Self-Enforcing Climate Change Treaties: A Generalized Differential Game Approach with Applications

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Abstract: *Based on recent proposals on non cooperative dynamic games for analysing climate negotiation outcomes, such as Dutta and Radner (2004, 2006a), we generalize a specific framework for modelling differential games of this type and describe the set of conditions for the existence of closed loop dynamics and its relation to adaptive evolutionary dynamics. We then show that the Dutta and Radner (2004, 2006a) discrete time dynamic setup is a specific case of that generalization and describe the dynamics both analytically and numerically for closed loop feedback and perfect state patterns. Our discussion is completed with the introduction of a cooperative differential framework for welfare analysis purposes, within our non cooperative proposal for climate negotiations.*

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1. Introduction

The scope of this paper is to put forward a generalized framework for analysing climate negotiations, at a country or region level, in a differential non-cooperative game setting. Departing from the original work by Dutta and Radner (2004, 2006a), on dynamical games applied to climate negotiations in a discrete time setup, we generalize their proposal in continuous time, assuming a closed loop memoryless perfect state feedback dynamic strategy, to allow for different configurations and hypotheses for future applied research proposals on this research field.

The main objectives driving this proposal are related to characterization of dynamical equilibria and bifurcation analysis. These issues are not straightforward in a differential game setup, as the implications arising from this framework do not have a straightforward strategy to allow for some sort of tractable analytical methodology. These analytical tractability issues, along with economic dynamic issues regarding both policy and economic qualitative period definition drive our option for a continuous time formalization. Assuming time as continuous, will allow the development of robust mathematical results for both equilibria characterization and numerical simulation analysis, within a closed loop memoryless feedback country/region strategy space, without imposing excessive limitations on theoretical modelling options. Last, the choice of a non cooperative game framework is related to the absence of a world regulator institution, which leads to what theorists understand as an unregulated public good game for climate decisions, at a country/region level, and to the quest on the existence (or not) of self enforcing climate change treaties.

In the next sections, we extend the research proposal discussed in the first two paragraphs to demonstrate how dynamic game theory in a continuous time optimal deterministic control formalization leads to the field of evolutionary dynamics. First, we will present a literature survey on the issues of dynamical game theory, evolutionary dynamics and optimal control differential games that we believe will help the reader to take into account the interdisciplinary and complexity issues that arise for this class of models. Second, we will put forward a generalized framework to model climate negotiations, which will allow for some sort of analytical tractability at a regional game level at least and will be general enough to accommodate a wide scope of reasonable modelling proposals. Finally, we discuss the dynamic implications of the Dutta and Radner (2004, 2006a) proposal and set up the main conditions for a theoretical world climate regulator, in order to allow for future welfare analysis on this generalized framework.

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2. *Literature review*

This section is divided in two parts. First, we put forward some literature to guide the reader between the links of differential games, optimal control and evolutionary dynamics. Next, we put forward a set of articles directly linked to our research proposal, in order to provide an overview of how the issue of climate negotiations and Green House Gas (GHG) emissions has evolved in the recent literature

On the subject of dynamical game theory and evolutionary dynamics, there is a wide variety of textbooks and articles available to provide an introduction to both these issues. On evolutionary dynamics, the article and book by Nowak (2004, 2006) and the Sandholm (2008) review, provide a simple starting point on this subject. For a more advanced overview on the topic, the reader should refer to Joosten (1996), Hofbauer and Sigmund (2003) and Smith (1999), for example. Regarding classic and dynamic game theory, there is also a wide variety of textbooks covering the topic, such as Fudenberg and Tirole (1991) and Weibull (1995). For an approach to the issue of differential games and its link with evolutionary dynamics, the early books by Basar and Olsder (1982), Mehlmann (1988) provide a good mathematical introduction to this interdisciplinary field of modelling, while the book by Petit (1990), provides a thorough overview for a more economic oriented audience. The paper by Yeung D. (1989) provides an interesting mathematical overview of the problems arising from closed loop state dependent feedback dynamics, which is an important issue on our proposal. Nevertheless, the referred books do not cover many advances in this recent field that have been scattered in various articles. For more recent literature on the subject of differential games and optimal control theory, which are closely related to our generalized proposal framework, the articles by Jean-Marie and Tidball (2005), Antoci et al (2008), Hofbauer et al (2009) and, especially the paper by Bressan (2009), provide an intuitive introduction and extensive discussion to this mathematics field.

The subject of climate change negotiations is a recent topic that is gaining ground among game theorists. Two main assumptions usually underlie their modelling strategies, a non cooperative and dynamic game approach. These theoretical options seek to accommodate the issue of no institutional regulation within a public good framework. For a straightforward introduction to this theoretical economics issues, the articles by Pfeiffer and Nowak (2006), Clemens and Riechman (2006), Sasaki et al (2007), Dreber and Nowak (2009) and Faber and Frenken (2009) provide a very simple and intuitive approach. Most of the existent articles in the field are based on these assumptions and usually solved in an open loop feedback Nash equilibria context, using the Hamilton-Jacobi-Bellman equation (H-J-B) for obvious analytical tractability issues. Examples of this approach are the articles by Dutta and Radner (2004, 2006a, 2006b), Tidball and Zacour (2009), Breton et al (2005), Kossioris et al (2008), Herbert et al (2005) and Haurie et al (2006), which we believe are specific extensions to the generalized framework we propose in the next section. An excellent outline of this modelling framework for both discrete and continuous time is given by Brock and Dechert (2008). Other approaches include extensive modelling of GHG emissions, such as the Integrated model of Climate and the Economy (RICE) modelling framework, found in Nordhaus and Yang (1996), Germain and Van Steenberghe (2003), among others. This line of research is always a good source of parameter values in the literature, as its main concern is model completeness for simulation purposes.

3. *Self-enforcing climate change treaties as differential non-cooperative games*

We start our generalization of dynamical climate games by stating the relevant functions and variables that we believe are the necessary ingredients to develop a meaningful differential games framework, which will still allow for some sort of analytical tractability in this issue. In the following sub-sections, we pose the generalized optimal control problem, closed loop feedback state dependent conditions and specific applications and extensions that we believe are relevant for a thorough analysis of GHG dynamics[2].

[2]The generalized framework for self-enforcing climate change treaties that we propose in section *3.* and *3.1.* is dependent on the specific form for the strategy function. This option allows us to obtain a complete generalization that includes a wide set of literature applications as discussed in section 2., but paves the way for different interpretations on what is the policy decision obtained from this generalization in an optimal control setup. This issue is best understood when strategy is considered to be the time derivative of technologic improvement, following the Dutta and Radner (2004, 2006a) proposal. In this specific application, our proposal defines policy to be a strategic policy decision that takes into account all feasible predetermined technological paths. Taking this into account, we prepared an appendix section, *4.*, where we propose a generic formalization hypothesis that restricts policy decision to be a dynamic policy strategic variable obtained from optimal control conditions, as

In a dynamic game framework we should start by defining the payoff function faced by each player in a discrete set of I players, $\mathrm{I}=\{1,...,I\}$. We consider that this payoff function is a function of output outcomes and related costs, $Y_i(t)$, and of a policy technology decisions function, $F_i(t)$, defining player strategy. All players face a global state condition defined by function $G(t)$, which describes the motion equation for GHG emissions. The extended form of these functions and a description of the variables follows:

$$Y_i(t)=Y_i\left(e_i(t),f_i(t),g(t)\right) \tag{1}$$

$$F_i(t)=F_i\left(f_i(t),f_j(t),g(t)\right),\ \ \forall\ i\neq j\ and\ i,j\in\mathrm{I} \tag{2}$$

$$\dot{g}(t)=G\left(A\left(\underset{\sim}{e}(t),\underset{\sim}{f}(t)\right),g(t)\right),\ \ \underset{\sim}{e}(t)=\left(e_1(t),...,e_I(t)\right)\ and\ \underset{\sim}{f}(t)=\left(f_1(t),...,f_I(t)\right) \tag{3}$$

Where, $e_i(t)$ is the player level of energy consumption used in production, which we shall consider to be given by a dynamic explicit form solution and not a policy decision, in order to restrict model complexity. $f_i(t)$ is the player technology level, $g(t)$ is the amount of GHG in the atmosphere and $A(t)$ represents the quantity of GHG emitted per unit of time by all the countries/regions. This function can be simplified to be given by $A(t)=\sum_{i=1}^{I}e_i(t)f_i(t)$. To finish this introduction, we need to impose two further conditions: (i) that no country goes backwards in its technology strategies, $F_i(t)\leq 0$, and (ii) non-negative bounded technology sets, $f_i(t)\in\left]0,f_0\right]$.

