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Abstract We consider communication situations games being the combination of a TU-game and a communication graph. We study the average tree (AT) solutions introduced by Herings *et al.* [9] and [10]. The AT solutions are defined with respect to a set, say $\mathscr{T}$, of rooted spanning trees of the communication graph. We characterize these solutions by efficiency, linearity and an axiom of $\mathscr{T}$-hierarchy. Then we prove the following results. Firstly, the AT solution with respect to $\mathscr{T}$ is a Harsanyi solution if and only if $\mathscr{T}$ is a subset of the set of trees introduced in [10]. Secondly, the latter set is constructed by the classical **DFS** algorithm and the associated AT solution coincides with the Shapley value when the communication graph is complete. Thirdly, the AT solution with respect to trees constructed by the other classical algorithm **BFS** yields the equal surplus division when the communication graph is complete.

Keywords Communication situations · average tree solution · Harsanyi solutions · **DFS** · **BFS** · Shapley value · equal surplus division

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1 Introduction

Game theoretical models regularly address the problem of cooperative behavior assuming that every agent is an autonomous acting individual. Under this assumption, every group of individuals has to be regarded as a formable coalition. If one discards the assumption that individuals are completely free in forming coalitions, one arrives at refinements which incorporate certain constraints in coalition formation. Examples are games with permission structures (Gilles *et al.* [6]), games with precedence constraints (Faigle and Kern [5]) and games on regular systems (Lange and Grabisch [12]). Such constraints can also be motivated by restrictions on communication. Myerson [15] is the first to model this feature by an undirected graph that describes the limited communication possibilities open to the agents. In his model, a coalition can only form if it is connected, i.e. if its members can communicate without the help of outside individuals. The combination of a cooperative game and a communication graph is called a communication situation.

This article studies solutions for communication situations. One such solution, called the average tree solution, has been recently introduced and characterized by Herings *et al.* [9] and Herings *et al.* [10]. As for the Shapley value (Shapley, [16]) for cooperative games with transferable utilities, this solution relies on specific marginal contribution vectors. Nevertheless, instead of considering orderings on the set of the agents, Herings *et al.* [9] construct rooted spanning trees that induce partial orders on the set of agents. Each rooted spanning tree can be seen as a communication hierarchy that singles out a unique agent, called the root, and assigns to each other agent a unique superior.[1] Any agent can communicate with the root by communicating iteratively with superiors. The more intermediary agents necessary to communicate with the root, the more delegated the communication hierarchy. The simplest communication hierarchies are the standard principal-agent model in which the principal is the root and is the unique superior of any other agent, and the chain that totally orders the agents. These types of communication hierarchies are the least and most delegated respectively.

A solution is called an average tree solution if it is the average of the contribution vectors over a nonempty set of rooted spanning trees. Herings *et al.* [9] restrict their analysis to cycle-free communication situations and consider the set of all rooted spanning trees. They show that the corresponding average tree solution is the only component efficient solution such that deleting a link between two agents yields for both resulting components the same average change in allocation, where the average is taken over the agents in the component. For arbitrary communication graph, Herings *et al.* [10] construct

[1] Following van den Brink [17], communication hierarchies and organizational hierarchies are distinguished in this article. He defines a (organizational) hierarchy as a permission structure on the agent set which determines the set of coalitions allowed to form depending on various requirements that can be imposed on the presence of superiors in the coalition. Communication hierarchies studied in this article are called communication situations in van den Brink [17] and corresponds to hierarchies as defined by Demange [4].

a specific set of admissible rooted spanning trees. The induced average tree solution coincides with the Shapley value when the underlying graph is complete and with the average tree solution as defined by Herings *et al.* [9] when the underlying graph is cycle-free.

In this article, we extend the work of Herings *et al.* [10] by allowing any nonempty set of rooted spanning trees $\mathscr{T}$ in the definition of the average tree solution. Our main objectives are to study the distribution of the Harsanyi dividends and to highlight a connection between communication hierarchies introduced in [10] and tree-growing algorithms. As a preliminary result, we provide the first characterization of the average tree solution for arbitrary communication situations. This result relies on the classical axioms of component efficiency and linearity, and on a hierarchical axiom. This third axiom states that for unanimity games, the difference of allocation between two agents is explained by the number of times their positions are decisive in a set of rooted spanning tree. From this characterization, we obtain a useful expression of the average tree solution. This expression is used to show that the average tree solution with respect to $\mathscr{T}$ is a Harsanyi solution (see van den Brink *et al.* [19]) if and only if $\mathscr{T}$ is a subset of the set introduced in [10].

Then, we study the connections between the average tree solutions and two well-known tree-growing algorithms called **DFS** (for Depth-First Search) and **BFS** (for Breadth-First Search) respectively. Such algorithms explore the communication graph so as to construct rooted spanning trees by growing a tree, one agent and one link at a time. In **DFS**, agents are explored out of the most recently visited agent who still has unvisited neighbors. Thus, the rooted spanning trees constructed by **DFS** are the most delegated communication hierarchies of a communication graph. Algorithm **BFS** systematically explores the links of the graph in order to visits every unexplored agent that is reachable from an initial agent. Therefore, **BFS** constructs the most centralized communication hierarchies of a graph. We prove that the set of trees in [10] is constructed by **DFS** and that the associated average tree solution coincides with the Shapley value when the communication graph is complete. We also show that the average tree solution with respect to trees constructed by **BFS** yields the equal surplus division when the communication graph is complete. In a sense, the difference between the Shapley value and the equal surplus division can be seen as a difference between delegated and and centralized communication hierarchies. While this difference is highlighted by comparable axiomatic characterizations in van den Brink [18], it is illustrated in terms of circulation of the information through a communication hierarchy in this article. This aspect is discussed in the last section of the article.

The rest of the article is organized as follows. Section 2 contains the definitions and notations. The axiomatic characterization of the average tree solutions is given in section 3. The results on the distribution of Harsanyi dividends are proved in section 4. We point out the connection between the average tree solutions and tree-growing algorithms in section 5. Section 6 concludes.

2 Preliminaries

2.1 TU-games

Let $N = \{1, \dots, n\}$ be a finite set of players. A *cooperative game with transferable utility* on N, or simply TU-game, is a *characteristic function* $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. For each $S \subseteq N$, $v(S)$ is the *worth* of *coalition* S. The set of all TU-games on N is denoted by Γ_N. A TU-game $v \in \Gamma_N$ is *additive* if $v(S) = \sum_{i \in S} v(\{i\})$ for any nonempty $S \in 2^N$. A *payoff vector* $x \in \mathbb{R}^n$ is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. A *solution* on Γ_N is a function $f : \Gamma_N \longrightarrow \mathbb{R}^n$ that assigns a payoff vector $f(v) \in \mathbb{R}^n$ to each $v \in \Gamma_N$. For any $S \in 2^N \backslash \{\emptyset\}$, the *unanimity game* u_S is given by $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. It is well-known that the collection of unanimity games forms a basis for Γ_N. Hence for each TU-game $v \in \Gamma_N$ we have $v = \sum_{S \in 2^N \backslash \{\emptyset\}} \Delta_v(S) u_S$, where the coefficients $\Delta_v(S)$ are called the *Harsanyi dividends* of v, see Harsanyi [8].

