^s \subsetneq A^{s-1}$ and $\overline{A}^{s-1} \subsetneq \overline{A}^s$. That is, there are more arcs paid completely.

This process finishes when $\overline{A}^s = t$. Since $a_i^s \in t$ for all agent i and all Step s, and $\overline{A}^{s-1} \subsetneq \overline{A}^s$, this process finishes in a finite number of steps (at most $|N|$), say γ.

Moreover, it is not difficult to check that $\sum\limits_{s=1}^{\gamma} p^s = 1$.

Definition 1. Given an order π of the set of arcs and a cost matrix C, we define the Boruvka's rule induced by the order π as

$$\beta_i^\pi (N_0, C) = \sum_{s=1}^{\gamma} f_i^s \text{ for each } i \in N.$$

We now compute β^π in Example 2 following this procedure.

Example 2. We have seen that, since all the cost of the arcs are different, $t^\pi = \{(0,1),(1,2),(1,3)\}$ for all π. Thus,

Step	$s=0$	$s=1$	$s=2$	$s=3$	TOTAL
a_1^s	$\emptyset$	$(1,2)$	$(2,3)$	$(0,1)$	
a_2^s	$\emptyset$	$(1,2)$	$(2,3)$	$(0,1)$	
a_3^s	$\emptyset$	$(2,3)$	$(2,3)$	$(0,1)$	
p^s	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	1
ϱ_{01}^s	0	0	0	1	
ϱ_{12}^s	0	1			
ϱ_{23}^s	0	$\frac{1}{2}$	1		
f_1^s	0	2	1	4	7
f_2^s	0	2	1	4	7
f_3^s	0	3	1	4	8
A^s	t	$\{(0,1),(2,3)\}$	$\{(0,1)\}$	$\emptyset$	
$\overline{A}^s$	$\emptyset$	$\{(1,2)\}$	$\{(1,2),(2,3)\}$	t	

We now see an example when the order π matters.

Example 3. Let (N_0, C) be the *mcstp* represented by Figure 3.

In this example we need to specify the order. There exist two possibilities.

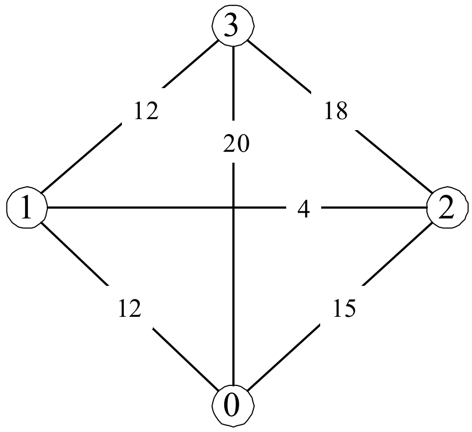

Figure 3: A three-node case with equal costs.

1. Let π be an order in which $\pi(1,3) < \pi(0,1)$. If we formally compute β^{π} , we realize that it is very similar to the one in Example 2. The only difference is the following:

Step	$s=0$	$s=1$	$s=2$	$s=3$	TOTAL
$f_1^{s,\pi}$	0	2	2	4	8
$f_2^{s,\pi}$	0	2	2	4	8
$f_3^{s,\pi}$	0	6	2	4	12

2. Let π' be an order in which $\pi'(0,1) < \pi'(1,3)$. We formally compute β^{π} .

Step	$s=0$	$s=1$	$s=2$	TOTAL
$a_1^{s,\pi'}$	$\emptyset$	$(1,2)$	$(0,1)$	
$a_2^{s,\pi'}$	$\emptyset$	$(1,2)$	$(0,1)$	
$a_3^{s,\pi'}$	$\emptyset$	$(1,3)$	$(1,3)$	
$p^{s,\pi'}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1
$\varrho_{01}^{s,\pi'}$	0	0	1	
$\varrho_{12}^{s,\pi'}$	0	1		
$\varrho_{23}^{s,\pi'}$	0	$\frac{1}{2}$	1	
$f_1^{s,\pi'}$	0	2	6	8
$f_2^{s,\pi'}$	0	2	6	8
$f_3^{s,\pi'}$	0	6	6	12
$A^{s,\pi'}$	t	$\{(0,1),(1,3)\}$	$\emptyset$	
$\overline{A}^{s,\pi'}$	$\emptyset$	$\{(1,2)\}$	t	

Notice that in this case the process finishes in two stages. Moreover Step 2 of π' is completely unrelated with stages 2 and 3 of π (agents pay different arcs, at each stage the proportion is different, ...).