3.1.Generalized climate differential games

Assuming the general functions (1) to (3), we can now define the climate negotiations game as a non-cooperative closed loop feedback state dependent optimal control differential game. In this game, each player faces the following optimization problem, where we assume a constant discount rate, δ:

$$\begin{aligned}&\underset{f_i}{MAX}\int_0^{\infty}e^{-\delta t}V_i\left(Y_i(t),F_i(t)\right)dt\\&subject\ to:\ \dot{g}(t)=G\left(A\left(\underset{\sim}{e}(t),\underset{\sim}{f}(t)\right),g(t)\right)\end{aligned} \tag{4}$$

This optimization problem follows a specific category of non cooperative differential games defined by the *Pontryagin's* maximum principle. The current value Hamiltonian and player specific optimal control conditions are[3]:

$$H_i^{*}=V_i\left(Y_i(t),F_i(t)\right)+\lambda_i G\left(A(t),g(t)\right) \tag{5}$$

Optimality Condition

$$\frac{dV_i(t)}{df_i(t)}+\frac{dG(t)}{df_i(t)}\lambda_i=0\Leftrightarrow\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{df_i(t)}+\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{df_i(t)}+\frac{dG(t)}{dA(t)}\frac{dA(t)}{df_i(t)}\lambda_i=0 \tag{6}$$

State Condition

$$\dot{g}(t)=G\left(A\left(\underset{\sim}{e}(t),\underset{\sim}{f}(t)\right),g(t)\right) \tag{7}$$

Co-state condition

$$\dot{\lambda}_i=\left(\delta-\frac{dG(t)}{dg(t)}\right)\lambda_i-\frac{dV_i(t)}{dg(t)}\Leftrightarrow\dot{\lambda}_i=\left(\delta-\frac{dG(t)}{dg(t)}\right)\lambda_i-\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{dg(t)}-\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{dg(t)} \tag{8}$$

Before going further on our analytical analysis of this differential game it is convenient to take some time to describe the option for a closed loop feedback solution and state in what conditions this option holds. First, we start

opposed to strategic decisions on feasible predetermined technology paths, following the common intuition usually obtained from optimal control policy modelling problems. This appendix section serves the purpose of providing the reader with an alternative overview of this discussion for proposals in the Dutta and Radner (2004, 2006a) fashion, by imposing stricter modelling assumptions on possible applications in this field, following this short footnote introduction.

[3]We discard the presentation here of the usual admissibility and transversality conditions of the maximum principle definition for simplification purposes.

by defining that closed loop feedback solution implies a state dependent control on our system for each of the players. It serves the purpose of guaranteeing that the result obtained is in accordance with adaptive evolutionary dynamics, as the technology function is usually given by a motion equation describing each specific strategy in a differential fashion. A full description of the conditions for existence of closed loop solutions for differential games is provided in section *1.* of the appendix, following Basar and Olsder (1982).

Basically, these conditions insure the existence of a unique state trajectory with an unique solution for the state condition of the system, as defined by (39), for every set of players strategies. Summarizing, it implies conceptually a set of conditions to guarantee Lipschitz-continuity over all the permissible strategies of the players. This property may however be relaxed, as players may prefer to undertake non-Lipschitz trajectories, because smooth strategies may be non-optimal in the specific differential game framework. If this is the case, then it may not be possible to consider such a game into a rigorous mathematical framework. Taking this into account, we shall now define the meaningful dynamical systems that describe the two information patterns considered, and assume that there are solutions possible of dynamic characterization to this problem, under the conditions stated above.

First, we shall start by defining the feedback perfect state (*FP*) information pattern dynamics. In this framework, we will consider that the players are able to adjust to the system state instantaneously, in other words, there is no marginal adjustment in the path of optimal strategy for each player, taking into account the dynamic information given by the state. In mathematical terms, it means that we will assume that each player co-state condition is in equilibrium, $\dot{\lambda}_i = 0 \Rightarrow \lambda_i = \bar{\lambda}_i$, and strategies will adjust instantaneously to state feedback outcomes perfectly in each period. This type of best response dynamics may be considered as an excessive simplification in an economic policy framework, where institutions are usually only able to adapt slowly to exogenous and endogenous events. However, this information pattern allows us to simplify our optimality condition, by substituting our state condition by its equilibrium solution and obtain a simplified closed loop framework to analyse the dynamic properties of this extended form dynamic game:

$$\dot{\lambda}_i = 0 \Rightarrow \bar{\lambda}_i = \frac{dV_i(t)}{dg(t)}\left(\delta - \frac{dG(t)}{dg(t)}\right)^{-1} \tag{9}$$

The feedback perfect state for each player is obtained by substituting the co-state equilibrium expression, (9), in the optimal control condition, (6). The best response in a perfect state feedback information pattern (*FB*) is then given by the following partial differential equation (PDE):

$$\frac{dV_i(t)}{df_i(t)} + \frac{dG(t)}{df_i(t)}\left(\delta - \frac{dG(t)}{dg(t)}\right)^{-1} \frac{dV_i(t)}{dg(t)} = 0 \tag{10}$$

In the case of closed loop perfect state (*CLPS*), we will no longer consider player strategies as instantaneously adapting to state conditions. This means that we will consider optimal control conditions for this problem, (6) to (8), in order to obtain the meaningful dynamical system for this differential game. The usual approach is to take the time derivative for (6) and then substitute the differential equation obtained and the optimal control, (6), in the co-state condition, (8). The PDE for the *CLPS* information structure pattern follows bellow in equation (12)[4]:

$$\lambda = -\frac{dV_i(t)}{df_i(t)}\frac{df_i(t)}{dG(t)} \Rightarrow \dot{\lambda} = \frac{d\left(-\frac{dV_i(t)}{df_i(t)}\frac{df_i(t)}{dG(t)}\right)}{dt} \Leftrightarrow \lambda = -\frac{dV_i(t)}{df_i(t)}\left(\frac{dG(t)}{df_i(t)}\right)^{-1} \Rightarrow \dot{\lambda} = \frac{d\left(-\frac{dV_i(t)}{df_i(t)}\left(\frac{dG(t)}{df_i(t)}\right)^{-1}\right)}{dt} \tag{11}$$

$$\frac{d\left(-\frac{dV_i(t)}{df_i(t)}\left(\frac{dG(t)}{df_i(t)}\right)^{-1}\right)}{dt} = \left(\delta - \frac{dG(t)}{dg(t)}\right)\left(-\frac{dV_i(t)}{df_i(t)}\left(\frac{dG(t)}{df_i(t)}\right)^{-1}\right) - \frac{dV_i(t)}{dg(t)} \tag{12}$$

Intuitively, these partial differential equations describe each country strategy as a dynamic policy rule that depends on the player technology and state condition outcome for each period. In each period, the players take into account

[4]The extend expressions for optimal control conditions (9) to (12) for both the *FB* and *CLPS* pattern are given in section *2.* of appendix in equations (43) to (46), respectively.

their specific economic conditions for technology adoption, (6), and GHG outcomes, (7), and formulate a decision based on this information. The joint decision process will then define global emissions, $A(t)$, which in turn will input new information on this decision rule. This sets the path for a continuous endogenous updating process, where the countries optimize their response and adapt their strategies assuming the rule described above. The meaningful dynamics for GHG will then be defined by the strategic global emissions outcome, which may be analysed in a dynamic evolutionary context, as an endogenous bifurcation parameter, given the dynamic interactions arising from a strategic policy game with I players.

This result can be better understood if we take into account simple mathematical differential theory. Although, the country specific control variable is the choice of emissions technology factor, $f_i(t)$, the dynamic control will be given by the composite derivative, given by the first expression in the right hand side of (11), defining the projection of specific technology policy decisions on the total country emissions during that period, expressed by function $A(t)$. One can consider that this differential projection based on the simple chain rule is the meaningful dynamic policy strategic variable, which allows for continuous adaptation to changes in the strategy space, where each country may only control their specific technology factor, but adjust this decision according to its impact on the global outcome. This result relates closely to evolutionary dynamics theory, since function $A(t)$ defines the fitness landscape, where strategic decisions are taken. To be more precise, the fitness landscape is defined by the state variable, but the quantity of interest is related to the players strategies. As strategic decisions change the fitness landscape, the differential framework we suggest takes into account these dynamics, as it allows for strategic decisions to adapt to changes in the fitness landscape. Therefore, the problem of climate change without cooperation enforcing institutions, under these specific assumptions, is well posed only if we consider an adaptive dynamics evolutionary game theory approach.

3.2. Uncoupled linear self-enforcing climate games: an applied example

In this sub-section, we evaluate the sub class of uncoupled and linear differential games arising from our generalized proposal. For this purpose, we will use a specific example based in the Dutta and Radner (2004, 2006a) discrete time dynamic game proposal, adapted to our generalized approach in continuous time.

Generically, the Dutta and Radner (2004, 2006a) proposal can be resumed, following our definition of (1) to (3), by the following set of equations:

$$Y\left(e_i(t), g(t)\right) = \phi_i K_i^{\gamma} L_i^{1-\gamma-\beta_i} e_i(t)^{\beta_i} - \rho_i e_i(t) - c_i g(t) \tag{13}$$

$$F_i\left(f_i(t), g(t)\right) = \kappa_i \dot{f}_i(t) \tag{14}$$

$$\dot{g}(t) = A(t) - (1-\sigma) g(t) \tag{15}$$

$$A(t) = \sum_{i=1}^{I} e_i(t) f_i(t) \tag{16}$$

Where the parameters, as described in Dutta and Radner (2004, 2006a) for this maximization problem, are the marginal cost of GHG emissions for country i, c_i, the marginal cost of adopting a cleaner technology, κ_i, and the price of energy faced by each country, ρ_i. The remaining parameters are, γ, β_i and ϕ_i, which refer to the country exogenous technology parameter and production factor elasticities, respectively. Finally, parameter σ refers to the persistence of GHG emissions on the atmosphere. Strategic decisions in this setup must always follow the rule $\dot{f}_i(t) \leq 0$ and $e_i(t) \succ 0$ dynamics are independent and explicitly given by a differential equation.

Given this short introduction to the Dutta and Radner (2004, 2006a) proposal, we can now use the generalized framework we proposed previously, to define the set of conditions that define this differential game.

