Extension of Spot Recovery Model for Gaussian Copula

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Abstract

Heightened systematic risk in the credit crisis has created challenges to CDO pricing and risk management. One important focus has been on the modeling of stochastic recovery. Different approaches within the Gaussian Copula framework have been proposed, but a consistent model was lacking until the recent paper of Bennani and Maetz [6] which shifted the modeling from period recovery to spot recovery. In this paper, we generalize their model to an arbitrary spot recovery distribution setup and extend the deterministic dependency on systematic factor to a random one. Besides, an extra parameter is introduced to control the correlation between default and recovery rate and the correlation between the recovery rates.

1. Introduction

The credit crisis has seen heightened systematic risk and a slew of corporate defaults. From time to time, the standard Gaussian Copula model [8] with fixed recovery rate assumption could not calibrate to the market, especially for senior tranches. Attempts have been made to extend the model framework to allow stochastic recovery rate [3, 6, 7, 10], with the emphasis that recovery rate should be positively correlated to the systematic factor, or loss rate should be negatively correlated to the systematic factor. This feature is based on the empirical observation [1] that, when default rates are high, recovery rates are normally lower. The focus has also shifted from modeling period recovery rate to modeling spot recovery rate [6, 10], as it was found that recent models of period recovery rate may lead to arbitrage and negative probability.

In a previous paper [9], we discussed some general ways to add correlated stochastic recovery model to Gaussian Copula base correlation framework and showed that the construction may lead to negative forward recovery rate and thus may not be arbitrage-free. A key feature of the discussion is that it works with period recovery rate and specifies the dependency of period recovery rate on period default probability through common systematic factors. The shortcomings of this kind of approach are also discussed in Y. Li [10] who suggested that a consistent model should be based on spot recovery rate. Recently, a new model of spot recovery rate for Gaussian Copula was proposed by Bennani and Maetz [6]. The model is flexible and tractable, easy to calibrate to index and tranche market. However, the form of the model does not lead to easy generalization to

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more factors and other Copula models. Besides, the model specifies the spot recovery rate as a deterministic function of the systematic factor, which may not be consistent with empirical evidence. As mentioned in their paper, recent recovery rates have a large spectrum of values, ranging from 1.5% (Tribune) to 91.51% (Fanny Mae). So even with high systematic risk, the recovery rate could still be high, although the average recovery rate would be lower than normal situation. The current paper will discuss a general way to add stochastic spot recovery rate to the model framework which avoids the deterministic dependency of recovery rate on systematic factor and can be easily extended to more factors and other copula models. We also introduce a parameter to control the correlation between default and recovery rate and the correlation between recovery rates. It is still not clear how the parameter should be calibrated.

The rest of the paper is organized as follows. In section 2, we introduce the Gaussian Copula model setup and discuss the conditional normal approximation in the context of the spot recovery rate model of Bennani and Maetz [6]. In section 3, we define a general model of stochastic spot recovery rate in a one factor setup and derive the important properties of the model that are useful for CDO valuation, such as the expected conditional portfolio loss and variance of the conditional portfolio loss. In section 4, we present two examples of spot recovery rate distribution and show that, under certain conditions, they both lead to the specific form of spot recovery in Bennani and Maetz [6]. In section 5, we present some numerical calibration results with a simple parsimonious spot recovery model. Section 6 concludes the paper.

2. Model setup

In the Gaussian Copula setup, a latent variable $V_i = \sqrt{\rho_d} Z + \sqrt{1-\rho_d}\, \varepsilon_i$ drives the default of obligor i of a credit portfolio, where Z and ε_i are independent normal random variables $\sim N(0,1)$ and Z is the systematic factor. The default event $1_{\tau_i \le t}$ can be characterized by $V_i \le v_i = \Phi^{-1}(p_i(t))$, where τ_i is the default time random variable, $p_i(t)$ is the cumulative default probability of the obligor i and $\Phi(x)$ is the standard cumulative normal distribution function. Conditional on the systematic factor Z, obligor defaults are independent and the conditional default probability for obligor i is given by

$$p_i(t,z) = p(\tau_i \le t \,|\, Z = z) = \Phi\left(\frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d}\, z}{\sqrt{1-\rho_d}} \right) \tag{1}$$

In the standard Gaussian Copula model, a constant recovery rate R^{MKT} is assumed (usually 40% for senior unsecured debt), which is the same as the one used in the single name CDS market to calibrate default probability of an obligor. The expected portfolio conditional period loss before time t would be

$$L_t(z) = \sum_i \omega_i \cdot E[L_t^i \mid Z = z] = \sum_i \omega_i \cdot p_i(t,z) \cdot (1 - R^{MKT}) \tag{2}$$

where ω_i is the weight of obligor i in the portfolio.

