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Arend, Mario

1 December 2007

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Abstract

In this research note I propose the use of the undetermined coefficients method as an alternative approach to solve the Central Bank optimization problem in a neo-keynesian economy. The advantage of using this method is that it provides a theory as to how rational expectations are constructed, and how shocks in the economy are propagated, in order to find an analytical solution for the interest rate reaction function in an economy with a forward-looking behavior.

1. An Alternative Analytical Solution for the Interest Rate Reaction Function

In general, the literature poses the Central Bank's optimization problem using the linear quadratic method.[1] The usefulness of this method is that it allows dynamic programming of the optimization problems and in some cases provides an analytical solution. Svensson (1998b) has developed a linear quadratic method for models considering a forward-looking behavior similar to the one proposed in the next section. However, this does not have an analytical solution. In the third section, using the undetermined coefficients method, an alternative procedure to solve the Central Bank optimization problem in an economy which has a neo-keynesian structure, is proposed. The advantage of this method is that it provides an analytical solution to the Central Bank's optimization problem, in an economy with a forward-looking behavior, incorporating the possibility of providing a theory into how the rational expectations are constructed, and how shocks in the economy are propagated. According to Rosende (2000), the hypothesis underlying this approach is that it identifies "the relevant theory in order to explain the variable's solution".

2. Economic Structure and the Central Bank's Optimization Process

According to Clarida, Gali and Gertler's (1999) neo-keynesian model, an economy can be described based on the aggregate demand function or IS curve (equation 1) and an inflation

[1] See, Collins et al (2004), Orphanides (1998), Rudebusch et al (1998), Svensson (1997), Svensson (1998a) and Svensson (1998b).

function or Phillips curve (equation 2). These can be represented in the following two equations:

$$y_t - y^* = E_t[y_{t+1} - y^*] - \varphi(i_t - E_t[\pi_{t+1}]) + g_t \quad (1)$$

$$\pi_t - \pi^* = \beta(E_t[\pi_{t+1} - \pi^*]) + \lambda(y_t - y^*) + \mu_t \quad (2)$$

In the first equation, y_t is defined as the logarithm of the product at time t, y^* as the logarithm of the trend product (henceforth, we will refer to the difference y_t- y^* as the output gap), E_t as the expectation operator in period t, φ as the coefficient of the monetary policy real interest rate, i_t as the monetary policy nominal interest rate, π_{t+1} as inflation in the period t +1, and g_t as a positive exogenous aggregate demand shock. In the second equation, π_t is defined as the inflation in period t, π^* as the inflation target (henceforth, we will refer to the difference π_t- π^* as the inflation deviation), β as the coefficient of the future inflation deviation, λ as the coefficient of the output gap and μ_t as a negative exogenous supply shock.

On the other hand, we can describe the Central Bank's behavior from an optimization process that seeks to minimize its loss function. This optimization process aims to establish the monetary policy interest rate that allows the location of inflation in its target and to stabilize the product. The optimization process described next, considers a loss function minimized by the Central Bank, keeping in mind the economic structure that was previously described using equations (1) and (2). This optimization process is described by the following equations:

$$\underset{\pi_t, y_t}{\text{Min}}\ E_t \sum_{t=0}^{\infty} \delta^t L_{T+t} \quad (3)$$

$$L_{T+t} = \frac{1}{2}\left\{ w(\pi_t - \pi^*)^2 + (1-w)(y_t - y^*)^2 \right\} \quad (4)$$

Equation 3 is the Central Bank intertemporal loss function. Where, δ^t is the Central Bank's discount rate at time t which is in the range [0,1], and L_{T+t} is the loss function in the T+t period, where T is the starting period in the optimization process. Equation 4 defines the loss function during the T+t period, according to a weighted average between inflation deviation squared and the output gap squared, where w weights the inflation deviation and 1-w weights the output gap.

There are two ways to solve this optimization problem: commitment and discretion. In the first case, the Central Bank attempts to guide its monetary policy according to a written rule.

In the second option, it periodically chooses which monetary policy rule it will use. Keeping in mind that reality is most similar to this last option; the problem will be solved under discretion.

3. Solving the Central Bank's Optimization Problem Using the Undetermined Coefficients Method

To solve the optimization problem under the undetermined coefficients method, 3 stages are needed:

Stage 1:

$$\underset{\pi_t}{Max} -\frac{1}{2}\left\{w(\pi_t - \pi^*)^2 + (1-w)(y_t - y^*)^2\right\} \tag{5}$$

Subject to the equation (2).
It is solved using a Lagrangian:

$$£ = -\frac{1}{2}\left\{w(\pi_t - \pi^*)^2 + (1-w)(y_t - y^*)^2\right\} + \theta\left\{\pi_t - \pi^* - \beta(E_t[\pi_{t+1} - \pi^*]) - \lambda(y_t - y^*) - \mu_t\right\} \tag{6}$$

Where θ is the Lagrange multiplier.