Optimality Condition

$$\frac{dV_i(t)}{df_i(t)} + \lambda \frac{dA(t)}{df_i(t)} = 0 \Leftrightarrow \kappa_i \frac{d\dot{f}_i(t)}{df_i(t)} + \lambda_i e_i(t) = 0 \tag{17}$$

State Condition

$$\dot{g}(t) = A(t) - (1-\sigma) g(t) \tag{18}$$

Co-State Condition

$$\dot{\lambda}_i = \lambda_i \left(\delta - \frac{dG(t)}{dg(t)} \right) - \frac{dV_i(t)}{dg(t)} \Leftrightarrow \dot{\lambda}_i = (\delta + 1 - \sigma)\lambda_i + c_i - \kappa_i \frac{d\dot{f}_i(t)}{dg(t)} \tag{19}$$

Again two possible solutions for this adaptive dynamic game framework may be considered as best response strategies. These strategies, as defined in the previous section, correspond either to adaptive instantaneous best response dynamics, assuming there is no marginal adjustment arising from the dynamics of the state variable, or to the case where the players define optimal strategic paths, taking into account the overall dynamics arising from the maximum principle. These two strategic decision functions are given bellow by the following PDE's for the linear uncoupled climate change game. First, we discuss the generalization for this case, which is as usual obtained by taking the time derivative from the optimal control condition and substituting in the co-state differential equation.

Generalized control strategic dynamics for the uncouple linear climate game

$$\frac{d\left(-\frac{dV_i(t)}{df_i(t)} \frac{df_i(t)}{dA(t)} \right)}{dt} = -\frac{dV_i(t)}{df_i(t)} \frac{df_i(t)}{dA(t)} \left(\delta - \frac{dG(t)}{dg(t)} \right) - \frac{dV_i(t)}{dg(t)} \tag{20}$$

The PDE's obtained in the specific Dutta and Radner (2004, 2006a) context, which defines the dynamic strategic country/region decisions are given by the following equations:

Instantaneous best response dynamics (FB pattern $\dot{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$)

$$\kappa_i \frac{d^2 f_i(t)}{df_i(t)dt} = -\left(\kappa_i \frac{d^2 f_i(t)}{dg(t)dt} - c_i \right) \frac{1}{(\delta + 1 - \sigma)} e_i(t) \tag{21}$$

Best response dynamic strategic path (CLPS pattern)

$$\frac{d^2 \dot{f}_i(t)}{dt df_i(t)} = \frac{d\dot{f}_i(t)}{df_i(t)} \left(\frac{\dot{e}_i(t)}{e_i(t)} + (\delta + 1 - \sigma) \right) - e_i(t) \left(\frac{c_i}{\kappa_i} - \frac{d\dot{f}_i(t)}{dg(t)} \right) \tag{22}$$

Both this policy functions may be rearranged, in order to define the integral differential equations that determine the strategic decisions of each of the players in the climate game:

Instantaneous best response dynamics strategies (FB pattern)

$$\dot{f}_i(t) = \int_{f_{best}}^{f_i(t)} -\left(\frac{d\dot{f}_i(t)}{dg(t)} - \frac{c_i}{\kappa_i} \right) \frac{e_i(t)}{(\delta + 1 - \sigma)} df_i(t) \tag{23}$$

Best response dynamic strategic path (CLPS pattern)

$$\dot{f}_i(t) = \int_{f_{best}}^{f(t)} \int_t^{\infty} \left[\frac{d\dot{f}_i(t)}{df_i(t)} \left(\frac{\dot{e}_i(t)}{e_i(t)} + (\delta + 1 - \sigma) \right) - e_i(t) \left(\frac{c_i}{\kappa_i} - \frac{d\dot{f}_i(t)}{dg(t)} \right) \right] dt df_i(t) \tag{24}$$

From now on, we shall concentrate on the dynamic analysis for the *FB* policy strategic expressions, given by (21) and (23), for reasons of simplification. Later in this section, it will be shown that the dynamics arising from the *CLPS* pattern follow closely the conditions defined for the *FB* case. Policy equation (21) states that each player is always available to perform technological improvements that take into account the strategic value attributed to global environment outcomes. We can extend this expression in its integral form, as we know the limits of integration, to obtain (23), where $f_{best} = \min(f(t))$ is the exogenous parameter defining the best available technology. Strategic equilibrium or no more best technologies policy is defined by the following set of conditions:

$$\dot{f}_i(t) = 0 \Rightarrow \begin{cases} \bar{f}_i(t) = f_{best} \\ \frac{d\dot{f}_i(t)}{dg(t)} \leq \frac{c_i}{\kappa_i} \Rightarrow \underbrace{\dot{f}_i(t)}_{=0} \leq \int_0^{\infty} \frac{c_i}{\kappa_i} dg \Leftrightarrow g_i^* \leq -\Psi_i \frac{\kappa_i}{c_i} \Rightarrow \bar{f}_i^*(t) \succ f_{best},\ \Psi_i \prec 0 \end{cases} \tag{25}$$

Taking into account the information provided by (21), (23) and (25), we can describe the dynamic adaptive behaviour of the *FB* pattern policy dynamics, as a function of distances between the technology level and global environment to best technology available and the relaxing policy threshold, respectively. This rate of adjustment will be greater the farther away existent player technology and GHG emissions are from the respective thresholds and it will depend exogenously on model parameters and endogenously on energy levels. Analogously, technology implementation will slower near the thresholds, until it eventually relaxes on the physical or policy constraints defined in (25). The environmental policy threshold will depend on the environmental economic indicator, given by the marginal cost of implementing new technologies relative to the marginal cost of GHG pollution. This threshold will also depend on the integrand constant, Ψ_i, which can be considered as an exogenous environmental policy parameter defining the player preferred level of GHG emissions.

In order to conclude the description of the Dutta and Radner (2004, 2006a) proposal, we require an upper boundary limit for admissible state trajectories, in order to fulfil condition *iii.*, related to the existence of infinite horizon differential games, as defined in section *1.* of the appendix. This condition should be related to some modelling assumption relating to catastrophe[5] outcomes or energy substitution dynamics, given the information available on global warming and its relation to resource use and environmental damage. Still, in the absence of specific modelling assumptions, we shall impose a set of assumptions in order to define the possible profile for energy and GHG emissions dynamics in the long run. First, we must assume that admissible trajectories in the limit must have dynamics governed by a fixed point, which is within a stable dynamic region, and where we can assume $\overline{f}_i^* = f_{best}$ and $\lim_{t\to\infty} e_i(t) = \overline{e}_i,\ \forall\, i \in \mathrm{I}$:

$$\lim_{t\to\infty} \dot{g}(t) = 0 \Rightarrow \overline{g} = \left(f_{best} \sum_{i=1}^{I} \overline{e}_i \right) (1-\sigma)^{-1},\ \forall\, g^*_{\max} \prec \overline{g} \prec \infty \tag{26}$$

This long run GHG equilibrium condition is only valid for outcomes defined in the cooperation region, as defined bellow in figure 1, where the player with a higher environmental policy threshold always cooperates.

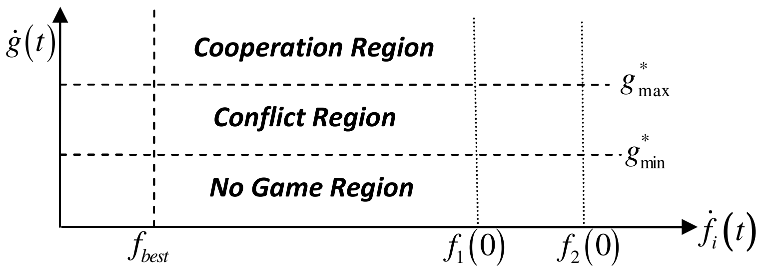

Fig. 1- Two dimensional phase space map and relevant dynamic game regions

By imposing the strict condition for stable equilibrium existence described in (26), we are able to analyse the linearized dynamics of the generalized multiplayer game under these conditions and set out the final conditions for bounded outcomes with feasible state trajectories. These set of conditions are expressed by the outcome of the generalized *Jacobian* matrix described in conditions (46) to (48) in section *3.* of the appendix, and imply the usual one dimensional fixed point stability condition for energy dynamics, $\left.\frac{d\dot{e}_i(t)}{de_i(t)}\right|_{e_i(t)=\overline{e}_i} \prec 0$. Further issues regarding the signs of partial derivatives of $\dot{f}_i$ do not alter this set of conditions, as the possibility of eigenvalues equal to zero are related to long run convergence issues. If we consider the technology dynamics partial derivatives to be all equal to zero under these conditions, then we may analyse just the linearized dynamics of energy and GHG emissions, which will imply the same strict stability differential game condition.

Having described the conditions for differential game existence and defined the relevant regions of cooperation, we can now discuss the implications of these constraints. First, the conditions imposed on energy dynamics, do not

[5]Recent research on the links between excess resource consumption, environmental damage and catastrophe outcomes suggests the need for introduction of threshold mechanisms to account for phase transition dynamics. Examples of this research in the field of environmental dynamic economics are based on the existent evidence on the collapse of the Easter Island society. Brander and Taylor (1998) article on Easter Island Dynamics is an interesting starting point on this subject.

allow for constant growth of energy in the long run, which implies that output dynamics, (13), will reach a steady state in the long run. This hypothesis, however, does not fit the usual setup defined by economic growth theory and imposes several limitations regarding the dynamics of economic development and its impact on GHG emissions. Second, for the same reasons stated above, we cannot consider long run energy decays, as this would imply economic decay, which to be considered as feasible, would have to involve some sort of energy substitution dynamics between clean and dirty energy consumption that would allow for economic expansion. Therefore, the most reasonable dynamics for energy to be considered in the Dutta and Radner (2004, 2006a) model, are in our opinion dynamics of the usual limited growth, such as logistic type dynamics.