In Bennani and Maetz [6], spot recovery rate is assumed to be a deterministic function of the systematic factor Z and applies to all obligors,

$$r(t,z) = E[r(\tau) \mid \tau = t, Z = z] \tag{3}$$

Then the expected portfolio conditional period loss before time t is

$$L_t(z) = \sum_i \omega_i \cdot L_t^i(z) = \sum_i \omega_i \cdot \int_0^t (1 - r(s,z)) \cdot dp_i(s,z) \tag{4}$$

Integration over the normal random variable Z will give us the expected portfolio unconditional period loss.

To price CDO tranches, we have to calculate portfolio loss distribution through time. This can be done using standard recursion method [4] or the conditional normal approximation [11]. The conditional normal approximation is very efficient for stochastic recovery models. It assumes that the portfolio conditional period loss follows a normal distribution. Besides the portfolio conditional expected loss, the only other information needed is the variance of portfolio conditional loss

$$V_t(z) = \sum_i \omega_i^2 \cdot V_t^i(z) = \sum_i \omega_i^2 \cdot \left[\int_0^t (1 - r(s,z))^2 \cdot dp_i(s,z) - L_t^i(z)^2 \right] \tag{5}$$

The loss $\hat{L}_t$ of a tranche with attachment a and detachment d can be written as

$$\hat{L}_t = \min(\max(L_t - a, 0), d - a) \tag{6}$$

This is a call spread on the portfolio loss variable L_t. Conditional on Z, L_t can be approximated by a normal distribution $\sim N(L_t(z), \sqrt{V_t(z)})$. We can easily derive the tranche conditional expected loss as

$$\begin{aligned} E[\hat{L}_t \mid Z = z] = {} & (L_t(z) - a) \cdot \Phi\left(\frac{L_t(z) - a}{\sqrt{V_t(z)}} \right) + \sqrt{V_t(z)} \cdot \phi\left(\frac{L_t(z) - a}{\sqrt{V_t(z)}} \right) \\ & - (L_t(z) - d) \cdot \Phi\left(\frac{L_t(z) - d}{\sqrt{V_t(z)}} \right) - \sqrt{V_t(z)} \cdot \phi\left(\frac{L_t(z) - d}{\sqrt{V_t(z)}} \right) \end{aligned} \tag{7}$$

where $\phi(x)$ is the standard normal density function. The tranche unconditional expected loss can be computed by numerical integration over the normal variable Z. Once the tranche expected loss term structure is know, it is straight forward to value the CDO. For other numerical approximation methods, see also the recent paper of Amraoui et al [2].

3. A general spot recovery model

Let us start with a generalized model of stochastic recovery where loss is driven by another latent variable $W_i = \sqrt{\rho_l} Z + \sqrt{1-\rho_l}\xi_i$ through a cumulative distribution function $F_L(l)$, where Z, ε_i, ξ_i are independent normal random variables. In a previous paper [9], we specify that loss given default is defined by $L_i = F_L^{-1}(\Phi(-W_i))$ conditional on $\tau_i \le t$ or $V_i \le \Phi^{-1}(p_i(t))$, where the negative sign in front of W_i is to ensure negative correlation between loss and the systematic factor. The loss defined this way is not the spot loss at default and may lead to arbitrage conditions. To build a consistent stochastic recovery model, we have to start with the spot loss or recovery upon default.

We need to define the condition on the latent variable such that default happens at a specific time instead of a time range. This depends on the continuity of the default probability curve. In general, $p_i(t)$ is a continuous and monotonically increasing function of time t. However, in certain situations, it is perfectly imaginable that the $p_i(t)$ curve has a jump. For example, if an obligor has a large debt due in 30 days, the default probability within 30 days will be low while the default probability over 30 days will be much higher. For now we assume that the $p_i(t)$ curve is continuous such that the mapping between cumulative default probability and default time is one-to-one. Thus we have

$$\tau_i = t \qquad \leftrightarrow \qquad V_i = \Phi^{-1}(p_i(t))$$

Under this condition, it is easy to prove that W_i follows a normal distribution with mean $\sqrt{\rho_d \rho_l}\Phi^{-1}(p_i(t))$ and standard deviation $\sqrt{1-\rho_d \rho_l}$. To ensure that $F_L(l)$ is indeed the marginal cumulative distribution for the spot loss given default, we define

$$L_i = F_L^{-1}\left(\Phi\left(-\frac{W_i - \sqrt{\rho_d \rho_l}\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_d \rho_l}}\right)\right) \tag{8}$$

Thus

$$P(L_i \le l \mid \tau_i = t) = P\left(F_L^{-1}\left(\Phi\left(-\frac{W_i - \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_d \rho_l}} \right)\right) \le l \mid \tau_i = t \right) \qquad (9)$$
$$= F_L(l)$$