A well-studied solution on Γ_N is the Shapley value (Shapley [16]). An *ordering* is a bijective function π on N, where $\pi(i)$ is the player at position $i \in \{1, \dots, n\}$ in π. For any ordering π on N define $S_i^\pi = \{\pi(1), \pi(2), \dots, \pi(i)\}$ and $S_0^\pi = \emptyset$. For any $v \in \Gamma_N$ consider the *marginal contribution* vector $m^\pi(v) \in \mathbb{R}^n$ defined as $m^\pi_{\pi(i)}(v) = v(S_i^\pi) - v(S_{i-1}^\pi)$ for each $i \in N$. The *Shapley value* is the solution Sh that assigns to each TU-game $v \in \Gamma_N$ the average of all $n!$ marginal contribution vectors $m^\pi(v)$:

$$\forall v \in \Gamma_N, \qquad \mathrm{Sh}(v) = \frac{1}{n!} \sum_{\pi} m^\pi(v) \tag{1}$$

Another solution on Γ_N is the *equal surplus division* ESD, which first assigns to each player $i \in N$ his stand-alone payoff $v(\{i\})$ and then distributes the remainder of $v(N)$ equally among all players in N:

$$\forall v \in \Gamma_N, \forall i \in N, \qquad \mathrm{ESD}_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n} \tag{2}$$

2.2 Communication situations

An *undirected graph* is a pair (N, L) where N is a set of nodes and L is a collection of *links*, i.e. $L \subseteq L^N$ where $L^N = \{\{i, j\} : i, j \in N, i \neq j\}$. For ease of notation we write ij instead of $\{i, j\}$. For each $S \subseteq N$, $L(S) = \{ij : i \in S, j \in S\}$ is the set of links between nodes of S. The graph $(S, L(S))$ is the subgraph of (N, L) induced by S. A sequence of distinct nodes $(i_1, \dots, i_k)$ is a *path* in (N, L) if $i_q i_{q+1} \in L$ for each $q = 1, \dots, k-1$. Two nodes i and j are *connected* in (N, L) if $i = j$ or there exists a path $(i_1, \dots, i_k)$ with $i_1 = i$ and $i_k = j$. A graph (N, L) is *connected* if any two nodes $i, j \in N$ are connected. A *tree* is a connected graph (N, L) such that for each link $ij \in L$, the graph $(N, L \backslash \{ij\})$ is not connected. A subset S of N is connected in (N, L)

if $(S, L(S))$ is a connected graph. The empty subset $\emptyset$ is trivially connected. Denote by $C(L)$ the set of connected subsets of N in (N, L). A subset $K \subseteq N$ is a *component* of (N, L) if $(K, L(K))$ is maximally connected, i.e. if $(K, L(K))$ is connected and for each $i \in N \backslash K$, $(K \cup \{i\}, L(K \cup \{i\}))$ is not connected. The collection of components of (N, L), denoted by N/L, forms a partition of N. A graph (N, L) is a *forest* if for each component $K \in N/L$, $(K, L(K))$ is a tree.

The combination of a TU-game and of a communication graph is a so-called *communication situation* (Myerson [15]), given by a triple (N, v, L) where N is the set of players, v is the characteristic function on N and L the set of links on N. In most of this article we consider communication situations on N such that (N, L) is a connected graph. Let $\mathcal{C}_N$ denote the set of all such communication situations. Also, let $\mathcal{C}_N^*$ be the set of all communication situations on N such that (N, L) is a forest. Denote by K_N the complete graph on N and by $\mathcal{C}_{K_N}$ the class of all communication situations on N with a complete communication graph. As for TU-games we omit N in our notation. Let $\mathcal{C}$ be any class of communication situations on N. A *solution* on $\mathcal{C}$ is a function f that assigns to each $(v, L) \in \mathcal{C}$ a payoff vector $f(v, L) \in \mathbb{R}^n$.

For any communication situation $(v, L) \in \mathcal{C}$, the Myerson restricted game $v^L \in \Gamma_N$ associated with (v, L) is defined as $v^L(S) = \sum_{T \in S/L(S)} v(T)$ for each $S \in 2^N$. Fix any communication graph (N, L). The collection $\{u_S : S \in C(L) \backslash \{\emptyset\}\}$ forms a basis for the vector space $\Gamma_N^L = \{v^L : v \in \Gamma_N\}$ of all graph-restricted games constructed from the communication graph (N, L), see Theorem 5.2.1 in Bilbao [2]. It follows that:

$$v^L = \sum_{S \in C(L) \backslash \{\emptyset\}} \Delta_{v^L}(S) u_S \tag{3}$$

In this article, we consider the solutions f on some classes of communication situations $\mathcal{C}$ such that for each possible L, $f(v, L) = g(v^L)$ for some solution g on Γ_N^L. We are interested in the class of Harsanyi solutions, introduced by Vasil'ev [21] for TU-games and studied by van den Brink *et al.* [19] for communication situations. A Harsanyi solution distributes the Harsanyi dividends to the players of the corresponding coalitions according to a sharing function which assigns to each coalition S a sharing vector specifying for each player in S its share in the dividend of S. A *sharing function* on N is a function z which assigns to each graph (N, L) a collection of vectors $z(L) = (z^S(L))_{S \in C(L) \backslash \{\emptyset\}}$ such that for each $S \in C(L) \backslash \{\emptyset\}$, the vector $z^S(L) \in \mathbb{R}^n_+$ satisfies $z_i^S(L) = 0$ for each $i \in N \backslash S$, $z_i^S(L) \geq 0$ for each $i \in S$ and $\sum_{i \in S} z_i^S(L) = 1$. For a given sharing function z and a given graph (N, L), the *Harsanyi payoff vector* $g^z(v^L)$ associated with $v^L \in \Gamma_N^L$ is given by:

$$\forall i \in N, \quad g_i^z(v^L) = \sum_{S \in C(L): i \in S} z_i^S(L) \Delta_{v^L}(S).$$

The *Harsanyi solution* on $\mathcal{C}$ with respect to z, denoted by f^z, is defined as:

$$\forall (v, L) \in \mathcal{C}, \quad f^z(v, L) = g^z(v^L) \tag{4}$$

2.3 The average tree solutions

Demange [4] adapts the marginal contribution vector m^π for TU-games in the context of communication situations by considering rooted spanning trees instead of orderings on the player set. First, for each component K of a graph (N, L), a *spanning tree* on K is a tree on K. A *rooted spanning tree* on K is a directed graph that arises from this spanning tree by selecting a player $r \in K$, called the *root*, and directing all links away from the root. For a given spanning tree on K, each player $r \in K$ is the root of exactly one rooted spanning tree denoted by t_r. For each t_r and each $j \in K\backslash\{r\}$, there is exactly one directed link (i, j). Player i is the unique *predecessor* of j and j is a *successor* of i in t_r. Denote by $s_r(i)$ the possibly empty set of successors of player $i \in K$ in t_r. A player j is a *subordinate* of i in t_r if there is a directed path from i to j, i.e. if there is a sequence of distinct players $(i_1, \ldots, i_k)$ such that $i_1 = i$, $i_k = j$, and, for each $q = 1, \ldots, k-1$, $i_{q+1} \in s_r(i_q)$. The set $S_r(i)$ denotes the union of all subordinates of i in t_r and $\{i\}$.

For each communication situation (v, L), each $K \in N/L$ and each rooted spanning tree t_r on K, Demange [4] defines the *marginal vector* as:

$$\forall i \in K, \qquad m_i^{t_r}(v, L) = v(S_r(i)) - \sum_{j \in s_r(i)} v(S_r(j)) \tag{5}$$

The marginal vector (5) is axiomatized by van den Brink *et al.* [20] for line-graph communication situations and by Khmelnitskaya [11] for forest-graph communication situations. Both articles also study economic applications. Herings *et al.* [9] introduce and characterize the *average tree solution* AT, a solution on $\mathcal{C}_N^*$ that assigns to each $(v, L) \in \mathcal{C}_N^*$, to each component $K \in N/L$ of the forest (N, L) and to each $i \in K$ the average of his contribution (5) over all the $|K|$ rooted spanning trees induced by $(K, L(K))$:

$$\forall (v, L) \in \mathcal{C}_N^*, \forall K \in N/L, \forall i \in K, \qquad \mathrm{AT}_i(v, L) = \frac{1}{|K|} \sum_{r \in K} m_i^{t_r}(v, L) \tag{6}$$