All the rules defined through Prim's algorithm, namely Bird (1976) [9], Dutta and Kar (2004) [14], and Bergantiños and Vidal-Puga (2007a) [6], depend on the order in which the arcs are selected. Two different orders can produce different allocations. Thus, these authors define a rule simply by taking the average over the allocation induced by the different orders. Feltkamp *et al* (1994a) [16] introduce a rule using Kruskal's algorithm. For each order in which the arcs are selected, they propose an allocation. Even though, this allocation could depend on the order, they prove that, it is actually independent. Thus, they define the rule as the allocation generated by each order. We believe that this fact makes the definition more interesting.

We have generated an allocation for each order of the arcs following Boruvka's algorithm. Even though this allocation could depend on the order, we prove that it is independent (as in Feltkamp *et al* (1994a) [16]). Moreover we prove that this allocation coincides with the folk solution φ. All these statements are proved in the following theorem.

Theorem 1. For each order π, β^{π} coincides with φ.
Proof. See Appendix.

Let us compare the definitions of the folk solution through Kruskal and Boruvka. An important difference is that the order in which we add the arcs could be different. Moreover, in Kruskal at each step we add an arc, which is paid completely. In Boruvka, at each step we can add several arcs. At least one of them is paid completely but others could be paid only partially. Above we have mentioned the four principles to divide the cost of an arc (i, j) following Kruskal. We see which of those principles are applied with Boruvka.

1. It is similar. In Boruvka agents already connected to the source, through completely paid arcs, pay nothing.

2. The same principle applies in the sense that agents outside $S_i \cup S_j$ pay nothing. Whereas in Kruskal all agents in $S_i \cup S_j$ pay something (when $0 \notin S_i \cup S_j$), in Boruvka it is possible that agents in S_i or S_j pay nothing.

3. It is the same.

4. It is different. In Boruvka all agents in $S_i \cup S_j$ pay the same.

4 Appendix

We prove Theorem 1.

We first introduce some properties of rules in *mcstp*. Let f be a rule.

Separability (SEP) For all *mcstp* (N_0, C) and all $S \subset N$ satisfying $m(N_0, C) = m(S_0, C) + m((N \backslash S)_0, C)$, we have

$$f_i(N_0, C) = \begin{cases} f_i(S_0, C) & \text{if } i \in S \\ f_i((N \backslash S)_0, C) & \text{if } i \in N \backslash S. \end{cases}$$

Equal Sharing of Extra Costs (ESEC) Let (N_0, C) and (N_0, C') be two *mcstp*. Let $c_0, c'_0 \geq 0$. Assuming $c_{0i} = c_0$ and $c'_{0i} = c_0$ for all $i \in N$, $c_0 < c'_0$, and $c_{ij} = c'_{ij} \leq c_0$ for all $i, j \in N$, we have

$$f_i(N_0, C') = f_i(N_0, C) + \frac{c'_0 - c_0}{|N|} \text{ for all } i \in N.$$

Continuity (CON) For all N, f is a continuous function on C^N.

Independence of Irrelevant Trees (IIT) Let (N_0, C) and (N_0, C') be two *mcstp*. Assuming t be a *mt* in both (N_0, C) and (N_0, C'), and $c_{ij} = c'_{ij}$ for all $(i, j) \in t$, we have

$$f(N_0, C) = f(N_0, C').$$

Bergantiños and Vidal-Puga (2007) prove that φ satisfies *SEP*, *ESEC*, *CON*, and *IIT*. We will use this result throughout the proof.

Let π be any order of the arcs in N_0, C a cost matrix, and $t^\pi = \{(i^0, i)\}_{i \in N}$ the *mt* in (N_0, C) obtained through Boruvka's algorithm. We will prove that $\beta^\pi(N_0, C) = \varphi(N_0, C)$.

We proceed by induction on the number of agents. For $|N| = 1$, the result is clear. Assume that the result holds for less than $|N|$ agents. We now prove it for $|N|$ agents.

We first prove that it is enough to prove that the result holds for matrices in $\mathcal{D}^N$, the set of matrices where all costs are different.

Lemma 1. Given an order π, if $\beta^\pi(N_0, C) = \varphi(N_0, C)$ for all $C \in \mathcal{D}^N$, then $\beta^\pi(N_0, C) = \varphi(N_0, C)$ for all $C \in \mathcal{C}^N$.