Taking into account the energy dynamic issues discussed above, we may move forward and define the main bifurcation condition arising in this model. As discussed previously, bifurcations in these setups are endogenous and time dependent, due to the adaptive dynamics considered. We start by defining the condition for no phase transitions to occur due to player environmental threshold non linearities. This condition is the generic condition for time dependent dynamic cooperation between players and is given by $g(t) \succ g^{*}_{\max}$. From this definition, we can define the generalized exogenous conditions that define cooperation dynamics always in the long run. This set of conditions is defined in expression (27):

$$\overline{g} = f_{best}\sum_{i=1}^{I}\overline{e}_i(1-\sigma)^{-1} \succ g^{*}_{\max} \Rightarrow f_{best} \succ g^{*}_{\max}(1-\sigma)\left(\sum_{i=1}^{I}\overline{e}_i\right)^{-1},\ g^{*}_{\max} = \max\left(-\Psi_i\frac{\kappa_i}{c_i}\right),\ \forall\ i \in \mathrm{I} \quad (27)$$

This condition relates the best attainable technology, worst policy threshold and energy steady states as the main bifurcation parameters, since if we consider logistic growth dynamics for energy, steady states for energy are defined exogenously. Still, bifurcations might arise during transitions due to initial conditions and convergence rate dynamics of energy and technology strategies, which in turn depend on the distance to always cooperation outcomes defined in (27). One further issue regarding the bifurcation condition defined in (27), relates some sort of substitution dynamics for energy, when best attainable technology parameter is defined in the region, $0 \prec f_{best} \prec 1$. As such substitution dynamics for energy have not been considered in this modelling framework, considerations of this nature may lead to ill defined conclusions. Nevertheless, the future introduction of such dynamics can be perfectly achieved in our generalized setup, in order to allow for further policy insight arising from this hypothesis.

So far, we have only discussed the dynamic features for the *FB* pattern, disregarding the dynamics of the *CLPS* pattern for reasons of simplification. The main difference between the two dynamic game patterns relates to marginal strategic adjustment to state conditions, as opposed to instantaneous adjustment in the *FB* pattern. This setup is more appropriate to model the dynamics of macro policy implementation, as it considers that a policy maker chooses paths of adjustment, where policy strategic paths always fulfil the strategy profile defined in (28), which are a continuous set of equilibrium strategic policies. Condition (28) is obtained by assuming technology boundary equilibrium conditions in (24) and taking into account differential game conditions for the *CLPS* pattern defined in section *3.1.* and section *1.* of the appendix.

$$\begin{gathered}\dot{f}_i(t) = \int_{f_i(t)}^{f_i(s)}\int_{s}^{t}\left[\frac{d\dot{f}_i(s)}{df_i(s)}\left(\frac{\dot{e}_i(s)}{e_i(s)}+(\delta+1-\sigma)\right)-e_i(t)\left(\frac{c_i}{\kappa_i}-\frac{d\dot{f}_i(s)}{dg(s)}\right)\right]ds\,df_i(s), \\ 0 \le s \le t \prec \infty,\ \dot{f}_i(t)=0 \Rightarrow s=t \Rightarrow f_i(s)=f_i(t) \Rightarrow \lim_{t\to\infty} f_i(t) = \begin{cases}\overline{f}_i \succ f_{best}\\ f_{best}\end{cases}\end{gathered} \quad (28)$$

Equilibrium game conditions for strategic policy paths in the *CLPS* pattern are obtained from the expression inside the double integral, which defines the set of feasible equilibrium dynamics for each strategic policy path:

$$\frac{d\dot{f}_i(t)}{df_i(t)} = -\left(\frac{d\dot{f}_i(t)}{dg(t)}-\frac{c_i}{\kappa_i}\right)e_i(t)\left(\frac{\dot{e}_i(t)}{e_i(t)}+(\delta+1-\sigma)\right)^{-1} \quad (29)$$

The main difference from the dynamics equilibrium expression for the *FB* pattern relates to the dependence on energy dynamics and levels. Now we have one additional policy relaxation rule, if we consider the possibility of energy dynamic decays, while the remaining policy relaxation rule is just the threshold policy rule defined in (25). We shall not consider this expression as relevant, since we will assume that energy follows logistic type dynamics.

Nevertheless, we shall consider this link on the size of energy decay rates as another feature of the model that already accommodates the possibility of energy substitution dynamics, following the inequality given in (30):

$$\frac{\dot{e}_i(t)}{e_i(t)} \succ -(\delta+1-\sigma) \tag{30}$$

You conclude the dynamic characterization of the Dutta and Radner (2004, 2006a) proposal with a final reference to what we consider are the most relevant findings in our analysis. First, the absence of upper boundary conditions on GHG emissions, imposes strict dynamic conditions for energy consumption, this issue is related to the subject of possible energy substitution dynamics that was discussed thoroughly throughout this section. It is our believe that such dynamics can be introduced within the generalized framework presented in section *3.*, in order to produce endogenous upper boundaries for GHG dynamics and account for other sources of climate related technology improvements. Second, we were able to define the generalized expression for bifurcation analysis in the long run, still the issue of endogenous bifurcations during transitions and the possibility of GHG overshoots in the medium run were not discussed here. We shall introduce these two issues in the next section, using for this purpose a dynamical system based on the dynamic characterization provided in this section. For this purpose, we shall assume that have environmental and development policies in accordance with the *environmental Kuznets curve* (EKC) hypothesis[6], which will able us to relate the most important features discussed in this section with a widely discussed theoretical proposal in environment economics literature.

3.2.1. Non cooperative climate games simulations

In this section, we put forward a generic simulation proposed for the Dutta and Radner (2004, 2006a) model in continuous time, based on the dynamic characterization presented in section *3.2.*. We shall assume technology adoption dynamics to follow a logistic decay differential equation with endogenous rates of adjustment interacting in a multiplicative fashion, following the intuition obtained for player technology boundaries and policy threshold for the *CLPS* pattern. Energy dynamics follow a logistic growth equation *a la Verhulst*, following the proof for existence of differential games given in section *3.* of the appendix and taking into account the discussion relating energy dynamics, economic growth and absence of energy substitution dynamics in section *3.2.*.

$$\dot{f}_i(t) = \underbrace{e_i(t)\left(\frac{\dot{e}_i(t)}{e_i(t)}+(\delta+1-\sigma)\right)^{-1}}_{\text{Energy dependent endogenous rate of technological adjustment}} f_i(t)\left(1-\frac{f_i(t)}{f_{best}}\right)\underbrace{\left(g(t)-g_i^*\right)}_{\substack{\text{Environmental policy dependent endogenous}\\ \text{rate of technological adjustment}}} H(t)\Phi$$

$$\dot{e}_i(t)=\alpha_i e_i(t)\left(1-\frac{e_i(t)}{\bar{e}_i}\right),\quad H(t)=\begin{cases}1 \text{ if } g(t)\succ g_i^* \\ 0 \text{ if } g(t)\le g_i^*\end{cases},\ g_i^*=\Psi_i\frac{\kappa_i}{c_i}\ \textit{and}\ \ \Phi\succ 0 \tag{31}$$

The function $H(t)$ in (31), refers to the *Heaviside* function to control for the dynamics of policy relaxation, while parameter Φ, is fitted with the purpose of fine-tuning the rate of technological adjustment. This parameter was included in order to accommodate the slow response dynamics associated with the *CLPS* pattern.

In the next sub sections, we shall illustrate some of the dynamics discussed in the main text. We shall focus on what we believe are the main features of the Dutta and Radner (2004, 2006a) proposal, when energy and technology dynamic issues are considered in different bifurcation contexts, taking into account the generic generalized bifurcation condition (27). First, we shall assume a simple setup to demonstrate overshoot and free riding dynamics in two and three player games, respectively. Table 1 and 2, bellow, summarizes the numerical parameters chosen for these simulations. Finally, we conclude by illustrating the complex transitions to equilibrium arising in a multiplayer game. For this purpose, we shall disregard short to medium run overshoot dynamics and take into account the dynamics of the system in the long run to illustrate equilibration dynamics[7].

[6]The EKC hypothesis states that a country maturity in relation to environmental damage depends on a country development level. Generically, this hypothesis states that the relation between environmental damage and development is linear quadratic and there is a maturity development maximum, which shifts policy goals from development to environment. For a recent review and survey on this literature see Dinda (2004).

[7]To simulate the proposed dynamical systems we used the Runge-Kutta 4[th] and 5[th] order routine provided by Matlab (ODE45).