Herings *et al.* [10] extend the definition of the average tree solution to arbitrary communication situations. For each graph, they consider a particular collection of rooted spanning trees which is presented and characterized in section 4.2. We consider a further extension by allowing any nonempty set of rooted spanning trees. Because the marginal vector (5) can be decomposed by the components of a graph, there is no loss of generality to focus on the class $\mathcal{C}_N$ of all communication situations with a connected communication graph. Also define a function $\mathscr{T}$ that assigns to each connected graph (N, L) a nonempty set $\mathscr{T}(L)$ of rooted spanning trees on N. The *average tree* solution $\mathrm{AT}^{\mathscr{T}}(v, L)$ with respect to $\mathscr{T}$ on $\mathcal{C}_N$ is defined as:

$$\forall (v, L) \in \mathcal{C}_N, \forall i \in N, \qquad \mathrm{AT}_i^{\mathscr{T}}(v, L) = \frac{1}{|\mathscr{T}(L)|} \sum_{t_r \in \mathscr{T}(L)} m_i^{t_r}(v, L) \tag{7}$$

3 Axiomatic characterization

Various characterizations of the average tree solution on the class forest-graph communication situations have been provided recently by Herings *et al.* [9], Mishra and Talman [14] and Béal *et al.* [1]. In this section we provide the first characterization the average tree solutions given by (7) on the class $\mathcal{C}_N$. We consider four axioms. The first two axioms are standard.

Efficiency For any $(v, L) \in \mathcal{C}_N$, it holds that $\sum_{i \in N} f_i(v, L) = v(N)$.

Linearity For any $(v, L) \in \mathcal{C}_N$, any $(w, L) \in \mathcal{C}_N$ and any $a \in \mathbb{R}$, it holds that $f(av, L) = af(v, L)$ and $f(v + w, L) = f(v, L) + f(w, L)$.

The third axiom states that the solution should give to each player his stand-alone payoff $v(\{i\})$ in a communication situation (v, L) if the corresponding Myerson restricted game v^L is additive.

Inessential restricted game property For any $(v, L) \in \mathcal{C}_N$ such that v^L is additive, it holds that $f_i(v, L) = v(\{i\})$ for each $i \in N$.

In order to state the fourth axiom, let $\mathscr{T}(L)$ be a nonempty set of rooted spanning trees associated with the communication graph of a communication situation $(v, L) \in \mathcal{C}_N$. For each $S \in 2^N \backslash \{\emptyset\}$ and each $t_r \in \mathscr{T}(L)$ denote the smallest subtree of t_r that contains S by $t_r(S)$ and its subroot by $i_{t_r(S)}$. If S is a connected coalition in t_r, then $t_r(S)$ is the subtree of t_r induced by S and so its subroot belongs to S. For a given function $\mathscr{T}$, define function $h^{\mathscr{T}}$ that assigns to each (N, L) the collection of vectors $h^{\mathscr{T}}(L) = (h^{\mathscr{T},S}(L))_{S \in 2^N \backslash \{\emptyset\}}$ such that for each $S \in 2^N \backslash \{\emptyset\}$, each vector $h^{\mathscr{T},S}(L) \in \mathbb{R}^n_+$ and each $i \in N$, $h_i^{\mathscr{T},S}(L)$ is equal to the average number of times player i is the subroot of a subtree that contains S among trees in $\mathscr{T}(L)$. Formally $h_i^{\mathscr{T},S}(L) = |\{t_r \in \mathscr{T}(L) : i = i_{t_r(S)}\}|/|\mathscr{T}(L)|$. The support of $h^{\mathscr{T},S}(L)$ is denoted by $D(h^{\mathscr{T},S}(L)) = \{i \in N : h_i^{\mathscr{T},S}(L) > 0\}$. Observe that the function $h^{\mathscr{T}} : (N, L) \longrightarrow (h^{\mathscr{T},S}(L))_{S \in C(L) \backslash \{\emptyset\}}$ is a sharing function if and only if $D(h^{\mathscr{T},S}(L)) \subseteq S$ for each $S \in C(L) \backslash \{\emptyset\}$. The fourth axiom states that the difference of allocation in (u_S, L) between two players i and j is only explained by the difference between $h_i^{\mathscr{T},S}(L)$ and $h_j^{\mathscr{T},S}(L)$.

$\mathscr{T}$-hierarchy For any $(u_S, L) \in \mathcal{C}_N$, $S \in 2^N \backslash \{\emptyset\}$, any $i \in N$ and any $j \in N$, it holds that $h_i^{\mathscr{T},S}(L) f_j(u_S, L) = h_j^{\mathscr{T},S}(L) f_i(u_S, L)$.

This axiom is inspired by the hierarchical strength axiom introduced by Faigle and Kern [5] in order to characterize a Shapley value for cooperative games with precedence constraints. The first result shows that a solution satisfying linearity and the inessential restricted game property assigns the same payoff vector to the communication situations (v, L) and (v^L, L).

Proposition 1 *If a solution f on $\mathcal{C}_N$ satisfies linearity and the inessential restricted game property, then for each $(v, L) \in \mathcal{C}_N$, $f(v, L) = f(v^L, L)$.*

Proof. Fix any connected communication graph (N, L) and pick any $(v, L) \in \mathcal{C}_N$. By definition of v^L, it holds that $(v - v^L)^L(S) = 0$ for each $S \in 2^N$ so that $(v - v^L)^L$ is additive. Therefore the inessential restricted game property yields $f_i(v - v^L, L) = 0$ for each $i \in N$. By linearity of f:

$$f(v, L) = f(v - v^L + v^L, L) = f(v - v^L, L) + f(v^L, L),$$

and we conclude that $f(v, L) = f(v^L, L)$. ■

The next result proves that for a given function $\mathscr{T}$, $\mathrm{AT}^{\mathscr{T}}$ is the unique solution on $\mathcal{C}_N$ that satisfies efficiency, linearity and $\mathscr{T}$-hierarchy.

Theorem 1 *For each function $\mathscr{T}$, the average tree solution $AT^{\mathscr{T}}$ is the unique solution on $\mathcal{C}_N$ that satisfies efficiency, linearity and $\mathscr{T}$-hierarchy.*

Proof. First we prove uniqueness of the solution. So, consider any function $\mathscr{T}$ and consider any solution f on $\mathcal{C}_N$ that satisfies efficiency, linearity and $\mathscr{T}$-hierarchy. Pick any $(v, L) \in \mathcal{C}_N$ and any $S \in 2^N \backslash \{\emptyset\}$. By definition, $u_S(N) = 1$. By $\mathscr{T}$-hierarchy, $h_i^{\mathscr{T},S}(L) f_j(u_S, L) = h_j^{\mathscr{T},S}(L) f_i(u_S, L)$ for each distinct pair of players i and j. Assume that $i \in D(h^{\mathscr{T},S}(L))$ and $j \notin D(h^{\mathscr{T},S}(L))$. Then, $0 = h_i^{\mathscr{T},S}(L) f_j(u_S, L)$ and so $f_j(u_S, L) = 0$. Thus, efficiency becomes $\sum_{j \in D(h^{\mathscr{T},S}(L))} f_j(u_S, L) = 1$. Combining this equation with the $\mathscr{T}$-hierarchy axiom, we first get for each $i \in D(h^{\mathscr{T},S}(L))$:

$$\sum_{j \in D(h^{\mathscr{T},S}(L))} f_j(u_S, L) = \sum_{j \in D(h^{\mathscr{T},S}(L))} f_i(u_S, L) \frac{h_j^{\mathscr{T},S}(L)}{h_i^{\mathscr{T},S}(L)} = 1,$$

which in turn gives $f_i(u_S, L) = h_i^{\mathscr{T},S}(L)$ for each $i \in D(h^{\mathscr{T},S}(L))$ and each $S \in 2^N \backslash \{\emptyset\}$. Because $\{u_S : S \in 2^N \backslash \{\emptyset\}\}$ is a basis for the vector space Γ_N, we have $v = \sum_{S \in 2^N \backslash \{\emptyset\}} \Delta_v(S) u_S$. By linearity of f, the solution $f(v, L) = \sum_{S \in 2^N \backslash \{\emptyset\}} \Delta_v(S) f(u_S, L)$ in $(v, L) \in \mathcal{C}_N$ is determined in a unique way.