The recovery distribution is related to the loss distribution by

$$F_R(r) = P(R \le r) = P(L \ge 1-r) = 1 - F_L(1-r) + P(L = 1-r) \qquad (10)$$

If the loss distribution is continuous, the last term will be zero

$$F_R(r) = P(R \le r) = 1 - F_L(1-r) \qquad (11)$$

We can equivalently define recovery as a function of W_i

$$R_i = F_R^{-1}\left(\Phi\left(\frac{W_i - \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_d \rho_l}} \right)\right) \qquad (12)$$

If the spot loss distribution $F_L(l)$ is time independent, then the distribution of loss to maturity is also time independent and is the same as $F_L(l)$. So if $F_L(l)$ has expected loss of $1-R^{MKT}$, then the model is automatically consistent with the single name CDS market.

Conditional on $V_i = \Phi^{-1}(p_i(t))$, Z follows a normal distribution with mean $\sqrt{\rho_d}\,\Phi^{-1}(p_i(t))$ and standard deviation $\sqrt{1-\rho_d}$, while ξ_i still follows the standard normal distribution. If we fix $Z = z$, then the conditional spot loss distribution will be

$$\begin{aligned}
&P(L_i \le l \mid \tau_i = t, Z = z) \\
&= P\left(\Phi\left(-\frac{W_i - \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_d \rho_l}} \right) \le F_L(l) \mid \tau_i = t, Z = z \right) \\
&= P\left(\xi_i \ge \frac{-\sqrt{\rho_l}\,z - \sqrt{1-\rho_d \rho_l}\,\Phi^{-1}(F_L(l)) + \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}} \mid \tau_i = t, Z = z \right) \qquad (13) \\
&= \Phi\left(\frac{\sqrt{\rho_l}\,z + \sqrt{1-\rho_d \rho_l}\,\Phi^{-1}(F_L(l)) - \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}} \right)
\end{aligned}$$

Thus the conditional spot recovery distribution is

$$
\begin{aligned}
&P(R_i \le r \mid \tau_i = t, Z = z) \\
&= 1 - P(L \le 1 - r \mid \tau_i = t, Z = z) + P(L = 1 - r \mid \tau_i = t, Z = z) \\
&= 1 - \Phi\left(\frac{\sqrt{\rho_l} z + \sqrt{1 - \rho_d \rho_l} \Phi^{-1}(F_L(1 - r)) - \sqrt{\rho_d \rho_l} \Phi^{-1}(p_i(t))}{\sqrt{1 - \rho_l}} \right) + P(L = 1 - r \mid \tau_i = t, Z = z) \\
&= \Phi\left(\frac{-\sqrt{\rho_l} z + \sqrt{1 - \rho_d \rho_l} \Phi^{-1}(F_R(r) - P(R = r)) + \sqrt{\rho_d \rho_l} \Phi^{-1}(p_i(t))}{\sqrt{1 - \rho_l}} \right) + P(R = r \mid \tau_i = t, Z = z)
\end{aligned}
\tag{14}
$$

Since $F_R(r)$ is right continuous, it is easy to show that

$$
\begin{aligned}
&P(R_i \le r \mid \tau_i = t, Z = z) \\
&= \Phi\left(\frac{-\sqrt{\rho_l} z + \sqrt{1 - \rho_d \rho_l} \Phi^{-1}(F_R(r)) + \sqrt{\rho_d \rho_l} \Phi^{-1}(p_i(t))}{\sqrt{1 - \rho_l}} \right)
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
&P(R_i = r \mid \tau_i = t, Z = z) \\
&= \Phi\left(\frac{-\sqrt{\rho_l} z + \sqrt{1 - \rho_d \rho_l} \Phi^{-1}(F_R(r)) + \sqrt{\rho_d \rho_l} \Phi^{-1}(p_i(t))}{\sqrt{1 - \rho_l}} \right) - \\
&\quad \Phi\left(\frac{-\sqrt{\rho_l} z + \sqrt{1 - \rho_d \rho_l} \Phi^{-1}(F_R(r) - P(R = r)) + \sqrt{\rho_d \rho_l} \Phi^{-1}(p_i(t))}{\sqrt{1 - \rho_l}} \right)
\end{aligned}
\tag{16}
$$

Next we derive the distribution for conditional period recovery rate defined as

$$
\begin{aligned}
&P(R_i \le r \mid \tau_i \le t, Z = z) \\
&= \frac{1}{p_i(t, z)} \cdot \int_0^t \Phi\left(\frac{-\sqrt{\rho_l} z + \sqrt{1 - \rho_d \rho_l} \Phi^{-1}(F_R(r)) + \sqrt{\rho_d \rho_l} \Phi^{-1}(p_i(s))}{\sqrt{1 - \rho_l}} \right) \cdot d p_i(s, z)
\end{aligned}
\tag{17}
$$