$g(0)$	δ	σ	f_{best} (2 players)	f_{best} (3 players)	Φ
60	0.03	0.6	1	0.1	10^(-7)

Table 1- Base parameters for simulations of overshoot and free riding dynamics

$e_1(0)$	$e_2(0)$	$e_2(0)$	$f_1(0)$	$f_2(0)$	$f_3(0)$	Ψ_1	Ψ_2	Ψ_3	α_1
5	3	2	5	7	10	20	40	50	0.015
α_2	α_3	κ_1	κ_2	κ_3	c_i	$\overline{e}_1$	$\overline{e}_2$	$\overline{e}_3$	
0.02	0.03	0.1	0.12	0.2	0.1	8	9	10	

Table 2-Player specific parameters for overshoot and free riding dynamics

3.2.2. *Overshoot dynamics in two player games*

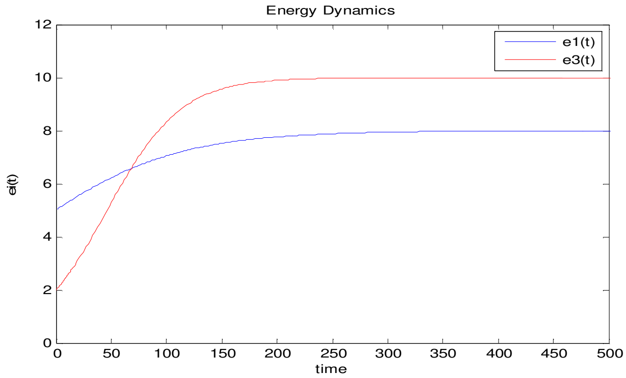

Fig. 2- Logistic dynamics for energy consumption

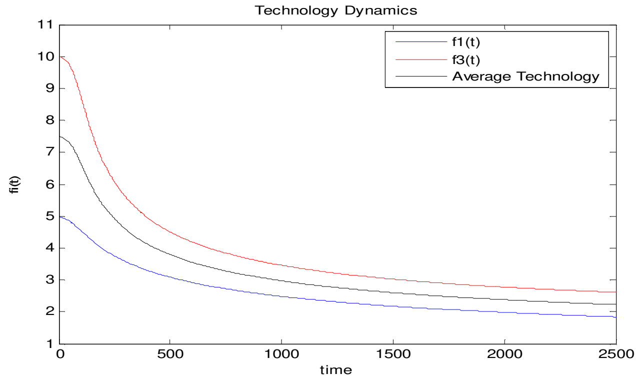

Fig. 3- Technology strategy paths

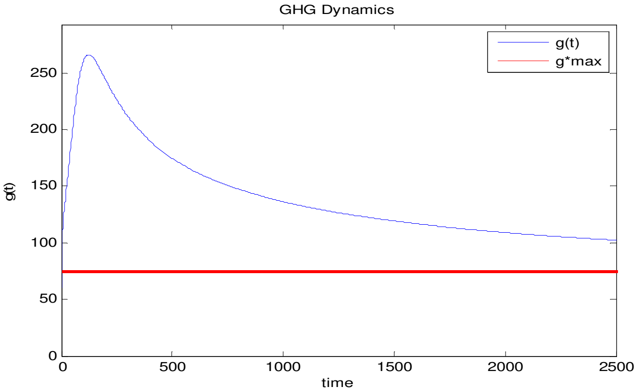

Fig. 4- GHG dynamics

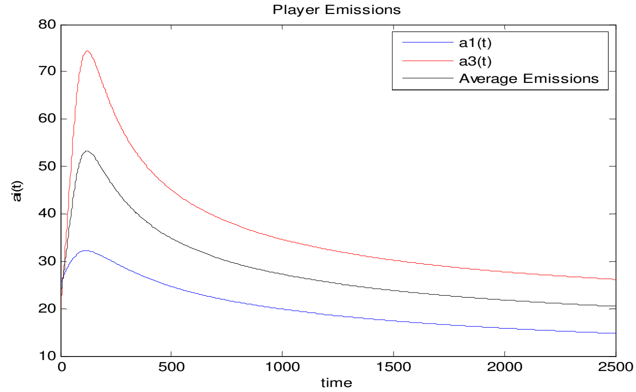

Fig. 5- Player emissions dynamics

3.2.3. *Free riding environmental dynamics in a three player game*

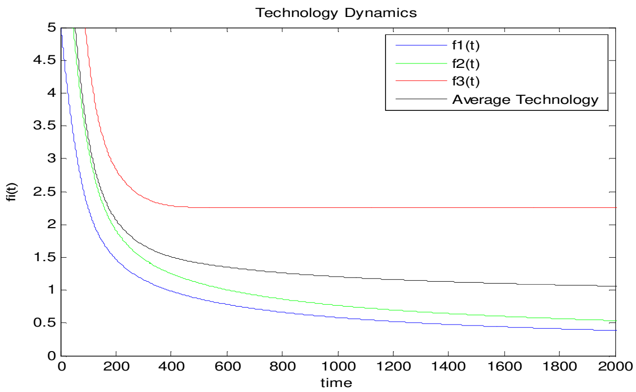

Fig. 6- Technology dynamics with free riding

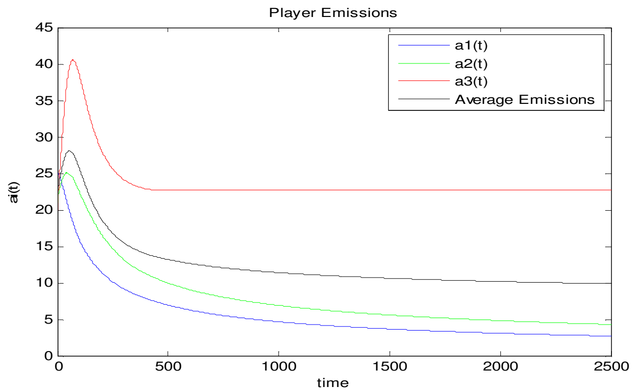

Fig. 7- Emissions dynamics with free riding

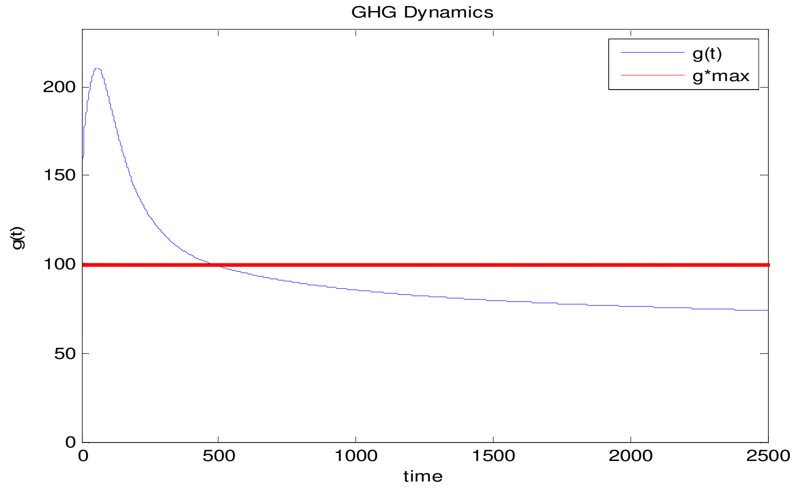

Fig. 8- GHG dynamics with free riding in conflict region

3.2.4. Complex transitions to equilibrium in multiplayer games

We finish this section by illustrating how complex transitions to equilibrium may arise. First, we illustrate complex transitions assuming that players environmental thresholds follow an ordered ranking pattern, as in the previous simulations. The set of parameter values for this simulation is described in tables 3 and 4, bellow, while figures 9 to 11 illustrate the GHG dynamics within this framework. Then, we assume $\Psi_3 = \Psi_4$ and $\kappa_3 = \kappa_4$, in order to reproduce some sort of coalition dynamics between the two players who have higher thresholds for environmental policy relaxation. This dynamics are portrayed in figures 12 to 14. Both these simulations serve the purpose of demonstrating that even one relevant player can drive GHG emissions outcomes according to its policy threshold for a relevant period of time, setting alone the self-enforcing of climate treaties, when the bifurcation condition (27) allows for conflict dynamics. Moreover, if worst scenario environment coalitions are relevant enough, GHG outcomes are determined by such coalitions and free riding better outcomes in the long run are not achieved.

$g(0)$	δ	σ	f_{best}	Φ
60	*0.03*	*0.6*	*10^(-7)*	*10^(-8)*

Table 3- Base parameters for long run transitions simulation

$e_1(0)$	$e_2(0)$	$e_3(0)$	$e_4(0)$	$f_1(0)$	$f_2(0)$	$f_3(0)$	$f_4(0)$	Ψ_1	Ψ_2	Ψ_3	Ψ_4	α_1
5	*4*	*3*	*2*	*5*	*7*	*9*	*10*	*20*	*30*	*40*	*50*	*0.001*
α_2	α_3	α_4	κ_1	κ_2	κ_3	κ_4	c_i	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_4$	
0.015	*0.002*	*0.0025*	*0.1*	*0.12*	*0.13*	*0.15*	*0.1*	*2250*	*2350*	*2200*	*2400*	