Second, we show that for any function $\mathscr{T}$, the average tree solution $\mathrm{AT}^{\mathscr{T}}(v, L)$ given by (7) satisfies the three axioms. Fix any $(v, L) \in \mathcal{C}_N$.

Efficiency For each $t_r \in \mathscr{T}(L)$, the vector m^{t_r} is efficient (see Theorem 3.7 in Herings *et al.* [9]). Thus the average tree solution $\mathrm{AT}^{\mathscr{T}}$ satisfies efficiency.
Linearity The average tree solution is linear as the average of $|\mathscr{T}(L)|$ contribution vectors.
$\mathscr{T}$-hierarchy Pick any $S \in 2^N \backslash \{\emptyset\}$, any $i \in N$ and any $t_r \in \mathscr{T}(L)$. Observe the following two facts:

1. If $i \neq i_{t_r(S)}$, then either $S_r(i) \not\supseteq S$ or $i_{t_r(S)} \in S_r(i) \backslash \{i\}$. In both cases $m_i^{t_r}(u_S, L) = 0$.

2. If $i = i_{t_r(S)}$, then $S_r(i) \supseteq S$ and so $u_S(S_r(i)) = 1$. Because $t_r(S)$ is the smallest subtree of t_r that contains S, we have $S_r(j) \not\supseteq S$ and so $u_S(S_r(j)) = 0$ for each $j \in s_r(i)$. Hence, $m_i^{t_r}(u_S, L) = 1$.

For each $S \in 2^N \backslash \{\emptyset\}$ and each $i \in N$, facts 1. and 2. imply that

$$\begin{aligned}
\mathrm{AT}_i^{\mathscr{T}}(u_S, L) &= \frac{1}{|\mathscr{T}(L)|}\left(\sum_{\substack{t_r \in \mathscr{T}(L): \\ i = i_{t_r(S)}}} m_i^{t_r}(u_S, L) + \sum_{\substack{t_r \in \mathscr{T}(L): \\ i \neq i_{t_k(S)}}} m_i^{t_r}(u_S, L)\right) \\
&= \frac{1}{|\mathscr{T}(L)|} \sum_{\substack{t_r \in \mathscr{T}(L): \\ i = i_{t_r(S)}}} 1 \\
&= h_i^{\mathscr{T},S}(L),
\end{aligned}$$

so that for any $i \in N$ and any $j \in N$, the equality $h_i^{\mathscr{T},S}(L)\mathrm{AT}_j^{\mathscr{T}}(u_S, L) = h_j^{\mathscr{T},S}(L)\mathrm{AT}_i^{\mathscr{T}}(u_S, L)$ holds. ■

By linearity, $\mathrm{AT}^{\mathscr{T}}$ on $\mathcal{C}_N$ can be written as follows:

$$\forall (v, L) \in \mathcal{C}_N, \forall i \in N, \qquad \mathrm{AT}_i^{\mathscr{T}}(v, L) = \sum_{\substack{S \in 2^N: \\ i \in D(h^{\mathscr{T},S}(L))}} h_i^{\mathscr{T},S}(L) \Delta_v(S) \qquad (8)$$

In order to determine whether $\mathrm{AT}^{\mathscr{T}}$ is a Harsanyi solution, we provide an alternative expression of this solution. For any $\mathscr{T}$, it is easy to see that $\mathrm{AT}^{\mathscr{T}}$ satisfies linearity and the inessential restricted game property.[2] In fact, consider any $(v, L) \in \mathcal{C}_N$ such that v^L is additive. For any $t_r \in \mathscr{T}(L)$ and any $i \in N$ we have:

$$m_i^{t_r}(v, L) = v^L(S_r(i)) - v^L(S_r(i) \backslash \{i\}) = v^L(\{i\}) = v(\{i\}),$$

so that (7) ensures that the axiom is satisfied. Therefore, we can use the previous results obtained in this section to prove the following statement.

Corollary 1 *For any $\mathscr{T}$, $AT^{\mathscr{T}}$ on $\mathcal{C}_N$ can be written as follows:*

$$\forall (v, L) \in \mathcal{C}_N, \forall i \in N, \qquad AT_i^{\mathscr{T}}(v, L) = \sum_{\substack{S \in C(L): \\ i \in D(h^{\mathscr{T},S}(L))}} h_i^{\mathscr{T},S}(L) \Delta_{v^L}(S) \qquad (9)$$

Moreover, $AT^{\mathscr{T}}$ is a Harsanyi solution on $\mathcal{C}_N$ if and only if, for each connected graph (N, L) and each $S \in C(L)$, it holds that $D(h^{\mathscr{T},S}(L)) \subseteq S$.

[2] Observe that the Myerson value (Myerson [15]) and the position value (Borm *et al.* [3]) also satisfy linearity and the inessential restricted game property.

Proof. Fix a function $\mathscr{T}$ and any $(v, L) \in \mathcal{C}_N$. By linearity of $\mathrm{AT}^{\mathscr{T}}$ and (3), we have $\mathrm{AT}^{\mathscr{T}}(v^L, L) = \sum_{S \in C(L)\setminus\{\emptyset\}} \Delta_{v^L}(S)\mathrm{AT}^{\mathscr{T}}(u_S, L)$. Because $\mathrm{AT}^{\mathscr{T}}$ satisfies the inessential restricted game property, the equality $\mathrm{AT}^{\mathscr{T}}(v, L) = \mathrm{AT}^{\mathscr{T}}(v^L, L)$ follows from Proposition 1. From the proof of Theorem 1 we have, for each $S \in 2^N\setminus\{\emptyset\}$, $\mathrm{AT}_i^{\mathscr{T}}(u_S, L) = h_i^{\mathscr{T},S}(L)$ if $i \in D(h^{\mathscr{T},S}(L))$ and $\mathrm{AT}_i^{\mathscr{T}}(u_S, L) = 0$ otherwise, which proves the first part of the result. The moreover part follows from expressions (4) and definition (9) of $\mathrm{AT}^{\mathscr{T}}$. ■

In the next two sections we will expose the properties on $\mathscr{T}$ such that the condition $D(h^{\mathscr{T},S}(L)) \subseteq S$ for each connected graph (N, L) and each $S \in C(L)$ is met. Before concluding this section, it is interesting to illustrate why expressions (8) and (9) coincide. In (8), $\mathrm{AT}^{\mathscr{T}}$ is obtained by summing on the set of all coalitions and considering coefficients of v. In (9) the sum is computed on the smaller set of all connected coalitions, but with coefficients of v^L: the changes in coefficients between games v^L and v compensate the smaller number of arguments in the sum. As an example, consider the player set $N = \{1, 2, 3\}$ and the communication situation $(v, L) \in \mathcal{C}_N$ such that:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	2	1	1	4	1	6

and $L = \{12, 23\}$. Assume that $\mathscr{T}(L)$ contains all the three rooted spanning trees of (N, L). Then we have:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$\Delta_v(S)$	0	2	1	-1	3	-2	3
$v^L(S)$	0	2	1	1	1	1	6
$\Delta_{v^L}(S)$	0	2	1	-1	–	-2	6
$h_1^{\mathscr{T},S}(L)$	3/3	0	0	1/3	1/3	0	1/3
$h_2^{\mathscr{T},S}(L)$	0	3/3	0	2/3	1/3	2/3	1/3
$h_3^{\mathscr{T},S}(L)$	0	0	3/3	0	1/3	1/3	1/3