Using the fact that (see [5])

$$
\int_{-\infty}^{c} \Phi(ax + b) \phi(x) dx = \Phi_2\left(\frac{b}{\sqrt{1 + a^2}}, c; \frac{-a}{\sqrt{1 + a^2}} \right) \tag{18}
$$

it is easy to prove that

$$
\begin{aligned}
&P(R_i \le r \mid \tau_i \le t, Z = z) \\
&= \frac{1}{p_i(t,z)} \cdot \int_0^t \Phi\left(\frac{-\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_R(r)) + \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(s))}{\sqrt{1-\rho_l}} \right) \cdot dp_i(s,z) \qquad (19) \\
&= \frac{1}{p_i(t,z)} \cdot \Phi_2\left(\frac{-(1-\rho_d)\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_R(r))}{\sqrt{1-\rho_l + \rho_d \rho_l - \rho_d^2 \rho_l}}, c(t,z); -\tilde{\rho} \right)
\end{aligned}
$$

where

$$
c(t,z) = \frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d}\, z}{\sqrt{1-\rho_d}} \text{ and } \tilde{\rho} = \frac{\sqrt{\rho_d \rho_l (1-\rho_d)}}{\sqrt{1-\rho_l + \rho_d \rho_l - \rho_d^2 \rho_l}}
$$

Note that $-\tilde{\rho}$ is essentially the negative correlation between default and recovery during the period conditional on Z. If $\rho_l = 0$, then recovery will be independent. As seen in equation (22), after integrating out Z, the marginal default and marginal recovery are independent as the marginal recovery distribution is time-independent.

Another way to look at the result is as follows

$$
\begin{aligned}
&P(1_{\{R_i \le r\}} \cdot 1_{\{\tau_i \le t\}} \mid Z = z) \\
&= E\left[\Phi\left(\frac{-\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_R(r)) + \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(\tau))}{\sqrt{1-\rho_l}} \right) \cdot 1_{\left\{\varepsilon_i \le \frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d}\, z}{\sqrt{1-\rho_d}}\right\}} \mid Z = z \right] \\
&= E\left[\Phi\left(\frac{-\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_R(r)) + \sqrt{\rho_d \rho_l}\, (\sqrt{\rho_d}\, z + \sqrt{1-\rho_d}\, \varepsilon_i)}{\sqrt{1-\rho_l}} \right) \cdot 1_{\left\{\varepsilon_i \le \frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d}\, z}{\sqrt{1-\rho_d}}\right\}} \mid Z = z \right] \\
&= \Phi_2\left(\frac{-(1-\rho_d)\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_R(r))}{\sqrt{1-\rho_l + \rho_d \rho_l - \rho_d^2 \rho_l}}, c(t,z); -\tilde{\rho} \right)
\end{aligned}
\qquad (20)
$$

such that

$$
P(R_i \le r \mid \tau_i \le t, Z = z) = \frac{P(1_{\{R_i \le r\}} \cdot 1_{\{\tau_i \le t\}} \mid Z = z)}{P(1_{\{\tau_i \le t\}} \mid Z = z)} = \frac{P(1_{\{R_i \le r\}} \cdot 1_{\{\tau_i \le t\}} \mid Z = z)}{p_i(t,z)} \qquad (21)
$$

The unconditional recovery distribution can be calculated as follows

$$
\begin{aligned}
P(R_i \le r \mid \tau_i \le t) &= \frac{E[1_{\{R_i \le r\}} 1_{\tau_i \le t}]}{P(\tau_i \le t)} \\
&= \frac{1}{p_i(t)} \int_{-\infty}^{\infty} P(1_{\{R_i \le r\}} 1_{\tau_i \le t} \mid Z = z) \cdot \phi(z) dz \\
&= \frac{1}{p_i(t)} \int_{-\infty}^{\infty} \Phi_2 \left(\frac{-(1-\rho_d)\sqrt{\rho_l} z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_R(r))}{\sqrt{1-\rho_l + \rho_d \rho_l - \rho_d^2 \rho_l}}, c(t,z); -\tilde{\rho} \right) \cdot \phi(z) dz \\
&= F_R(r)
\end{aligned} \tag{22}
$$

using the fact that (see [9])

$$
\int_{-\infty}^{+\infty} \Phi_2(az + b, cz + d; \rho) \cdot \phi(z) dz = \Phi_2 \left(\frac{b}{\sqrt{1+a^2}}, \frac{d}{\sqrt{1+c^2}}; \frac{ac + \rho}{\sqrt{(1+a^2)(1+c^2)}} \right) \tag{23}
$$

So the marginal distribution of period recovery rate is the same as marginal distribution of spot recovery and is time-independent.