It is derived with regards to $\pi_t - \pi^*$ y $y_t - y^*$, obtaining the following first order conditions:

$$\frac{\partial £}{\partial(\pi_t - \pi^*)} = -w(\pi_t - \pi^*) + \theta = 0 \tag{7}$$

$$\frac{\partial £}{\partial(y_t - y^*)} = -(1-w)(y_t - y^*) - \theta\lambda = 0 \tag{8}$$

Matching (7) and (8):

$$y_t - y^* = -\frac{w\lambda}{1-w}(\pi_t - \pi^*) \tag{9}$$

Replacing (9) in (2):

$$\pi_t - \pi^* = \frac{1-w}{(1-w)+\lambda^2 w}\beta\left(E_t\left[\pi_{t+1} - \pi^*\right]\right) + \mu_t \tag{10}$$

Stage 2:

To solve the problem using the undetermined coefficients method, rational expectations of $\pi_{t+1} - \pi^*$ must be constructed, which is done as follows:

$$\pi_t - \pi^* = \phi_1\mu_{t-1} + \phi_2\varepsilon_t \tag{11}$$

Where μ_{t-1} is the supply shock in t-1 and ε_t is the white noise error in t, which arises from the following autoregressive process of order 1 (AR[1]):

$$\mu_t = \rho\mu_{t-1} + \varepsilon_t \tag{12}$$

That satisfies the following conditions:

$$\varepsilon_t \sim \mathrm{N}(0,\sigma_\varepsilon^2) \tag{13}$$

$$|\rho| < 1 \tag{14}$$

$$E_t\varepsilon_{t+1} = 0 \tag{15}$$

$$E_t[\varepsilon_t\varepsilon_{t+1}] = 0 \tag{16}$$

Bringing forward the equation (11) one period and getting its rational expectations you obtain:

$$E_t[\pi_{t+1} - \pi^*] = \phi_1\mu_t = \phi_1(\rho\mu_{t-1} + \varepsilon_t) \tag{17}$$

Replacing (17) in (10) and matching with equation (11):

$$\phi_1\mu_{t-1} + \phi_2\varepsilon_t = \frac{1-w}{(1-w)+\lambda^2 w}\{\beta\phi_1(\rho\mu_{t-1} + \varepsilon_t) + \mu_t\}$$

$$\phi_1\mu_{t-1} + \phi_2\varepsilon_t = \frac{1-w}{(1-w)+\lambda^2 w}\{\beta\phi_1(\rho\mu_{t-1} + \varepsilon_t) + \rho\mu_{t-1} + \varepsilon_t\} \tag{18}$$

Ordering the right side of equation (18) you find value for ϕ_1 y ϕ_2:

$$\phi_1 = \frac{\rho(1-w)}{(1-w)+\lambda^2 w + \beta\rho(1-w)} \tag{19}$$

$$\phi_2 = \frac{(1-w)}{(1-w)+\lambda^2 w + \beta\rho(1-w)} \tag{20}$$

Replacing (19) and (20) in the equation (11):

$$\pi_t - \pi^* = \frac{\rho(1-w)}{(1-w)+\lambda^2 w + \beta\rho(1-w)}\mu_{t-1} + \frac{(1-w)}{(1-w)+\lambda^2 w + \beta\rho(1-w)}\varepsilon_t \tag{21}$$

Using the equation (12):

$$\pi_t - \pi^* = \frac{(1-w)}{(1-w)+\lambda^2 w + \beta\rho(1-w)}\mu_t \tag{22}$$

Replacing (22) in the first order condition solution (equation (9)), you get:

$$y_t - y^* = -\frac{w\lambda}{1-w}\left(\frac{(1-w)}{(1-w)+\lambda^2 w+\beta\rho(1-w)}\mu_t\right)$$

$$y_t - y^* = -\frac{w\lambda}{(1-w)+\lambda^2 w+\beta\rho(1-w)}\mu_t \tag{23}$$

Considering an autoregressive process, one can express the future inflation deviation as follows:

$$E_t[\pi_{t+1} - \pi^*] = \rho(\pi_t - \pi^*) \tag{24}$$

Replacing (22) in (24):

$$E_t[\pi_{t+1} - \pi^*] = \rho\frac{(1-w)}{(1-w)+\lambda^2 w+\beta\rho(1-w)}\mu_t \tag{25}$$

Working out the value μ_t:

$$\mu_t = \frac{(1-w)+\lambda^2 w+\beta\rho(1-w)}{\rho(1-w)}E_t[\pi_{t+1} - \pi^*] \tag{26}$$

Stage 3:

Obtaining i_t from the aggregate demand curve (1):

$$\mathrm{i}_t = \frac{1}{\varphi}\left\{\mathrm{E}_t[\mathrm{y}_{t+1} - \mathrm{y}^*] - (\mathrm{y}_t - \mathrm{y}^*) + \varphi\mathrm{E}_t[\pi_{t+1}] + \mathrm{g}_t\right\} \tag{27}$$

Using the expectation of the equation (9) going forward one period and the equation (23) in the equation (27), you obtain:

$$\mathrm{i}_t = \frac{1}{\varphi}\left\{-\frac{w\lambda}{1-w}E_t(\pi_{t+1} - \pi^*) + \frac{w\lambda}{(1-w)+\lambda^2 w+\beta\rho(1-w)}\mu_t + \varphi\mathrm{E}_t[\pi_{t+1}] + \mathrm{g}_t\right\}$$