Table 4– Player specific parameters for long run transitions simulation

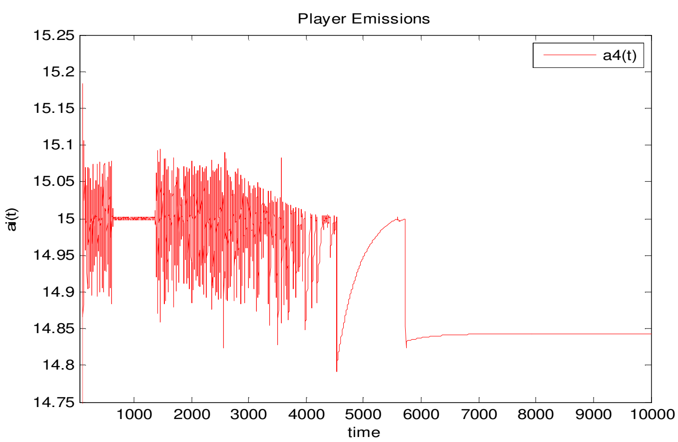

Fig. 9- Emissions transition dynamics for player 4

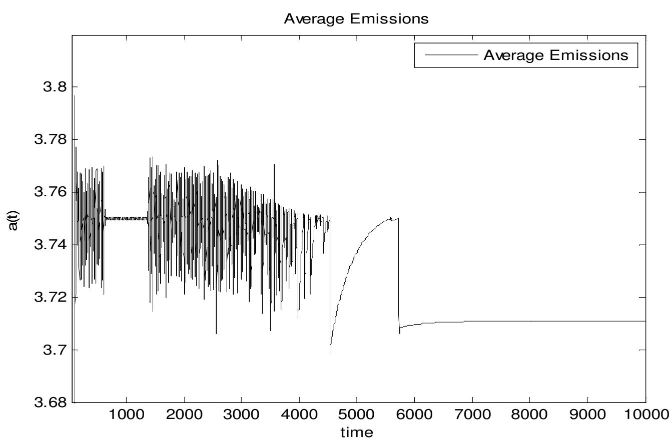

Fig. 10- Average emissions during transition

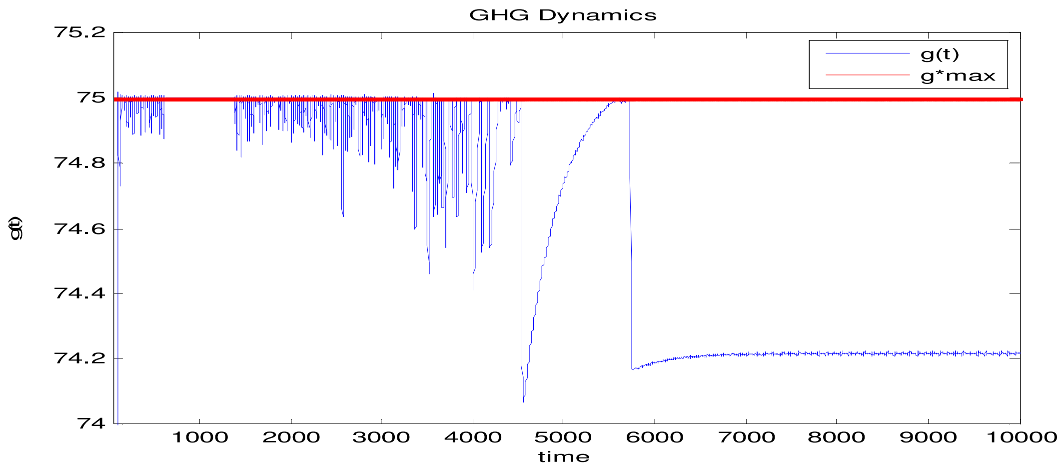

Fig. 11- GHG dynamic transitions from highest threshold to conflict region

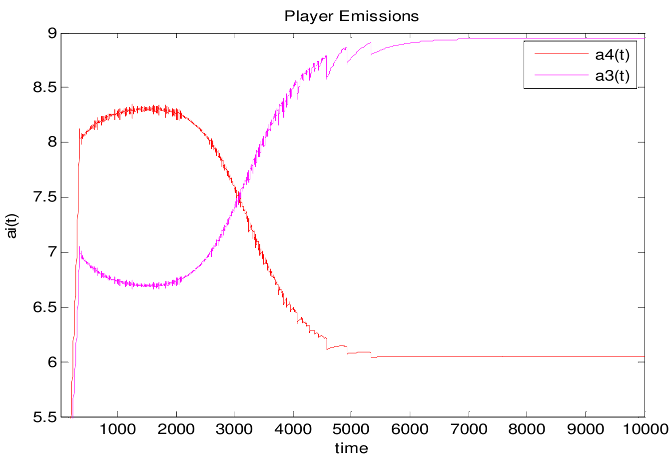

Fig. 12- Emission transition dynamics for coalition members

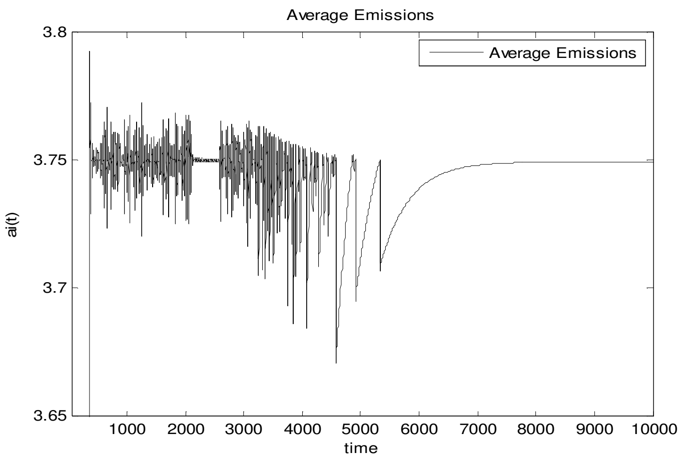

Fig. 13- Average emission transition dynamics with coalitions

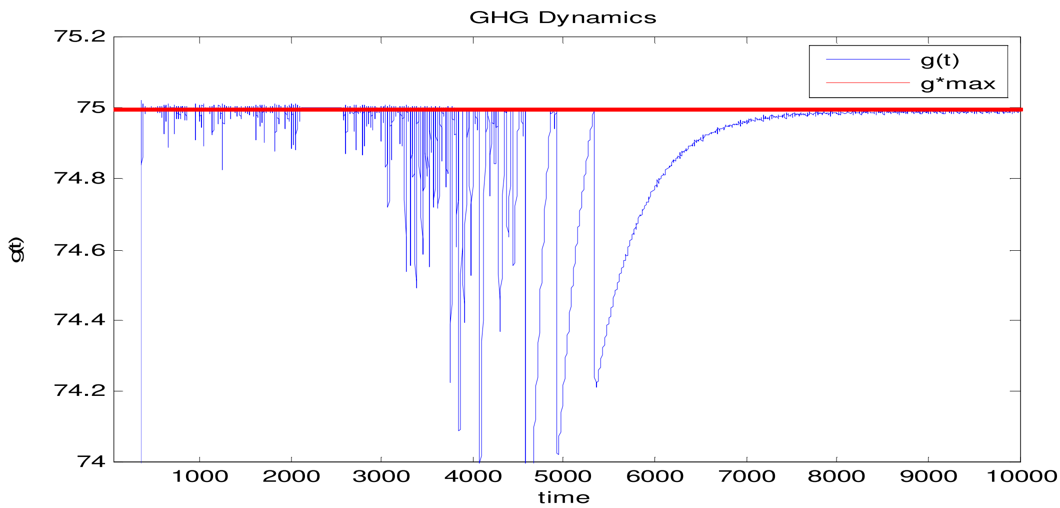

Fig. 14- GHG transition dynamics with coalition threshold

4. Further relevant extensions: the global GHG emissions regulator

In the previous sections, we dealt with climate games in a non cooperative framework. An important extension to this framework should be to consider a regulator type of global institution, which is able to maximize the players payoffs taking in consideration this specific non cooperative framework. Although, we discussed the lack of present viability for such a global enforcing institution, modelling a reasonable cooperative dynamic game will allow for a better understanding of the dynamics of the non cooperative case, described in section *3.1.*, and may also be used as a benchmark for welfare analysis.

Let's consider that there exists a world regulation institution for climate negotiations that is able to take into account all the players payoffs functions. The payoff function of this regulator would be, generically, given by the following expression: $\mathrm{V}(t)=\sum_{i=1}^{I} V_i(t)$. This simple generalization provides the ground for modelling a regulator that will maximize all the players payoffs. However, there are two issues that should be further taken into account. First, this formalization is only relevant when you consider the coupled technologies case, where this regulator impact on the strategic decision process, by taking into account players strategic reply to other players strategies. In the uncoupled technology case, this sort of regulator would have no impact in the strategic decision process and the optimal control problem would reduce to the non cooperative case described in section *3.1.*[8]. Second, the regulator should only be able to manage GHG emissions, by shaping country/region strategic behaviour, therefore any further extensions assuming any sort of regulation on the state dynamics of the system, to solve the lack of regulation power in the uncoupled technologies case, would not be a feasible modelling assumption. To solve this issue, the regulator may enforce some sort of instrument to manage players strategies. This instrument may be exogenous, in this case it could be viewed as an emission regulation policy based on some predefined quota system, in the fashion of the *Kioto Protocol* negotiations. On the other hand, one may consider a more sophisticated mechanism that takes into account system dynamics, in order to develop a reasonable endogenous instrument that adapts to the GHG outcomes and shapes country policy. Linking such an instrument to the model time varying variables, such as country emissions, would provide a feasible endogenous global policy instrument regulation purposes.