Now consider player 2. All coalitions except $\{1, 3\}$ are connected in (N, L). Thus, among the coalitions that contain player 2, only N has a different dividend in v and v^L. However the compensation occurs because 2 has a positive weight $h_2^{\mathscr{T},\{1,3\}}$ for the disconnected coalition $\{1, 3\}$ to which he does not belong. While this coalition has no dividend in v^L, it has a dividend of 3 in v. Therefore, player 2 does not benefit anymore from the share 1/3 of the dividend 3 when AT is computed from the dividends of v^L. However the reduction of the worth of $\{1, 3\}$ in v^L also increases the dividend of coalition N in v^L of the same amount, which compensates exactly the loss of player 2 on the dividend of coalition $\{1, 3\}$. Formally, $\mathrm{AT}_2(v, L)$ is:

$$\frac{3}{3}2 + \frac{2}{3}(-1) + \frac{2}{3}(-2) + \frac{1}{3}3 + \frac{1}{3}3 = 3,$$

and is equal $\mathrm{AT}_2(v^L, L)$:

$$\frac{3}{3}2 + \frac{2}{3}(-1) + \frac{2}{3}(-2) + \frac{1}{3}6 = 3.$$

4 Distribution of the Harsanyi dividends

For simplicity, we assume that the communication graph is connected. We first recall the expression of the average tree solution for tree-graph communication situations provided by Herings *et al.* [9] in terms of distribution of the Harsanyi dividends of the restricted game. Then we consider the set of rooted spanning trees introduced by Herings *et al.* [10] in their extension of the average tree solution to arbitrary communication situations. We provide a characterization of this set and use it to prove that the average tree solution $\mathrm{AT}^{\mathscr{T}}$ is a Harsanyi solution if and only if, for each (N, L), $\mathscr{T}(L)$ is a subset of this set.

4.1 Trees

The approach taken by Herings *et al.* [9] for tree-graph communication situations in definition (6) is to consider the function $\mathscr{T}^a$ that assigns the set $\mathscr{T}^a(L)$ of all possible rooted spanning trees of a graph (N, L). Each player i induces exactly one rooted spanning tree t_i on N. Hence $\mathscr{T}^a(L)$ contains exactly n elements. It has been shown (Herings *et al.* [9], Theorem 5.1) that, for each $i \in N$, the corresponding average tree solution, given by (6), can be written as:

$$\mathrm{AT}_i(v, L) = \mathrm{AT}_i^{\mathscr{T}^a}(v, L) = \sum_{\substack{S \in C(L): \\ i \in S}} \frac{1 + p_S^L(i)}{|S| + \sum_{j \in S} p_S^L(j)} \Delta_{v^L}(S)$$

where $p_S^L(j)$, $j \in S$, is the number of players outside S that j represents. Player $j \in S$ represents player k outside S if k is connected to j and, on the unique path connecting j and k, all players between j and k are outside S. Because (N, L) is a tree, it holds that $D(h^{\mathscr{T}^a, S}(L)) = S$, and it is easy to check that:

$$\forall i \in S, \qquad h_i^{\mathscr{T}^a, S}(L) = \frac{1 + p_S^L(i)}{|S| + \sum_{j \in S} p_S^L(j)} \tag{10}$$

Firstly, note that $\{p_S^L(j)\}_{j \in S}$ forms a partition of $N \backslash S$, i.e. $p_S^L(j) \cap p_S^L(i) = \emptyset$ for each $i, j \in S$, $i \neq j$, and $\cup_{j \in S} p_S^L(j) = N \backslash S$. It follows that $|S| + \sum_{j \in S} p_S^L(j) = n$. Secondly, for each $i \in S$ there is a unique $t_i \in \mathscr{T}^a(L)$, and player i is such that $i_{t_i(S)} = i$. This corresponds to the unit in the numerator of (10). Thirdly, for each $t_k \in \mathscr{T}^a(L)$, $t_k(S)$ is the subtree of t_k induced by S. If i represents k, then S is a subset of the set of subordinates of i in t_k so that $i_{t_k(S)} = i$. If i does not represent k, there exists a player $j \in S$ who represents k and so i is a subordinate of j in t_k. This implies that $i \neq i_{t_k(S)}$. Conclude that $i_{t_k(S)} = i$ if and only if i represents k. Therefore, $|\{t_k \in \mathscr{T}^a(L) : i = i_{t_k(S)}\}| = 1 + p_S^L(i)$. Thus, (10) holds.

4.2 Harsanyi trees of arbitrary graphs

In [10], Herings *et al.* [10] consider the average tree solution with respect to a specific set of rooted spanning trees constructed as follows. Fix any communication graph (N, L). Let $B = \{B_i\}_{i \in N}$ be a collection of coalitions satisfying:

1. For each $i \in N$, it holds that $i \in B_i$ and for some $j \in N$, $B_j = N$;
2. For each $i \in N$ and each component K of the subgraph of (N, L) induced by $B_i \backslash \{i\}$, it holds that $K = B_j$ and $ij \in L$ for some $j \in N$.

For a given graph (N, L), any collection B satisfying the above two conditions has the following property (see Lemma 3.2 in Herings *et al.* [10]):

(a) For all $i, j \in N$, $i \neq j$, it holds that either $B_i \subseteq B_j \backslash \{j\}$ or $B_j \subseteq B_i \backslash \{i\}$, or both $B_i \cap B_j = \emptyset$ and $B_i \cup B_j \notin C(L)$.

In addition B induces a unique rooted spanning tree, say t_r^B, such that (i, j) is a directed link of t_r^B if and only if B_j is a component of $B_i \backslash \{i\}$. Therefore, t_r^B is such that $S_r(i) = B_i$ for each $i \in N$. Denote by $\mathscr{T}^B$ the function that assigns to each connected graph (N, L) the set $\mathscr{T}^B(L)$ of all such rooted spanning trees. For reasons that will appear subsequently, each element of $\mathscr{T}^B(L)$ will be called a *Harsanyi tree* of (N, L). The following result provides a simple and useful characterization of $\mathscr{T}^B(L)$.

Proposition 2 *Let (N, L) be any connected communication graph on N. A rooted spanning tree t_r belongs to $\mathscr{T}^B(L)$ if and only if, for each $ij \in L$, it holds that either $i \in S_r(j)$ or $j \in S_r(i)$.*

Proof. Consider any connected graph (N, L), any $t_r \in \mathscr{T}^B(L)$ and any $ij \in L$. We have to show that either $i \in S_r(j)$ or $j \in S_r(i)$. Consider the unique player $k \in N$ such that both $ij \subseteq B_k = S_r(k)$ and for any other player $q \in N$ where $ij \subseteq B_q = S_r(q)$, we have $k \in S_r(q)$. Assume that $k \notin ij$. Because $ij \in L$, condition 2 in the definition of B implies that there exists a successor of k, say $i_k \in s_r(k)$, such that $ij \subseteq S_r(i_k) = B_{i_k}$. This contradicts the definition of k. Conclude that $k \in ij$, which gives the result.

For the converse part, pick any rooted spanning tree t_r of (N, L) such that for each $ij \in L$, it holds that either $i \in S_r(j)$ or $j \in S_r(i)$. We have to show that the collection of coalitions $\{S_r(1), \ldots, S_r(n)\}$ satisfies conditions 1 and 2 described above. Condition 1 follows from definition of $S_r(i)$, $i \in N$.

In order to prove that condition 2 is satisfied, we first show that the collection $\{S_r(1), \ldots, S_r(n)\}$ satisfies property (a). By definition of a rooted spanning tree, for each pair of distinct players $\{i, j\}$, only one of the three possibilities holds: $S_r(i) \subseteq S_r(j) \backslash \{j\}$, $S_r(j) \subseteq S_r(i) \backslash \{i\}$ or $S_r(i) \cap S_r(j) = \emptyset$. The first two possibilities guarantee that the first part of property (a) is satisfied. For the proof of the second part of (a), assume that there is a pair of distinct players $\{i, j\}$ such that $S_r(i) \cap S_r(j) = \emptyset$. Then, for each $i_c \in S_r(i)$ and each $j_c \in S_r(j)$, we have $i_c \notin S_r(j_c)$ and $j_c \notin S_r(i_c)$ and so $i_c j_c \notin L$. Therefore, $S_r(i) \cup S_r(j)$ cannot be a connected coalition of (N, L). We conclude that property (a) holds.