Proof of Lemma 1. Notice that $\mathcal{D}^N$ is a dense subset of $\mathcal{C}^N$. Let $C \in \mathcal{C}^N \backslash \mathcal{D}^N$ and t^π the tree obtained through Boruvka's algorithm. We can find a sequence of matrices $\{C^m\}_{m=1}^{\infty}$ such that

1. $C^m \in \mathcal{D}^N$ for all m,
2. t^π is an mt in C^m for all m,
3. if $c_{ii'} = c_{jj'}$ and $\pi(i,i') < \pi(j,j')$, then $c^m_{ii'} < c^m_{jj'}$ for all m, and
4. C^m approaches C as m increases.

Under conditions 2 and 3 when we compute $\beta^\pi(N_0,C)$ and $\beta^\pi(N_0,C^m)$, we have that for any m: $\gamma(N_0,C) = \gamma(N_0,C^m)$; $a_i^{s,\pi}(N_0,C) = a_i^{s,\pi}(N_0,C^m)$ for any $i \in N$ and any $s = 0,...,\gamma(N_0,C)$; $p^{s,\pi}(N_0,C) = p^{s,\pi}(N_0,C^m)$ for any $s = 0,...,\gamma(N_0,C)$; $\varrho_{ij}^{s,\pi}(N_0,C) = \varrho_{ij}^{s,\pi}(N_0,C^m)$ for any $(i,j) \in t^\pi$ and any $s = 0,...,\gamma(N_0,C)$; and $A^{s,\pi}(N_0,C) = A^{s,\pi}(N_0,C^m)$ for any $s = 0,...,\gamma(N_0,C)$.

Let $i \in N$. Thus, for each m

$$\begin{aligned}\beta_i^\pi(N_0,C^m) &= \sum_{s=1}^{\gamma(N_0,C^m)} p^s(N_0,C^m)\, c^m_{a_i^s(N_0,C^m)} \\ &= \sum_{s=1}^{\gamma(N_0,C)} p^s(N_0,C)\, c^m_{a_i^s(N_0,C)}.\end{aligned}$$

Now,

$$\lim_{m\to\infty} \beta_i^\pi(N_0,C^m) = \sum_{s=1}^{\gamma(N_0,C)} p^s(N_0,C)\, c_{a_i^s(N_0,C)} = \beta_i^\pi(N_0,C).$$

Since $(N_0,C^m) \in \mathcal{D}^N$, $\beta_i^\pi(N_0,C^m) = \varphi_i(N_0,C^m)$. Since φ satisfies CON,

$$\lim_{m\to\infty} \beta_i^\pi(N_0,C^m) = \lim_{m\to\infty} \varphi_i(N_0,C^m) = \varphi_i(N_0,C). \blacksquare$$

Hence, we prove the result assuming that $C \in \mathcal{D}^N$. Then, $t^\pi = t^{\pi'}$ and $\beta^\pi(N_0,C) = \beta^{\pi'}(N_0,C)$ for any pair of orders π and π'. Thus, it is enough to prove that $\beta^\pi(N_0,C) = \varphi(N_0,C)$ for some order π.

Let π be an order and $t = t^\pi$. Let $N^0 = \{i \in N : i^0 = 0\}$ and (j^0,j) the most expensive arc in t. We consider several cases:

Case 1. $|N^0| \geq 2$. For any $i \in N^0$, let F^i be the set of agents $j \in N$ such that $(0,i)$ is in the unique path in t from j to 0. Then, $\{F^i\}_{i\in N^0}$ is a partition of N satisfying that $\sum\limits_{i\in N^0} m(F_0^i,C) = m(N_0,C)$ and $t_{F_0^i}$ is a tree in (F_0^i,C) for all $i \in N^0$.

Since φ satisfies SEP, for all $i \in N^0$ and $k \in F^i$, we have $\varphi_k(N_0,C) = \varphi_k(F_0^i,C)$. We just need to prove $\beta_k^\pi(N_0,C) = \beta_k^\pi(F_0^i,C)$ for all $i \in N^0$ and $k \in F^i$ and apply the induction hypothesis.