To value a CDO tranche, we need to look at the expected loss of the CDO tranche at each time horizon. As discussed in section 2, in the conditional normal approximation, only conditional portfolio expected loss and conditional portfolio loss variance will be important. The conditional expected loss for obligor i before time t would be

$$
\begin{aligned}
L_t^i(z) &= \int_0^1 (1-r) \cdot d_r P(1_{\{R_i \le r\}} \cdot 1_{\{\tau_i \le t\}} \mid Z = z) \\
&= \int_0^1 P(1_{\{R_i \le r\}} \cdot 1_{\{\tau_i \le t\}} \mid Z = z) \cdot dr
\end{aligned} \tag{24}
$$

The loss variance for obligor i is

$$
\begin{aligned}
V[L_t^i \mid Z = z] &= \int_0^1 (1-r)^2 \cdot d_r P(1_{\{R_i \le r\}} \cdot 1_{\{\tau_i \le t\}} \mid Z = z) - L_t^i(z)^2 \\
&= 2\int_0^1 P(1_{\{R_i \le r\}} \cdot 1_{\{\tau_i \le t\}} \mid Z = z) \cdot (1-r) \cdot dr - L_t^i(z)^2
\end{aligned} \tag{25}
$$

Under conditional independence, the portfolio expected loss and loss variance will be the sum of individual expected losses and loss variances. If the loss or recovery distribution is continuous, we have to use numerical integration to calculate the conditional loss expectation and loss variance unless the loss distribution takes some special form as discussed in the next section.

If the marginal spot recovery distribution is discrete, R takes values r_j with probability p_j where $j = 1,\dots,J$. Let $F_j = \sum_{k=1}^{j} p_k$, $F_0 = 0$ and $F_J = 1$, then the conditional expected loss for obligor i before time t is

$$
\begin{aligned}
L_t^i(z) &= \sum_{j=1}^{J} (1-r_j) \cdot \left(P(1_{\{R_i \le r_j\}} \cdot 1_{\{\tau \le t\}} \mid Z = z) - P(1_{\{R_i \le r_{j-1}\}} \cdot 1_{\{\tau \le t\}} \mid Z = z) \right) \\
&= \sum_{j=1}^{J} (1-r_j) \cdot \left(\Phi_2\left(\frac{-(1-\rho_d)\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_j)}{\sqrt{1-\rho_l + \rho_d \rho_l - \rho_d^2 \rho_l}}, c(t,z); -\tilde{\rho} \right) \right. \\
&\quad \left. - \Phi_2\left(\frac{-(1-\rho_d)\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_{j-1})}{\sqrt{1-\rho_l + \rho_d \rho_l - \rho_d^2 \rho_l}}, c(t,z); -\tilde{\rho} \right) \right)
\end{aligned}
\tag{26}
$$

and the conditional loss variance is

$$
\begin{aligned}
& V[L_t^i \mid Z = z] \\
&= \sum_{j=1}^{J} (1-r_j)^2 \cdot \left(\Phi_2\left(\frac{-(1-\rho_d)\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_j)}{\sqrt{1-\rho_l + \rho_d \rho_l - \rho_d^2 \rho_l}}, c(t,z); -\tilde{\rho} \right) \right. \\
&\quad \left. - \Phi_2\left(\frac{-(1-\rho_d)\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_{j-1})}{\sqrt{1-\rho_l + \rho_d \rho_l - \rho_d^2 \rho_l}}, c(t,z); -\tilde{\rho} \right) \right) - L_t^i(z)^2
\end{aligned}
\tag{27}
$$

With the knowledge of the expected conditional loss and the variance of conditional loss, we can easily apply the conditional normal approximation to value CDO tranches.

In the case $\rho_l = 1$, the model reduces to deterministic spot recovery conditional on Z, which gives the general form for the Bennani-Maetz model [6].

4. Two Examples of spot recovery distribution

First we start with a special discrete distribution, which is similar to recovery mark down but still keeps the expected recovery at R^{MKT}. It was used in [9] to show the relationship between the Krekel model [7] and the Amraoui-Hitier model [3]. The recovery rate can be 1 or $\tilde{R}$ with probability $\frac{R^{MKT} - \tilde{R}}{1-\tilde{R}}$ and $\frac{1-R^{MKT}}{1-\tilde{R}}$ respectively, such that the expected recovery rate is R^{MKT}. The conditional spot recovery rate distribution is

$$P(R=\tilde{R}\mid \tau_i=t, Z=z)=\Phi\left(\frac{-\sqrt{\rho_l}\,z+\sqrt{1-\rho_d\rho_l}\,\Phi^{-1}(\frac{1-R^{MKT}}{1-\tilde{R}})+\sqrt{\rho_d\rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}}\right)$$

$$P(R=1\mid \tau_i=t, Z=z)=1-\Phi\left(\frac{-\sqrt{\rho_l}\,z+\sqrt{1-\rho_d\rho_l}\,\Phi^{-1}(\frac{1-R^{MKT}}{1-\tilde{R}})+\sqrt{\rho_d\rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}}\right)$$

$$=\Phi\left(\frac{\sqrt{\rho_l}\,z+\sqrt{1-\rho_d\rho_l}\,\Phi^{-1}(\frac{R^{MKT}-\tilde{R}}{1-\tilde{R}})-\sqrt{\rho_d\rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}}\right) \tag{28}$$