Now pick any $i \in N$ and consider the subgraph $(S_r(i)\backslash\{i\}, L(S_r(i)\backslash\{i\}))$ of (N, L) induced by $S_r(i)\backslash\{i\}$. Assume, for the sake of contradiction, that there exists a connected component K of $(S_r(i)\backslash\{i\}, L(S_r(i)\backslash\{i\}))$ such that $K \neq S_r(j)$ for each $j \in s_r(i)$. Then there necessarily exists a set of distinct players $\{j_1, j_2, \ldots, j_q\}$ included in $s_r(i)$ such that $\{S_r(j_1), S_r(j_2), \ldots, S_r(j_q)\}$ forms a partition of K. Hence the union of at least one pair of elements in $\{S_r(j_1), S_r(j_2), \ldots, S_r(j_q)\}$ must be a connected coalition in (N, L) since K is connected in $(S_r(i)\backslash\{i\}, L(S_r(i)\backslash\{i\}))$. This is a contradiction with property (a), which implies that condition 2 holds. ■

We have the following corollary.

Corollary 2 *If (N, L) is a tree, then the set $\mathscr{T}^B(L)$ of Harsanyi trees coincides with the set $\mathscr{T}^a(L)$ of all rooted spanning trees. If (N, L) is the complete graph K_N, then the set $\mathscr{T}^B(L^N)$ of Harsanyi trees coincides with the set of all $n!$ line-trees, i.e. the set of all rooted spanning trees where each player has at most one successor.*

From Corollary 2, Herings *et al.* [10] exhibit the following two properties of the corresponding average tree solution.

Proposition 3 *(Herings et al. [10], Theorems 3.2 and 3.3) If (N, L) is a tree, then, for each $(v, L) \in \mathcal{C}_N$, the average tree solution defined with respect to $\mathscr{T}^B(L)$ and given by (7) is the average of n contribution vectors and coincides with (6). If (N, L) is the complete graph K_N, then, for $(v, L^N) \in \mathcal{C}_{K_N}$, the average tree solution defined with respect to $\mathscr{T}^B(L^N)$ and given by (7) is the average of $n!$ contribution vectors and coincides with the Shapley value given by (1).*

Theorem 2 below points out another advantage of considering Harsanyi trees. It states that the average tree solution $\mathrm{AT}^{\mathscr{T}}$ is a Harsanyi solution on $\mathcal{C}_N$ if and only if, for each connected communication graph (N, L), $\mathscr{T}(L)$ is a subset of the set $\mathscr{T}^B(L)$ of Harsanyi trees of (N, L).

Theorem 2 *Consider any function $\mathscr{T}$. The average tree solution $AT^{\mathscr{T}}$ is a Harsanyi solution on $\mathcal{C}_N$ if and only if, for each connected communication graph (N, L), $\mathscr{T}(L) \subseteq \mathscr{T}^B(L)$.*

Proof. Consider any function $\mathscr{T}$. Given that the average tree solution $\mathrm{AT}^{\mathscr{T}}$ can be written as (9), Theorem 2 can be proved by showing for each connected graph (N, L), that $D(h^{\mathscr{T},S}(L) \subseteq S$ holds for each $S \in C(L)\backslash\{\emptyset\}$ if and only if $\mathscr{T}(L) \subseteq \mathscr{T}^B(L)$.

Firstly, let $\mathscr{T}$ be such that $\mathscr{T}(L) \subseteq \mathscr{T}^B(L)$ for each connected communication graph (N, L). Fix any connected graph (N, L) and assume, for the sake of contradiction, that there is $i \in D(h^{\mathscr{T},S}(L))\backslash S$ for some nonempty $S \in C(L)$. Then, there is $t_r \in \mathscr{T}(L) \subseteq \mathscr{T}^B(L)$ such that $i = i_{t_r(S)} \in N\backslash S$. Consider the subgraph of t_r induced by S. It follows that this subgraph is a forest.

Moreover for any pair of players in S belonging to distinct components of this subgraph, one player of this pair cannot be the subordinate of the other in t_r. Because $S \in C(L)$, there is at least one such pair of players incident to the same link in (N, L). By Proposition 2, $t_r \notin \mathscr{T}^B(L)$, a contradiction. Conclude that $D(h^{\mathscr{T},S}(L)) \subseteq S$.

Secondly, let $\mathscr{T}$ be such that $\mathscr{T}(L) \nsubseteq \mathscr{T}^B(L)$ for some connected communication graph (N, L). Then there exists $t_r \in \mathscr{T}(L)$ not in $\mathscr{T}^B(L)$. By Proposition 2, there is $ij \in L$ such that neither $i \in S_r(j)$ nor $j \in S_r(i)$. Therefore consider coalition $\{i, j\} \in C(L)$ and $t_r(\{i, j\})$, the smallest subtree of t_r that contains $\{i, j\}$. It follows that $i_{t_r(\{i,j\})}$, the subroot of $t_r(\{i, j\})$, does not belong to coalition $\{i, j\}$. Hence $D(h^{\mathscr{T},\{i,j\}}(L)) \nsubseteq S$ as desired. ■

5 Constructing communication hierarchies

In this section a general algorithm, called **Tree-Growing**, is given for constructing spanning trees of a given graph. It is borrowed from computer science (see Gross and Yellen [7]) and consists in growing a subtree, one link and one player at a time. Then, two particular instances of this algorithms will be considered and connected to the average tree solutions. The associated sets of rooted spanning trees have a meaningful interpretation, which is discussed in the concluding section.

5.1 Tree-growing algorithms

Consider a communication graph (N, L) which is assumed to be connected for the sake of simplicity. The algorithm introduced in this section can be easily applied to the connected components of a non-connected graph. A pair (S, L_S) with $S \in 2^N \backslash \{\emptyset\}$ and $L_S \subseteq L(S)$ is a *subtree* of (N, L) if (S, L_S) is a tree on S. Denote by G any such subtree. For any given subtree G of a graph (N, L), the links and players of G are called *tree links* and *tree players*, and the links and players in (N, L) that are not in G are called *non-tree links* and *non-tree players*. A *frontier link* for G is a non-tree link with one endpoint in G, called its *tree endpoint*, and one endpoint not in G, its *non-tree endpoint*. By definition, the graph resulting from adding any frontier link of G and its associated non-tree endpoint to the subtree G is still a subtree of (N, L).

An essential component of algorithm **Tree-Growing** is the rule `nextLink` which selects a frontier link to add to the current subtree. For any subtree G of a graph (N, L), let F denote the set of frontier links for G. Then the function `nextLink`$((N, L), F)$ chooses and returns as its value a frontier link in F that is to be added to subtree G. Then, the selected frontier link and its non-tree endpoint are added to the subtree G. Note that the rule `nextLink` may not be deterministic, depending on how it has been specified to select a frontier link in F. After a frontier link is added to the current subtree, the function `updateFrontier`$((N, L), F)$ removes from F those links that are no longer frontier links and adds to F those links that have become frontier links. The pseudocode of **Tree-Growing** is given by Algorithm 1.

Algorithm 1 – Tree-Growing

Input: a finite connected graph (N, L) and a starting player $r \in N$.
Output: a spanning tree G of (N, L).
Initial conditions: $G = (\{r\}, \emptyset)$, $F = \{ri \in L : i \in N\}$.