We need to prove that for each $i \in N^0$, the cost of the arcs in $t_{F_0^i}$ is paid only by the agents in F^i. Suppose not. Then, there exist $i \in N^0$ and $k \in F^i$ such that k selects in step $s+1$ an arc $a_k^{s+1} = \left(i^T, j^T\right) \in t \backslash t_{F_0^i}$ for some $T \in \mathcal{P}\left(N_0, \overline{A}^s\right)$ with $k \in T$. Let s be the first stage in which we can find such $i \in N^0$ and $k \in F^i$. Thus, $a_l^{s+1} = \left(i^T, j^T\right)$ for all $l \in T$. Since $\left(i^T, j^T\right) \notin t_{F_0^i}$, we deduce that the arcs in t_{T_0} have been paid in Step s, namely, $t_{T_0} \subset \overline{A}^s$. By definition, all agents in T are connected through arcs in t. Thus, t_{T_0} is a tree in T_0. Since in t_{T_0} there are exactly $|T|$ arcs, the cost of the arcs in t_{T_0} is paid only by agents in T (s is the first stage in which an agent $k \in F^i$ is paying an arc outside $t_{F_0^i}$), and each agent pays the same proportion p^r at each step r, we deduce that $\sum_{r=1}^{s} p^r = 1$. This means that the procedure is already finished in Step s. Hence, there is no Step $s+1$, which is a contradiction. ■

Case 2. $|N^0| = 1$ and $j^0 \neq 0$ (the most expensive arc does not connect to the source). Let F be the set of agents $i \in N$ such that arc (j^0, j) is in the unique path in t from i to 0. Let $\overline{F} = N \setminus F$. Notice that $F \neq \emptyset$ and $\overline{F} \neq \emptyset$ because $j \in F$ and $j^0 \in \overline{F}$.

We first prove that agents in $\overline{F}$ only pay the cost of the arcs in $t_{\overline{F}_0}$. Suppose not. Then, there exists $k \in \overline{F}$ such that $a_k^{s+1} = (j^0, j)$ for some step s. Let s be the first stage where this happens. Let $T \in \mathcal{P}\left(N_0, \overline{A}^s\right)$ with $k \in T$. Thus, $a_i^{s+1} = (j^0, j)$ for all $i \in T$. Since $c_{j^0 j} > c_{ii'}$ for all $(i, i') \in t_{\overline{F}_0}$ and $t_{\overline{F}_0}$ is a tree in $\overline{F}_0$, we deduce that $T = \overline{F}$ and $t_{\overline{F}_0} \subset \overline{A}^s$. Since there are exactly $\left|\overline{F}\right|$ arcs in $t_{\overline{F}_0}$, and all the agents pay the same proportion p^r at each Step r, we deduce that $\sum_{r=1}^{s} p^r = 1$. This means that the procedure is already finished in Step s. Hence, there is no Step $s+1$, which is a contradiction.

Similarly, we can prove that agents in F only pay the cost of arcs in $t_{F \cup \{j^0\}}$.

Take the matrix $C' \in \mathcal{D}^N$ defined as $c'_{0j} = c_{j^0 j}$, $c'_{j^0 j} = c_{0j}$, and $c'_{il} = c_{il}$ otherwise. It is clear, following the above reasoning, that $\beta^\pi(N_0, C) = \beta^\pi(N_0, C')$.

Since t is the unique mt in (N_0, C), $t' = (t \backslash \{(j^0, j)\}) \cup \{(0, j)\}$ is the unique mt in (N_0, C'). Thus, C' is in Case 1. Hence, $\beta^\pi(N_0, C') = \varphi(N_0, C')$.

Take now the matrix $C'' \in \mathcal{C}^N$ defined as $c''_{0j} = c_{j^0 j}$ and $c''_{il} = c_{il}$ otherwise. It is straightforward to check that both t and t' are mt in C''. Since φ satisfies IIT,

$$\varphi(N_0, C') = \varphi(N_0, C'') = \varphi(N_0, C). \blacksquare$$

Case 3. $|N^0| = 1$ and $j^0 = 0$. Let $(k^0, k) \in t \backslash \{(0, j)\}$ be the most expensive arc in $t \backslash \{(0, j)\}$. Under our hypothesis, $k^0 \neq 0$.

We define a new matrix $C' \in \mathcal{C}^N$ from C by reducing the cost of the arcs in $\{(0, i)\}_{i \in N}$ to the same cost as arc (k^0, k). Namely, for each $i, l \in N$, $c'_{0i} = c_{k^0 k}$, and $c'_{il} = c_{il}$. Of course $C' \notin \mathcal{D}^N$.