(28)

Using (26) in (28), you get to:

$$\mathrm{i}_t = \frac{1}{\varphi}\left\{-\frac{w\lambda}{1-w}E_t(\pi_{t+1} - \pi^*) + \frac{w\lambda}{\rho(1-w)}E_t[\pi_{t+1} - \pi^*] + \varphi\mathrm{E}_t[\pi_{t+1}] + \mathrm{g}_t\right\}$$

(29)

Rearranging, you have:

$$\mathrm{i}_t = \frac{1}{\varphi}\left\{-\frac{w\lambda\rho - w\lambda}{\rho(1-w)}E_t(\pi_{t+1} - \pi^*) + \varphi\mathrm{E}_t[\pi_{t+1}] + \mathrm{g}_t\right\}$$

(30)

Splitting the elements of (30) and factoring, you obtain:

$$\mathrm{i}_t = \frac{1}{\varphi}\left\{-\frac{w\lambda\rho - w\lambda - \varphi\rho(1-w)}{\rho(1-w)}E_t(\pi_{t+1}) + \frac{w\lambda\rho - w\lambda}{\rho(1-w)}\pi^* + \mathrm{g}_t\right\}$$

(31)

Subtracting and adding $\varphi\pi^*$ inside the parenthesis of the equation (31) and reordering, you get:

$$\mathrm{i}_t = \frac{1}{\varphi}\left\{-\frac{w\lambda\rho - w\lambda - \varphi\rho(1-w)}{\rho(1-w)}E_t(\pi_{t+1} - \pi^*) + \varphi\pi^* + \mathrm{g}_t\right\}$$

(32)

You can rewrite equation (32) as follows:

$$\mathrm{i}_t = \frac{\varphi\rho(1-w) + w\lambda - w\lambda\rho}{\varphi\rho(1-w)}E_t(\pi_{t+1} - \pi^*) + \pi^* + \frac{1}{\varphi}\mathrm{g}_t$$

(33)

Subtracting $E_t(\pi_{t+1})$ at both sides of equation (33):

$$\mathrm{i}_t - E_t(\pi_{t+1}) = \frac{\varphi\rho(1-w) + w\lambda - w\lambda\rho}{\varphi\rho(1-w)}E_t(\pi_{t+1} - \pi^*) + \pi^* + \frac{1}{\varphi}\mathrm{g}_t - E_t(\pi_{t+1})$$

(34)

Where i_t- $E_t(\pi_{t+1})$ corresponds to the real interest rate of monetary policy that was defined as r_t.

Reordering the equation (34):

$$r_t = \frac{w\lambda - w\lambda\rho}{\varphi\rho(1-w)}E_t(\pi_{t+1} - \pi^*) + \frac{1}{\varphi}\mathrm{g}_t \qquad (35)$$

Making the following definitions:

$$\gamma_1 = \frac{1}{\varphi} \qquad (36)$$

$$\gamma_2 = \frac{w\lambda - w\lambda\rho}{\varphi\rho(1-w)} \qquad (37)$$

The reaction function could be expressed as follows:

$$r_t = \gamma_1\mathrm{g}_t + \gamma_2 E_t(\pi_{t+1} - \pi^*) \qquad (38)$$

5. Conclusion

In this research note an alternative solution for the Central Bank's optimization problem is proposed using the undetermined coefficients method. In this case, the interest rate reaction function is the same as that obtained by other procedures (see, Clarida, Gali and Gertler (1999)), but has the advantage of providing a theory of how rational expectations are constructed, and how shocks in the economy are propagated. The undetermined coefficient procedure leaves open the possibility that the rational expectations differ from the structure of the economy and this way new elements can be introduced to the analysis of the Central Bank's optimization problem.

References

Clarida, R., Galí, J. & Gertler, M. (1999), "The Science of Monetary Policy: A New Keynesian Perspective", Journal of Economic Literature, Volume 37, Nº 4, December 2004.

Collins, S. & Siklos, P., "Optimal Monetary Policy Rules and Inflation Targets: Are Australia, Canada, and New Zeland Different from the US?", Open Economies Review, March 2004.

Orphanides, A. (1998), "Monetary Policy Evaluation With Noisy Information", Board of Governors of Federal Reserve System, October 1998.

Rosende, F. (2000), "Teoría Macroeconómica: Ciclos Económicos, Crecimiento e Inflación", Universidad Católica de Chile Editions, March 2000.

Rudebusch, G. & Svensson, L. (1998), "Policy Rules for Inflation Targeting", NBER Conference on Monetary Policy Rules, Febraury 1998.

Svensson, L. (1997), "Inflation Targeting: Some Extensions", Institute for International Economics Studies, Stockholm University, Febraury 1997.

Svensson, L. (1998a), "Inflation Targeting as a Monetary Policy Rule", National Bureau of Economic Research, Working Paper 6790, November 1998.

Svensson, L. (1998b), "Open Economy Inflation Targeting", Institute for International Economics Studies, Stockholm University, April 1998.