Considering now that the regulator payoff function is given by $\mathrm{V}(t,\theta)=\sum_{i=1}^{I}\theta_i V_i(t)$, where θ may be considered as an exogenous parameter for the regulator problem or a function of $e_i(t)$ and $f_i(t)$ to account for country specific emissions, for example. In both of these cases, the regulator maximization intertemporal problem will be given, without loss of generalization by:

$$\begin{aligned}&\underset{f_1,\dots,f_I}{MAX}\int_0^{\infty} e^{-\delta t}\sum_{i=1}^{I}\theta_i V_i(t)\,dt\\ &subject\ to:\ \dot{g}=G(t)\end{aligned} \tag{32}$$

The current value Hamiltonian is given by:

$$H^{*}=\sum_{i=1}^{I}\left[\theta_i V_i(t)\right]+\lambda G(t) \tag{33}$$

Exogenous regulator instrument- generalized optimality condition

$$\frac{dV_i(t)}{df_i(t)}\theta_i+\theta_j\sum_{j=1}^{I}\frac{dV_j(t)}{df_i(t)}+\lambda\frac{dG(t)}{df_i(t)}=0,\ \forall\ i\neq j\ and\ i,j\in \mathrm{I} \tag{34}$$

Endogenous regulator instrument- generalized optimality condition

$$\frac{dV_i(t)}{df_i(t)}\theta_i(t)+\frac{d\theta_i(t)}{df_i(t)}V_i(t)+\theta_j(t)\sum_{j=1}^{I}\frac{dV_j(t)}{df_i(t)}+\lambda\frac{dG(t)}{df_i(t)}=0,\ \forall\ i\neq j\ and\ i,j\in \mathrm{I} \tag{35}$$

General co-state condition for exogenous and endogenous regulator instruments

$$\dot{\lambda}=\left(\delta-\frac{dG(t)}{dg(t)}\right)\lambda-\frac{d\left[\sum_{i=1}^{I}\left(V_i(t)\theta_i\right)\right]}{dg(t)} \tag{36}$$

This cooperation enforcing formalization, as discussed, has interesting similarities with the non cooperative case described in the previous sections. In the simplified case of uncoupled technologies and linear arguments, the controls of the regulator without no instrument, match those of the non cooperative differential game considered in section *3.2.*, although the co-state condition is now a measure of all the marginal adjustments and not of specific player dynamics. Still, the most interesting feature arising from this generalization is the fact that it

[8]Bressan (2009) extends analytically this class of uncoupled differential games for the two player case.

allows for a benchmark case to compare with existing global cooperation policy frameworks and to introduce the issue of endogenous policy instruments. Of course, the usefulness of these two possible welfare benchmarks should always be taken into consideration within our base hypothesis of non cooperative dynamic environmental games.

5. *Conclusion and further research*

In this paper, we introduced a fully generalized framework for climate negotiations in a differential game setup. We discussed its links with evolutionary dynamic game theory and extended our setup to show the relation of our proposal with some of the models proposed in recent literature, such as the Dutta and Radner (2004, 2006a) proposal, for which we provided a dynamic analysis based on analytical characterization and dynamical simulations that we believe are relevant for both academic and policy purposes. Further, we provided a generic framework for the existence of a regulator in this setup, which can be used to analyse existent institutions, in order to evaluate welfare outcomes arising from cooperative and non-cooperative games. Future research should focus on such applied proposals, in order to obtain meaningful results for dynamic characterization. Furthermore, it is also fundamental to bear in mind the need for improved simulation procedures to take into account more complex game dynamics, within setups that allow for the possibility of coalition formation among blocks of countries that share the same common objectives, such as the simplified EKC hypotheses with two and four players that we present in section *3.2.*, one near environmental policy maturity and another one far away. Both these goals will certainly drive this research field in the future.

Appendix

1. *Conditions for existence of closed loop solutions in differential games*

Following Basar and Olsder (1982), the conditions for the existence of a closed loop feedback solution, considering a set valued function $\eta_i(.,t)$, with a state trajectory of the system defined by $x(t)$ and considering a terminal infinite state $T=\infty$ ($T=\min\{t\in\mathbb{R}^+:(x(t),t)\in\Lambda\}$), where $\Lambda\subset S^0\times\mathbb{R}^+$, is a closed subset, usually called terminating target in infinite set of time, given I-tuple strategies), are[9]:

Closed loop information structure definition:

i. *Closed-loop perfect state (CLPS) pattern if:*

$$\eta_i(t)=\{x(s),0\le s\le t\},\ t\in[0,T] \tag{37}$$

ii. *Feedback perfect state (FB) pattern if:*

$$\eta_i(t)=\{x(t)\},\ t\in[0,T] \tag{38}$$

Conditions for a I-player differential game with infinite horizon in continuous time:

i. *An index set* $\mathrm{I}=\{1,...,I\}$ *called the players set*

ii. *A time interval* $[0,T]$, *which is specified a priori and which denotes the duration of the evolution of the game (In this case* $T\in\mathbb{R}^+$*)*

iii. *An infinite set* S_0 *with some topological structure, called the trajectory of the game. Its elements are denoted as* $\eta_i(t)=\{x(t)\}$, $t\in[0,T]$ *and constitute the permissible state trajectories of the game. Furthermore, for each fixed* $t\in[0,T]$, $x(t)\in S^0$, *where* S^0 *is a subset of a finite dimensional vector space, say* $\mathbb{R}^n$

[9]The following definitions are described in pages 210 to 212 of Basar and Olsder (1982) book. Except for some slight changes to adapt to this framework the set of definitions in italic follows closely the text and analytics described in this reference.

iv. *An infinite set* U_i *with some topological structure, defined for each* $i \in \mathrm{I}$ *and which is called the control action space of player* i *, whose elements,* $\{u_i(t),\ 0 \le t \le T\}$ *are the control functions for each player. Furthermore, there exists a set* $S^i \subseteq \mathbb{R}^{m_i}$, $(i \in \mathrm{I})$ *so that, for each fixed* $t \in [0,T]$, $u_i(t) \in S^i$

v. *A differential equation,*

$$\frac{dx(t)}{dt} = f(t, x(t), u_1(t), ..., u_I(t)),\ x(0) = x_0 \tag{39}$$

whose solution describes the state trajectory of the game corresponding to the I-tuple of control functions $\{u_i(t),\ 0 \le t \le T\}$ $(i \in \mathrm{I})$ *and the given initial state* x_0

vi. *A set value function* $\eta_i(.)$ *defined for each* $i \in \mathrm{I}$ *as*

$$\eta_i(t) = \{x(s),\ 0 \le s \le \varepsilon_t^i\},\ 0 \le \varepsilon_t^i \le t \tag{40}$$

Where ε_t^i *in non decreasing in* t *and* $\eta_i(t)$ *determines the state information gained and recalled by each player at time* $t \in [0,T]$. *This specification is the information structure of the game for the CLPS case*

vii. *A sigma field* I_t^i *in* S_0 *generated for each* $i \in \mathrm{I}$ *by the cylinder sets* $\{x \in S_0,\ x(s) \in B\}$ *where B is a Borel set in* S^0 *and* $0 \le s \le \varepsilon_t$. N_t^i, $t \ge t_0$, *is called the information of each player*

viii. *A prespecified class* Γ^i *of mappings (strategy space of player i)* $\gamma^i : [0,T] \times S_0 \to S^i$, *with the property that* $u_i(t) = \gamma^i(t,x)$ *is* N_t^i*-measurable (strategies are adapted to the information field)*

ix. *Two functionals* $q^i : S^0 \to \mathbb{R},\ g^i : [0,T] \times S^0 \times S^1 \times ... \times S^I \to \mathbb{R}$ *defined for each* $i \in \mathrm{I}$, *so that the composite functional:*

$$L^i(u_1, ..., u_I) = \int_0^T g^i(t, x(t), u_1(t), ..., u_I(t))dt + q^i(x(T)) \tag{41}$$

is well defined for every $u_j(t) = \gamma^j \in \Gamma^j\ (j \in \mathrm{I})$ *and for each* $i \in \mathrm{I}$.

To these set of definitions one must add the following additional set of definitions to cover feedback perfect state family of strategies, taking into account the definitions stated above. Let the information structure for each player be any of the information patterns stated above. Furthermore, let $S_0 = C^i[0,T]$. *Then if:*

x. $f(t, x, u_1, ..., u_I)$ *is continuous in* $t \in [0,T]$ *for each* $x \in S^0$, $u_i \in S^i$, $i \in \mathrm{I}$

xi. $f(t, x, u_1, ..., u_I)$ *is uniformly Lipschitz in* $x, u_1, ..., u_I$, *i.e., for some* $k \succ 0$,

$$\|f(t, x, u_1, ..., u_I) - f(t, \bar{x}, \bar{u}_1, ..., \bar{u}_I)\| \le k \max_{0 \le t \le T} \left\{ \|x(t) - \bar{x}(t)\| + \sum_{i \in \mathrm{I}} \|u_i(t) - \bar{u}_i(t)\| \right\},$$

$$x(.), \bar{x}(.) \in C^i[0,T]; u_i(.) \in U_i\ \ (i \in \mathrm{I})$$

xii. *For* $\gamma^i \in \Gamma^i$ $(i \in \mathrm{I})$, $\gamma^i(t,x)$ *is continuous in* t *for each* $x(.) \in C^i[0,T]$ *and uniformly Lipschitz in* $x(.) \in C^i[0,T]$, *the differential equation describing the state of the system, (39), admits a unique solution, and this unique trajectory is continuous*

2. *Optimal control extended conditions for generalized climate differential games*

Feedback perfect state extended conditions (FB pattern)

$$\bar{\lambda}_i = \left(\frac{dV_i(t)}{dY_i(t)} \frac{dY_i(t)}{dg(t)} + \frac{dV_i(t)}{dF_i(t)} \frac{dF_i(t)}{dg(t)} \right) \left(\delta - \frac{dG(t)}{dg(t)} \right)^{-1} \tag{42}$$

$$\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{df_i(t)}+\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{df_i(t)}+\frac{dG(t)}{dA(t)}\frac{dA(t)}{df_i(t)}\left(\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{dg(t)}+\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{dg(t)}\right)\left(\delta-\frac{dG(t)}{dg(t)}\right)^{-1}=0 \quad (43)$$