We consider an order π such that for each $i, i', i'' \in N$, $\pi(i', i'') < \pi(0, i)$. Namely, the arcs $\{(0, i)\}_{i \in N}$ are the last according with π. Moreover, $\pi(0, j) < \pi(0, k) < \pi(0, i)$ for all $i \in N \setminus \{j, k\}$.

We now proceed by a series of claims:

Claim 1: $\beta_i^{\pi}(N_0, C) = \beta_i^{\pi}(N_0, C') + \frac{c_{0j} - c_{k^0 k}}{|N|}$ for all $i \in N$.

Proof of Claim 1. When computing $\beta_i^{\pi}(N_0, C)$ and $\beta_i^{\pi}(N_0, C')$, we realize:

- $t^{\pi}(N_0, C') = t^{\pi}(N_0, C) = t$.
- Both procedures coincide until step $\gamma - 1$ where all the arcs in $t \setminus \{(0, j)\}$ are completely paid in both procedures and $(0, j)$ is not paid at all. Namely, $\overline{A}^{\gamma-1,\pi}(N_0, C) = \overline{A}^{\gamma-1,\pi}(N_0, C') = t \setminus \{(0, j)\}$ and $\varrho_{0j}^{\gamma-1,\pi}(N_0, C) = \varrho_{0j}^{\gamma-1,\pi}(N_0, C) = 0$.

  Thus, $f_i^{s,\pi}(N_0, C) = f_i^{s,\pi}(N_0, C')$ for all $i \in N$ and all $s = 1, ..., \gamma - 1$.
- In Step γ, all the players choose arc $(0, j)$. Namely, $a_i^{s,\pi}(N_0, C) = a_i^{s,\pi}(N_0, C') = (0, j)$ for all $i \in N$. Hence, the cost of arc $(0, j)$ is shared equally among all agents. Namely, $p^{\gamma} = \frac{1}{|N|}$.

  Thus, for all $i \in N$, $f_i^{\gamma,\pi}(N_0, C) = \frac{c_{0j}}{|N|}$ and $f_i^{\gamma,\pi}(N_0, C) = \frac{c'_{0j}}{|N|}$.
- $\gamma(N_0, C) = \gamma(N_0, C') = \gamma$.
- Now, for all $i \in N$,

$$\begin{aligned}
\beta_i^{\pi}(N_0, C) - \beta_i^{\pi}(N_0, C') &= \sum_{s=1}^{\gamma^{\pi}(N_0,C)} f_i^{s,\pi}(N_0, C) - \sum_{s=1}^{\gamma^{\pi}(N_0,C')} f_i^{s,\pi}(N_0, C') \\
&= \frac{c_{0j}}{|N|} - \frac{c'_{0j}}{|N|} = \frac{c_{0j} - c_{k^0 k}}{|N|}. \blacksquare
\end{aligned}$$

We consider an order π' such that for each $i, i', i'' \in N$, $\pi'(0, i) < \pi'(i', i'')$. Namely, the arcs $\{(0, i)\}_{i \in N}$ are the first according with π'. Moreover, $\pi'(0, j) < \pi'(0, k) < \pi'(0, i)$ for all $i \in N \setminus \{j, k\}$.

Claim 2. $\beta^{\pi}(N_0, C') = \beta^{\pi'}(N_0, C')$.

Proof of Claim 2. Since the *mcstp* is the same we omit (N_0, C') from the notation.

Let G be the set of agents $i \in N$ such that arc (k^0, k) is in the unique path in t from i to 0. Let $\overline{G} = N \setminus G$. Notice that $G \neq \emptyset$ and $\overline{G} \neq \emptyset$ because $k \in G$ and $k^0 \in \overline{G}$.

We prove that $\beta_i^{\pi} = \beta_i^{\pi'}$ for all $i \in G$. The case $i \in \overline{G}$ can be proved in a similar way and we omit it.

We know that $t^{\pi} = t$. Because of the definition of β^{π} there exist r^1 and r^2 such that

1. From Step 1 to Step r^1 agents in G select arcs in t_G, namely $a_i^{s,\pi} \in t_G$ for all $s = 1, ..., r^1$ and all $i \in G$.

   All arcs in t_G have been paid completely in Step r^1, namely $t_G \subset \overline{A}^{r^1,\pi}$.