Then the expected conditional spot recovery rate is

$$r_i(t,z)=1-(1-\tilde{R})\cdot\Phi\left(\frac{-\sqrt{\rho_l}\,z+\sqrt{1-\rho_d\rho_l}\,\Phi^{-1}(\frac{1-R^{MKT}}{1-\tilde{R}})+\sqrt{\rho_d\rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}}\right) \tag{29}$$

The expected conditional loss for obligor i up to time t is

$$L_t^i(z)=\int_0^t (1-r_i(s,z))\cdot dp_i(s,z)$$

$$=\int_0^t (1-\tilde{R})\cdot\Phi\left(\frac{-\sqrt{\rho_l}\,z+\sqrt{1-\rho_d\rho_l}\,\Phi^{-1}(\frac{1-R^{MKT}}{1-\tilde{R}})+\sqrt{\rho_d\rho_l}\,\Phi^{-1}(p_i(s))}{\sqrt{1-\rho_l}}\right)\cdot dp_i(s,z) \tag{30}$$

$$=(1-\tilde{R})\cdot\Phi_2\left(c(t,z),d(z);-\tilde{\rho}\right)$$

where

$$d(z)=\frac{-(1-\rho_d)\sqrt{\rho_l}\,z+\sqrt{1-\rho_d\rho_l}\,\Phi^{-1}(\frac{1-R^{MKT}}{1-\tilde{R}})}{\sqrt{1-\rho_l+\rho_d\rho_l-\rho_d^2\rho_l}}$$

The variance of conditional loss for obligor i up to time t is

$$V_t^i(z) = \int_0^t (1-\tilde{R})^2 \cdot \Phi\left(\frac{-\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}\left(\frac{1-R^{MKT}}{1-\tilde{R}}\right) + \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(s))}{\sqrt{1-\rho_l}} \right) \cdot dp_i(s,z) - L_t^i(z)^2$$

$$= (1-\tilde{R})^2 \cdot \left(\Phi_2\left(c(t,z), d(z); -\tilde{\rho}\right) - \Phi_2\left(c(t,z), d(z); -\tilde{\rho}\right)^2 \right) \tag{31}$$

By taking the expectation of the conditional spot recovery and using it as the deterministic spot recovery rate conditional on Z, we generate a new set of spot recovery models that only depend on the systematic factor Z, see our previous paper [9]. These will be special forms of the Bennani-Maetz model [6]. The expected conditional loss is still the same as the above model, but the variance will be smaller.

The variance of conditional loss for the new model is

$$V_t^i(z) = \int_0^t (1 - r_i(s,z))^2 \cdot dp_i(s,z) - L_t^i(z)^2$$

$$= \int_0^t (1-\tilde{R})^2 \cdot \Phi\left(\frac{-\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}\left(\frac{1-R^{MKT}}{1-\tilde{R}}\right) + \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(s))}{\sqrt{1-\rho_l}} \right)^2 dp_i(s,z) - L_t^i(z)^2 \tag{32}$$

We will use a general formula

$$\int_{-\infty}^{z_0} \Phi_2(az+b, cz+d; \rho) \cdot \phi(z) dz$$

$$= \Phi_3\left(z_0, \frac{b}{\sqrt{1+a^2}}, \frac{d}{\sqrt{1+c^2}}; \begin{pmatrix} 1 & \frac{-a}{\sqrt{1+a^2}} & \frac{-c}{\sqrt{1+c^2}} \\ \frac{-a}{\sqrt{1+a^2}} & 1 & \frac{ac+\rho}{\sqrt{(1+a^2)(1+c^2)}} \\ \frac{-c}{\sqrt{1+c^2}} & \frac{ac+\rho}{\sqrt{(1+a^2)(1+c^2)}} & 1 \end{pmatrix} \right) \tag{33}$$