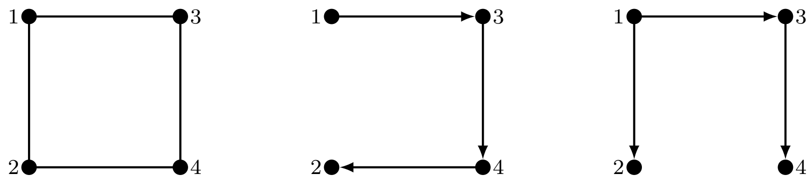
While $F \neq \emptyset$
  $e \longleftarrow$ `nextLink`$((N, L), F)$
  Let i be the non-tree endpoint of e
  Add link e and player i to G.
  `updateFrontier`$((N, L), F)$
Return tree G.

Each different specification of rule `nextLink` creates a different instance of **Tree-Growing**. In the remaining part of this section, we describe two well-known instances of **Tree-Growing** called Depth-First Search (**DFS**) and Breadth-First Search (**BFS**). Both algorithms rely on the discovery order. For each subtree G of (N, L) induced by **Tree-Growing**, the *discovery order* is a listing of players in N in the order in which they are added as subtree G is grown. Once the spanning tree G has been returned by **Tree-Growing**, one can easily consider its oriented version t_r, where the root is the starting player r specified as input in **Tree-Growing**. Henceforth, we will refer to t_r as the output of algorithm **Tree-Growing**. For any output t_r of **Tree-Growing**, the position of player i in the discovery order, starting with 0 for player r, is called the *discovery number* of i in t_r.

In algorithm **DFS**, `nextLink` selects a frontier link in F whose tree endpoint has the largest discovery number. In other words, **DFS** chooses a frontier link incident to the most recently discovered player. If such a link fails to exist, then **DFS** "backtracks" to the second most recently discovered player and tries again, and so on. Therefore, **DFS** discovers players "deeper" in the graph whenever possible. In this way, **DFS** creates spanning trees containing maximal directed paths starting at the root r. Let **DFS**(L) denote the set of all rooted spanning trees of graph (N, L) that **DFS** creates. For any $t_r \in$ **DFS**(L), the discovery number of a player $i \in N$ is denoted `dfnumber`(t_r, i). Since `nextLink` is not necessarily a deterministic function, several different executions of **DFS** on a graph (N, L) can create the same rooted spanning tree t_r. In such a situation, `dfnumber`(t_r, i) can take different values depending on which region of the graph (N, L) is first explored.

In algorithm **BFS**, `nextLink` selects a frontier link in F whose tree endpoint has the smallest discovery number. In other words, algorithm **BFS** chooses a frontier link incident to the less recently discovered player. If such an link fails to exist, then **BFS** considers the second less recently discovered player and tries again, and so on. Therefore, **BFS** explores the graph by selecting frontier links incident to players as close to the root as possible. In this way, **BFS** creates shortest directed paths from the root to any other player (see Proposition 4.2.4 in Gross and Yellen [7]). Let $\mathbf{BFS}(L)$ denote the set of all rooted spanning trees of graph (N, L) that **BFS** creates.

As an example, consider the undirected graph (N, L) on the left hand side of the picture below, where $N = \{1, 2, 3, 4\}$ and $L = \{12, 13, 24, 34\}$. Assume that **DFS** and **BFS** both have graph (N, L) and player 1 as input. Without any loss of generality, assume further that after one step of exploration, the tree is grown by adding player 3 and link 13. At this step of the execution of both algorithms, the current set of frontier links is $\{12, 34\}$. Because player 3 is the most recently discovered agent, **DFS** will select link 34 to grow the tree. Because player 1 is the least recently discovered agent, **BFS** will select link 12 to grow the tree and so on. As a consequence, the rooted spanning tree t_1 can be constructed by **DFS** but not by **BFS**, whereas the rooted spanning tree t'_1 can be constructed by **BFS** but not by **DFS**.



(N, L) t_1 t'_1

The next two sections compare the average tree solutions with respect to the set of spanning trees created by **DFS** and **BFS** respectively. When the communication graph is complete the resulting AT solutions are shown to coincide with the Shapley value and the equal surplus division on Γ_N.

5.2 **DFS**, Harsanyi Trees and the Shapley value

We start this section by proving that the set $\mathbf{DFS}(L)$ of all rooted spanning trees of the connected graph (N, L) that algorithm **DFS** creates coincides with the set of Harsanyi trees introduced in Herings *et al.* [10].

Proposition 4 *Let (N, L) be a connected graph. Then $t_r \in \mathbf{DFS}(L)$ if and only if t_r is a Harsanyi tree of (N, L).*

Proof. By Proposition 2, it is sufficient to show that $t_r \in \mathbf{DFS}(L)$ if and only if, for each $ij \in L$, either $j \in S_r(i)$ or $i \in S_r(j)$. Proposition 2.4.1 in Gross and Yellen [7] establishes the only if part of this claim. Thus, it remains to show that if part.

So, consider any rooted spanning tree t_r of (N, L) such that $ij \in L$ implies either $j \in S_r(i)$ or $i \in S_r(j)$. Let $(N, L(t_r))$ denote the underlying graph of t_r, i.e. the undirected graph on N with link set $L(t_r) = \{ij : (i, j)$ is a directed link of $t_r\}$. Since $(N, L(t_r))$ is a tree, any execution of **DFS** on $(N, L(t_r))$ with initial conditions $G = (\{r\}, \emptyset)$ and $F = \{ri \in L(t_r) : i \in N\}$ will constructs t_r. Consider any such execution and denote by π its discovery number, i.e. $\pi(i) = \texttt{dfnumber}(t_r, i)$ for each $i \in N$. Now, consider the deterministic version $\texttt{nextLink}^\pi$ of rule `nextLink` in **DFS** that is obtained by breaking ties according to π. More specifically, if F contains several frontier links incident to the most recently discovered player, then $\texttt{nextLink}^\pi$ chooses the link such that the non-tree endpoint is the player with the minimal rank in π. Denote by $\mathbf{DFS}^\pi$ the deterministic algorithm that is obtained by replacing `nextLink` by $\texttt{nextLink}^\pi$ in **DFS**. By construction, if the (unique) execution of $\mathbf{DFS}^\pi$ on (N, L) with initial condition $G = (\{r\}, \emptyset)$ and $F = \{ri \in L : i \in N\}$ constructs t_r, then there exists an execution of **DFS** on (N, L) that constructs t_r. In the rest of the proof, we omit to mention that algorithms $\mathbf{DFS}^\pi$ and **DFS** are both run on (N, L) with initial conditions $G = (\{r\}, \emptyset)$ and $F = \{ri \in L : i \in N\}$. We show that the execution $\mathbf{DFS}^\pi$ on (N, L) returns the same output than the execution of $\mathbf{DFS}^\pi$ on $(N, L(t_r))$, that is output t_r.