2. From Step r^1+1 to Step r^2 all agents in G select arc (k^0, k), namely $a_i^s = (k^0, k)$ for all $s = r^1+1, ..., r^2$ and all $i \in N$.

   All arcs in $t_G \cup \{(k^0, k)\}$ have been paid completely in Step r^2, namely $t_G \cup \{(k^0, k)\} \subset \overline{A}^{r^2,\pi}$.

3. Hence, $a_i^{s,\pi} = (0, j)$ for all $i \in N$ and $\gamma^\pi = r^2 + 1$.

It is easy to see that $t^{\pi'} = (t \backslash \{(k^0, k)\}) \cup \{(0, k)\}$. Because of the definition of $\beta^{\pi'}$,

1. From Step 1 to Step r^1 agents in G select arcs in t_G, namely $a_i^{s,\pi'} \in t_G$ for all $s = 1, ..., r^1$ and all $i \in G$.

   Moreover, $a_i^{s,\pi'} = a_i^{s,\pi}$ and $p^{s,\pi} = p^{s,\pi'}$ for all $s = 1, ..., r^1$ and all $i \in G$.

   All arcs in t_G have been paid completely in Step r^1, namely $t_G \subset \overline{A}^{r^1,\pi}$.

2. From Step r^1+1 to Step $\gamma^{\pi'}$ all agents in G select arc $(0, k)$, namely $a_i^s = (0, k)$ for all $s = r^1+1, ..., \gamma^{\pi'}$ and all $i \in N$.

Let $i \in G$. Then,

$$\begin{aligned}
\beta_i^\pi &= \sum_{s=1}^{r^1} p^{s,\pi} c'_{a_i^{s,\pi}} + \sum_{s=r^1+1}^{r^2} p^{s,\pi} c'_{a_i^{s,\pi}} + p^{\gamma^\pi,\pi} c'_{a_i^{\gamma^\pi,\pi}} \\
&= \sum_{s=1}^{r^1} p^{s,\pi} c_{a_i^{s,\pi}} + \left(1 - \sum_{s=1}^{r^1} p^{s,\pi}\right) c_{k^0 k}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\beta_i^{\pi'} &= \sum_{s=1}^{r^1} p^{s,\pi'} c'_{a_i^{s,\pi'}} + \sum_{s=r^1+1}^{\gamma^{\pi'}} p^{s,\pi'} c'_{a_i^{s,\pi'}} \\
&= \sum_{s=1}^{r^1} p^{s,\pi} c_{a_i^{s,\pi}} + \left(1 - \sum_{s=1}^{r^1} p^{s,\pi}\right) c_{k^0 k}. \blacksquare
\end{aligned}$$

Claim 3. $\beta^{\pi'}(N_0, C') = \varphi(N_0, C')$.

Proof of Claim 3. The proof is analogous to the proof of Case 1 and hence we omit it. ■

Claim 4: $\varphi_i(N_0, C) = \varphi_i(N_0, C') + \frac{c_{0j} - c_{k^0 k}}{|N|}$ for all $i \in N$.

Proof of Claim 4. Let $C'' \in \mathcal{C}^N$ defined as $c''_{0i} = c_{0j}$ and $c''_{il} = c_{il}$ for all $i, l \in N$. Since φ satisfies $ESEC$, or all $i \in N$,

$$\varphi_i(N_0, C'') = \varphi_i(N_0, C') + \frac{c_{0j} - c_{k^0 k}}{|N|}$$

Since t is an mt in (N_0, C'') and (N_0, C) and φ satisfies IIT, $\varphi(N_0, C'') = \varphi(N_0, C)$. ■

We now prove that $\beta^\pi(N_0, C) = \varphi(N_0, C)$ in Case 3. For all $i \in N$

$$\begin{aligned}
\beta_i^\pi(N_0, C) &\overset{\text{Claim 1}}{=} \beta_i^\pi(N_0, C') + \frac{c_{0j} - c_{k^0 k}}{|N|} \\
&\overset{\text{Claim 2}}{=} \beta_i^{\pi'}(N_0, C') + \frac{c_{0j} - c_{k^0 k}}{|N|} \\
&\overset{\text{Claim 3}}{=} \varphi_i(N_0, C') + \frac{c_{0j} - c_{k^0 k}}{|N|} \\
&\overset{\text{Claim 4}}{=} \varphi_i(N_0, C).
\end{aligned}$$

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