If we define $X = \sqrt{\rho}\varepsilon + \sqrt{1-\rho}\varepsilon_1 - aZ$, $Y = \sqrt{\rho}\varepsilon + \sqrt{1-\rho}\varepsilon_2 - cZ$, where ε, ε_1, ε_2, Z are standard normal variables, then

$$\int_{-\infty}^{z_0} \Phi_2(az+b, cz+d; \rho) \cdot \phi(z) dz = E[1_{Z \le z_0} \cdot 1_{X \le b} \cdot 1_{Y \le d}]$$

which leads to the result (33). When $\rho = 0$, $a = c$ and $b = d$, we have

$$\int_{-\infty}^{z_0} \Phi(az+b)^2 \cdot \phi(z)dz \tag{34}$$

$$= \Phi_3\left(z_0, \frac{b}{\sqrt{1+a^2}}, \frac{b}{\sqrt{1+a^2}}; \begin{pmatrix} 1 & \frac{-a}{\sqrt{1+a^2}} & \frac{-a}{\sqrt{1+a^2}} \\ \frac{-a}{\sqrt{1+a^2}} & 1 & \frac{a^2}{1+a^2} \\ \frac{-a}{\sqrt{1+a^2}} & \frac{a^2}{1+a^2} & 1 \end{pmatrix} \right)$$

Thus

$$V_t^i(z) = (1-\tilde{R})^2 \cdot \Phi_3\left(c(t,z), d(z), d(z); \begin{pmatrix} 1 & -\tilde{\rho} & -\tilde{\rho} \\ -\tilde{\rho} & 1 & \tilde{\rho}^2 \\ -\tilde{\rho} & \tilde{\rho}^2 & 1 \end{pmatrix} \right) - L_t^i(z)^2 \tag{35}$$

Here we will show that the specific form of the Bennani-Maetz model [6] is actually related to a spot recovery distribution with fixed mean R^{MKT} and maximum variance $R^{MKT}(1-R^{MKT})$, just like how the Amraoui-Hitier model is related to the same distribution in a Krekel model (see [9]). A spot recovery distribution with mean R^{MKT} will have maximum variance when the recovery only takes the extreme values of 0 and 1 with probabilities $1-R^{MKT}$ and R^{MKT}. The conditional spot recovery distribution is

$$P(R=0 \mid \tau_i = t, Z=z) = \Phi\left(\frac{-\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\,\Phi^{-1}(1-R^{MKT}) + \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}} \right)$$

$$P(R=1 \mid \tau_i = t, Z=z) = 1-\Phi\left(\frac{-\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\,\Phi^{-1}(1-R^{MKT}) + \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}} \right)$$

$$= \Phi\left(\frac{\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\,\Phi^{-1}(R^{MKT}) - \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}} \right) \tag{36}$$

Then the expected conditional spot recovery rate is

$$r(t,z)=\Phi\left(\frac{\sqrt{\rho_l}\,z+\sqrt{1-\rho_d\rho_l}\,\Phi^{-1}(R^{MKT})-\sqrt{\rho_d\rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}}\right) \tag{37}$$

Comparing with equation (20) of Bennani and Maetz [6], we have

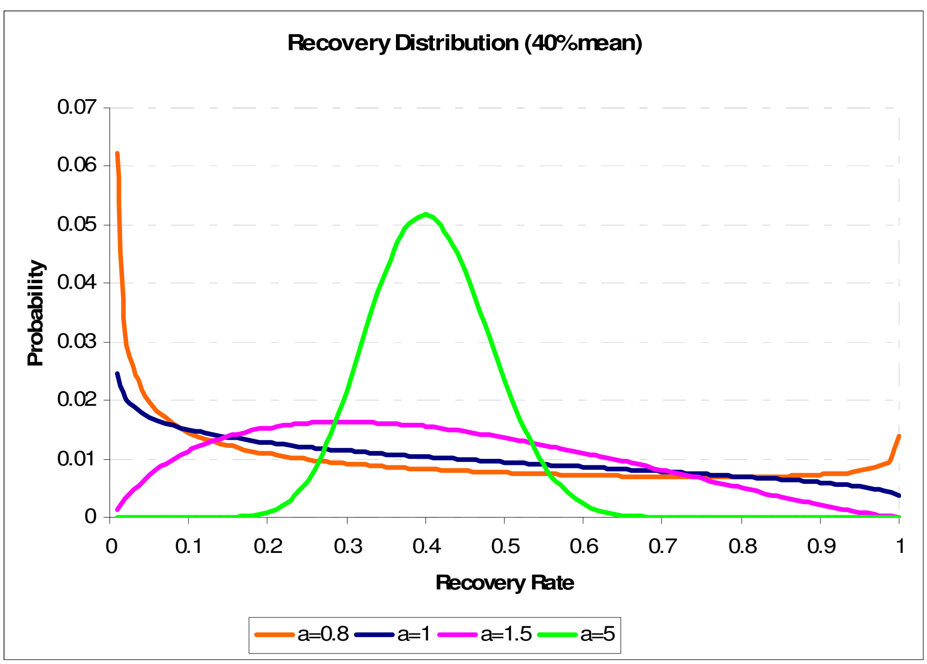
$$r(t,z)=\Phi\left(\gamma_\rho\left(\frac{z-\sqrt{\rho_d}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_d}}\right)+\delta\right)=\Phi\left(\alpha_\rho z+\beta_i\right) \tag{38}$$

where

$$\gamma_\rho=\sqrt{\frac{\rho_l(1-\rho_d)}{1-\rho_l}},\ \alpha_\rho=\sqrt{\frac{\rho_l}{1-\rho_l}}\ \text{ and }\ \delta=\sqrt{\frac{1-\rho_d\rho_l}{1-\rho_l}}\Phi^{-1}(R^{MKT})$$