Because $L \supseteq L(t_r)$, the execution of $\mathbf{DFS}^\pi$ on (N, L) is identical to the one of $\mathbf{DFS}^\pi$ on $(N, L(t_r))$ until no frontier link is incident to the most recently discovered player, say player i, during the execution of $\mathbf{DFS}^\pi$ on $(N, L(t_r))$. At that time, $\mathbf{DFS}^\pi$ on $(N, L(t_r))$ backtracks to the second most recently discovered player, whereas $\mathbf{DFS}^\pi$ on (N, L) can grow the current tree by adding a frontier link incident to i if such a link exists in F. We show that F cannot contain such a link during the execution of $\mathbf{DFS}^\pi$ on (N, L). By way of contradiction, assume that any such link $ij \in L \backslash L(t_r)$ belongs to F. By assumption, either $j \in S_r(i)$ or $i \in S_r(j)$. If $i \in S_r(j)$, then player j has already been discovered by $\mathbf{DFS}^\pi$ on $(N, L(t_r))$ (and so on (N, L)) so that ij has no non-tree endpoint at that step of the execution of the algorithm. This means that ij cannot be a frontier link, a contradiction. If $j \in S_r(i)$, there exists $k \in s_r(i)$ on the unique path between i and j in t_r and k is not yet discovered. Thus ik is a frontier link, and $\mathbf{DFS}^\pi$ on $(N, L(t_r))$ cannot backtrack from player i, another contradiction. Conclude that $ij \notin F$ and cannot be chosen by $\texttt{nextLink}^\pi$ to grow the tree. Because ij was an arbitrary link incident to i in $L \backslash L(t_r)$, we obtain that $\mathbf{DFS}^\pi$ on (N, L) backtracks from i. Continuing in this fashion for any other discovered player i during the execution of $\mathbf{DFS}^\pi$ on $(N, L(t_r))$, we obtain two cases:

(1) the execution of $\mathbf{DFS}^\pi$ on $(N, L(t_r))$ backtracks from i. Then, at this step of the execution of $\mathbf{DFS}^\pi$ on (N, L), F will never contain links in $L \backslash L(t_r)$,
(2) the execution of $\mathbf{DFS}^\pi$ on $(N, L(t_r))$ does not backtrack from i. Then, at this step of the execution of $\mathbf{DFS}^\pi$ on (N, L), function $\texttt{nextLink}^\pi$ will select a frontier link ij in F such that $ij \in L(t_r)$.

In both situations, the execution of $\mathbf{DFS}^\pi$ on (N, L) will not use links in $L \backslash L(t_r)$ to grow the tree. Therefore, $\mathbf{DFS}^\pi$ on (N, L) returns the output t_r and we can conclude that $t_r \in \mathbf{DFS}(L)$, as desired. ■

Combining this result with Proposition 3 and Theorem 2 immediately yields the following result.

Theorem 3 *If (N, L) is a tree, then, for each $(v, L) \in \mathcal{C}_N$, the average tree solution defined with respect to* $\mathbf{DFS}(L)$ *and given by (7) is the average of n contribution vectors and coincides with (6). If (N, L) is the complete graph K_N, then, for $(v, L^N) \in \mathcal{C}_{K_N}$, the average tree solution defined with respect to* $\mathbf{DFS}(L^N)$ *and given by (7) is the average of $n!$ marginal vectors and coincides with the Shapley value given by (1). Moreover, the average tree solution $AT^{\mathscr{T}}$ is a Harsanyi solution on $\mathcal{C}_N$ if and only if $\mathscr{T}(L) \subseteq \boldsymbol{DFS}(L)$ for each connected graph (N, L).*

5.3 **BFS** and the equal surplus division

In this section, we show that for each communication situations on $\mathcal{C}_N$ with a complete graph K_N, the average tree solution with respect to $\mathbf{BFS}(L^N)$ coincides with the equal surplus division given by (2).

Theorem 4 *If (N, L) is a tree, then, for each $(v, L) \in \mathcal{C}_N$, the average tree solution defined with respect to* $\mathbf{BFS}(L)$ *and given by (7) is the average of n contribution vectors and coincides with (6). If (N, L) is the complete graph K_N, then, for $(v, L^N) \in \mathcal{C}_{K_N}$, the average tree solution defined with respect to* $\mathbf{BFS}(L^N)$ *and given by (7) is the average of n marginal vectors and coincides with the equal surplus division given by (2).*

Proof. The proof of the first statement in Theorem 4 is immediate and is omitted. For the proof of the second statement, consider the complete graph K_N and any $(v, L^N) \in \mathcal{C}_{K_N}$. Note that for each $r \in N$, any player $i \in N \backslash \{r\}$ is at distance 1 of r since K_N is the complete graph. Hence, for any $r \in N$, the execution of **BFS** on K_N starting at r yields a unique spanning tree t_r in which r is the predecessor of all other players. The set $\mathbf{BFS}(L^N)$ contains n such rooted spanning trees, one for each $r \in N$. The vector of marginal contributions in t_r is then given by $m_r^{t_r}(v, L^N) = v(N) - \sum_{j \in N \backslash \{r\}} v(\{j\})$ and $m_i^{t_r}(v) = v(\{i\})$ for each $i \in N \backslash \{r\}$. Therefore, for each $i \in N$, we can write

$$\begin{aligned} \mathrm{AT}_i^{\mathbf{BFS}}(v, L^N) &= \frac{1}{n} \sum_{r \in N} m_i^{t_r}(v, L^N) \\ &= \frac{1}{n} \Big(v(N) - \sum_{j \in N \backslash \{i\}} v(\{j\}) + (n-1) v(\{i\}) \Big) \\ &= \mathrm{ESD}_i(v), \end{aligned}$$

which gives the result. ■

6 Concluding remarks

In [18], van den Brink provides a characterization of the equal surplus division that is comparable to the classical characterization of the Shapley value in the sense that both results differ only with respect to a property concerning null or nullifying players. In this article, we proved that these two solutions can also be related to each other by mean of the AT operator for communication situations with a complete graph. The proof of Theorem 4 provides a characterization of the equal surplus division in terms of an average of marginal contributions. The interpretation is similar to the usual interpretation of the Shapley value, except that the involved marginal contribution vectors are different. We already pointed out that these contribution vectors have a significant meaning from a hierarchical point of view.

On the one hand, when the communication graph is complete, algorithm **DFS** can always go on by visiting some unexplored player. Any created spanning tree is shaped like a directed line, and $\mathbf{DFS}(L^N)$ coincides with the set of all $n!$ such directed lines. The AT solution on $\mathcal{C}_{K_N}$ with respect to $\mathbf{DFS}(L^N)$ coincides with the Shapley value on Γ_N. For any directed line, a player in position k in the line benefits from the information held and reported to him by the player in position $k+1$, who in turn benefits from the information held and reported to him by the player in position $k+2$ and so on. Therefore a player in position k in the line benefits from the information of each of the $n-k$ players located downstream from him. Thus, the line architecture defines the most delegated type of communication hierarchy.

On the other hand, when the communication graph is complete, algorithm **BFS** starts from an initial player and can always visit directly any other player. Any created spanning tree is shaped like a (outward pointing) star and $\mathbf{BFS}(L^N)$ coincides with the set of all n such directed stars. The AT solution on $\mathcal{C}_{K_N}$ with respect to $\mathbf{BFS}(L^N)$ coincides with the equal surplus division on Γ_N. For any star, the player at the center of the star benefits from the information held and reported to him by each other player. Thus, the star architecture defines the most centralized type of communication hierarchy.

The average tree solutions offers a new way of looking at the old economic debate about centralization versus decentralization. Advantages claimed for the two solutions have been extensively studied (see for instance Marschak [13]). From the regulator point of view, the allocation choice between the Shapley value and the equal surplus division can be seen as a choice between delegation and centralization. The regulator knows the worth produced by the grand coalition that should be allocated among the players. Because communication can be costly, only hierarchical structures of communication can be considered to be efficient. No doubt that the worth ultimately achieved by any communication hierarchy is the same, because all agents eventually communicate and cooperate with each other. Nevertheless, the share of this worth that a particular agent can claim will typically depend on the communication hierarchy under consideration and on his/her position in the hierarchy. The regulator might not know which particular communication hierarchy has been used by the players for producing the worth of the grand coalition. In order to redistribute this worth among the players, he can focus on a particular class of communication hierarchies that he considers as plausible, and compute the average marginal contribution of each player over the communication hierarchies in this class. This is the viewpoint as reflected in definition of the Shapley value, where each player obtains the average over all orderings of the players of his marginal contribution. If the regulator considers that the communication between players has been established by a delegated channel but does not know which particular one, he will allocate the value of the grand coalition among the players according to the Shapley value as the average marginal contribution vector over all these $n!$ delegated channels. If the regulator considers that the communication between players has been established by a centralized channel but does not know who was the central authority, he will allocate the value of the grand coalition among the players according to the equal surplus sharing as the average marginal contribution vector over all these n centralized channels.

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