Closed loop perfect state extended conditions (CLPS pattern)

$$\lambda_i=-\left(\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{df_i(t)}+\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{df_i(t)}\right)\left(\frac{dG(t)}{dA(t)}\frac{dA(t)}{df_i(t)}\right)^{-1}\Rightarrow\dot{\lambda}_i=\frac{d\left[-\left(\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{df_i(t)}+\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{df_i(t)}\right)\left(\frac{dG(t)}{dA(t)}\frac{dA(t)}{df_i(t)}\right)^{-1}\right]}{dt} \quad (44)$$

$$\frac{d\left[-\left(\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{df_i(t)}+\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{df_i(t)}\right)\left(\frac{dG(t)}{dA(t)}\frac{dA(t)}{df_i(t)}\right)^{-1}\right]}{dt}= \quad (45)$$

$$=\left(-\left(\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{df_i(t)}+\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{df_i(t)}\right)\left(\frac{dG(t)}{dA(t)}\frac{dA(t)}{df_i(t)}\right)^{-1}\right)\left(\delta-\frac{dG(t)}{dg(t)}\right)-\frac{dV_i(t)}{dY_i(t)}\frac{dY_i(t)}{dg(t)}-\frac{dV_i(t)}{dF_i(t)}\frac{dF_i(t)}{dg(t)}$$

3. Linearized dynamics and characteristic equation for the uncoupled linear climate game

Generic Jacobian matrix for linearized Dutta and Radner (2004, 2006a) dynamical system

$$J=\begin{bmatrix}
-(1-\sigma) & f_{best} & \cdots & f_{best} & \bar{e}_1 & \cdots & \bar{e}_I \\
0 & \left.\frac{d\dot{e}_1(t)}{de_1(t)}\right|_{e_1(t)=\bar{e}_1} & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \iddots & \vdots \\
0 & \cdots & 0 & \left.\frac{d\dot{e}_I(t)}{de_I(t)}\right|_{e_I(t)=\bar{e}_I} & 0 & \cdots & 0 \\
\left.\frac{d\dot{f}_1(t)}{dg(t)}\right|_{\substack{g(t)=\bar{g}\\ e_1(t)=\bar{e}_1\\ f_1(t)=f_{best}}} & 0 & \cdots & 0 & \left.\frac{d\dot{f}_1(t)}{df_1(t)}\right|_{\substack{g(t)=\bar{g}\\ e_1(t)=\bar{e}_1\\ f_1(t)=f_{best}}} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & 0 \\
\left.\frac{d\dot{f}_I(t)}{dg(t)}\right|_{\substack{g(t)=\bar{g}\\ e_I(t)=\bar{e}_I\\ f_I(t)=f_{best}}} & 0 & \cdots & 0 & \cdots & 0 & \left.\frac{d\dot{f}_I(t)}{df_I(t)}\right|_{\substack{g(t)=\bar{g}\\ e_I(t)=\bar{e}_I\\ f_I(t)=f_{best}}}
\end{bmatrix}_{2I+1} \quad (46)$$

Characteristic equation

$$\left(-(1-\sigma)-\lambda\right)\left(\left.\frac{d\dot{e}_1(t)}{de_1(t)}\right|_{e_1(t)=\bar{e}_1}-\lambda\right)\cdots\left(\left.\frac{d\dot{e}_I(t)}{de_I(t)}\right|_{e_I(t)=\bar{e}_I}-\lambda\right)\left(\left.\frac{d\dot{f}_1(t)}{df_1(t)}\right|_{\substack{g(t)=\bar{g}\\ e_1(t)=\bar{e}_1\\ f_1(t)=f_{best}}}-\lambda\right)\cdots\left(\left.\frac{d\dot{f}_I(t)}{df_I(t)}\right|_{\substack{g(t)=\bar{g}\\ e_I(t)=\bar{e}_I\\ f_I(t)=f_{best}}}-\lambda\right)=0 \quad (47)$$

Eigenvalues

$$\lambda=-(1-\sigma)\prec 0,\ \lambda=\left.\frac{d\dot{e}_i(t)}{de_i(t)}\right|_{e_i(t)=\bar{e}_i},\ \lambda=\left.\frac{d\dot{f}_i(t)}{df_i(t)}\right|_{\substack{g(t)=\bar{g}\\ e_i(t)=\bar{e}_i\\ f_i(t)=f_{best}}}\leq 0,\ \forall\ i\in\mathrm{I}\ and\ \bar{g}\succ g^*_{\max} \quad (48)$$

4. Generalized climate differential games with strict dynamic policy strategic decisions

Following the discussion introduced in section *3.* of the main text, relating to the interpretation of the player policy decision, arising from our general proposal for an optimal control problem setup for climate change differential games in the Dutta and Radner (2004, 2006a) fashion, we set up in this appendix section an alternative formalization for defining this proposal, within the optimal control usual definition of dynamic policy decisions. This alternative setup sets policy decisions to be a strategic dynamic variable, as opposed to a strategic policy decision on feasible predetermined technology paths that is obtained when considering the technology function as $F_i\left(\dot{f}_i\left(t\right)\right)$, following the Dutta and Radner (2004, 2006a) discrete time setup transposed to a continuous time optimal control framework. This formalization may be relaxed when a strategic policy function is given by a functional form, following our definition in (2), and the optimal control conditions for the generalized problem are then given by the framework discussed in section *3.1.*.

The main issue regarding this discussion is related to the additional assumptions that one must undertake to ensure that the Dutta and Radner (2004, 2006a) formalization follows the intuition usually obtained from optimal control policy problems, which generically imposes an additional state condition, in order to obtain the usual control policy interpretation in this specific set of applications. Alternatively, one can consider the different interpretation for policy decisions when formalizing problems in this fashion and characterize policy dynamics taking into account this interpretation, following the generic setup we propose in the main text. Still, one cannot assure that the sort of policy rules obtained in our proposed characterization match the policy dynamics obtained from the formalization that we present in this section. Further, some of the differential game sufficient conditions that were imposed throughout our analysis in section *3.2.* might be altered in this setup, since we need to consider an additional state dimension for each player intertemporal optimization problem. All these issues are better understood when the specific optimal control conditions for this differential game are considered. Nevertheless, a careful overview of these modelling implications must be taken into account before any conclusions on the differences arising from these distinct policy interpretations. We finish this appendix section with a generic proposal and brief overview of our discussion in an analytical framework related to the Dutta and Radner (2004, 2006a) proposal and leave an in depth analysis of these issues for a future opportunity.

Generic maximization problem

$$\begin{aligned} &\underset{u}{MAX}\int_0^\infty e^{-\delta t}V_i\left(\underline{x},u,t\right)dt \\ &subject\ to:\ \dot{\underline{x}} = g\left(\underline{x},u,t\right) \end{aligned} \tag{49}$$

Present value Hamiltonian and generic optimal control conditions

$$H\left(\underline{x},u,\underline{\mu},t\right) = f\left(\underline{x},u,t\right) + \underline{\mu}g\left(\underline{x},u,t\right) \tag{50}$$

$$\frac{dH}{du} = 0,\ \frac{dH}{d\underline{x}} = -\dot{\underline{\mu}} + \delta\underline{\mu},\ \frac{dH}{d\underline{\mu}} = \dot{\underline{x}} \tag{51}$$

Definitions and optimal control conditions

Conditions (47) to (49) define the Dutta and Radner (2004, 2006a) proposal in a generic fashion, where we assumed a formalization using vector variables for reasons of simplification. The correspondence of these variables, taking in consideration the definitions we put forward in section *3.* to *3.2.* is given in table 5:

Definition	Control Variable	State Variables	Co-State Variables
Generic	u	$\underline{x}$	$\underline{\mu}$
Correspondence	$F_i\left(t\right)$	$\left(f_i\left(t\right), g\left(t\right)\right)$	$\left(\lambda_i, \lambda_{i,g}\right)$

Table 5- Optimal control definitions and variable correspondences

This alternative formalization sets our control variable to be given by the strategy function, which allows us to formalize this differential game assuming technology paths motion equation as an additional state condition, which is defined by the strategic function, following the original definition (14) , scaled by the marginal cost of implementing new technology parameter:

$$\dot{\underline{x}} = \underline{g}\left(\underline{x},u,t\right) \Rightarrow \begin{cases} \dot{f}_i\left(t\right) = \dfrac{F_i\left(t\right)}{\kappa_i} \\ \dot{g}\left(t\right) = G\left(t\right) \end{cases} \tag{52}$$

Still, by imposing an additional state condition defined by the strategic paths motion equation, we have to consider an additional multiplier condition relating the marginal adjustment of strategy to the optimal policy path or policy state. Following this short description, we can define the set of non cooperative optimal control conditions for the Dutta and Radner (2004, 2006a) climate differential game, which are now given by:

$$H = V_i\left(F_i, f_i, g, G, t\right) + \lambda_i \frac{F_i\left(t\right)}{\kappa_i} + \lambda_{i,g} G\left(t\right) \tag{53}$$

Optimal Control Condition

$$\frac{\lambda_i}{\kappa_i} = -\frac{dV_i\left(...\right)}{dF_i\left(t\right)} \tag{54}$$

Co-State conditions

$$\begin{cases} \dot{\lambda}_i = \delta\lambda_i - \dfrac{dV_i\left(...\right)}{df_i\left(t\right)} - \lambda_{i,g}\dfrac{dG\left(t\right)}{df_i\left(t\right)} \\ \dot{\lambda}_{i,g} = \left(\delta - \dfrac{dG\left(t\right)}{dg\left(t\right)}\right)\lambda_{i,g} - \dfrac{dV_i\left(...\right)}{dg\left(t\right)} \end{cases} \tag{55}$$

We shall not go further on possible interpretations arising from this formalization, since the introduction of an additional co-state condition as both implications on system dimension and differential game pattern interpretation. Moreover, there are issues related to analytical tractability of this specific formalization that can only be accounted after a careful analysis of the dynamical relations arising from optimal control conditions we considered in this appendix section. It is not the scope of this paper to develop further this specific modelling proposal, however, we decided to introduce this discussion in appendix because we believe the policy interpretation issues we introduced are relevant for future research on the field of climate negotiations and differential games.

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