This shows that γ_ρ is determined by the correlations ρ_d and ρ_l. A natural choice would be to assume $\rho_d=\rho_l=\rho$ and then $\alpha_\rho=\sqrt{\frac{\rho}{1-\rho}}$ instead of $\alpha_\rho=\frac{\rho}{1-\rho}$ in Bennani and Maetz [6].

Next we consider a continuous distribution which is not used often but actually is similar to the beta distribution, as shown in the Figure below.



It was discussed by Andersen and Sidenius [5], and is equivalent to the distribution used by Bennani and Maetz [6] as in equation (38). It has the following form

$$F_R(r) = P(R \le r) = \Phi(a \cdot \Phi^{-1}(r) - \sqrt{1+a^2}\,\Phi^{-1}(r_0)) \tag{39}$$

or for the density function

$$f_R(r) = a \cdot \frac{\phi(a \cdot \Phi^{-1}(r) - \sqrt{1+a^2}\,\Phi^{-1}(r_0))}{\phi(\Phi^{-1}(r))} \tag{40}$$

where $a \ge 0$ and $0 \le r_0 \le 1$. This distribution will simplify calculation for Gaussian Copula model. It is easy to prove that the expected recovery rate is r_0 and the variance of recovery rate is

$$V(r) = 1 - 2 \cdot \Phi_2\left(-\Phi^{-1}(r_0), 0; \frac{a}{\sqrt{2(1+a^2)}} \right) - r_0^2 \tag{41}$$

Assume $r_0 = R^{MKT}$. When a goes to zero, the variance goes to the maximum value $R^{MKT}(1-R^{MKT})$, which corresponds to the case where r takes the value 0 or 1 just discussed in the previous example. When a goes to infinity, the variance goes to zero and the distribution reduces to a constant recovery R^{MKT}.

The original spot recovery equation can be written as

$$\begin{aligned}
R &= F_R^{-1}\left(\Phi\left(\frac{W_i - \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_d \rho_l}} \right) \right) \\
&= \Phi\left(\frac{W_i - \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{a\sqrt{1-\rho_d \rho_l}} + \sqrt{1+\frac{1}{a^2}}\,\Phi^{-1}(R^{MKT}) \right) \\
&= \Phi\left(\frac{\sqrt{\rho_l}Z + \sqrt{1-\rho_l}\,\xi_i - \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{a\sqrt{1-\rho_d \rho_l}} + \sqrt{1+\frac{1}{a^2}}\,\Phi^{-1}(R^{MKT}) \right)
\end{aligned} \tag{42}$$

Then we have

$$
\begin{aligned}
&P(R_i \le r \mid \tau_i = t, Z = z) \\
&= \Phi\left(\frac{-\sqrt{\rho_l}\, z - \sqrt{1-\rho_d \rho_l}\, \Phi^{-1}(F_R(r)) + \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}} \right) \\
&= \Phi\left(\frac{-\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l} \cdot (a\Phi^{-1}(r) - \sqrt{1+a^2}\, \Phi^{-1}(R^{MKT})) + \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l}} \right)
\end{aligned}
\tag{43}
$$

The expected conditional spot recovery is

$$
\begin{aligned}
r_i(t,z) &= \int_0^1 r \cdot d_r P(R_i \le r \mid \tau_i = t, Z = z) \\
&= \Phi\left(\frac{\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l} \cdot \sqrt{1+a^2}\, \Phi^{-1}(R^{MKT}) - \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l + a^2(1-\rho_d \rho_l)}} \right)
\end{aligned}
\tag{44}
$$

The expected conditional loss up to time t is

$$
\begin{aligned}
L_t^i(z) &= \int_0^t (1 - r_i(s,z)) \cdot dp_i(s,z) \\
&= \int_0^t \left(1 - \Phi\left(\frac{\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l} \cdot \sqrt{1+a^2}\, \Phi^{-1}(R^{MKT}) - \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(s))}{\sqrt{1-\rho_l + a^2(1-\rho_d \rho_l)}} \right) \right) \cdot dp_i(s,z) \\
&= \Phi_2(c(t,z), b(z); -\hat{\rho})
\end{aligned}
\tag{45}
$$

where

$$
b(z) = -\frac{(1-\rho_d)\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l}\sqrt{1+a^2}\, \Phi^{-1}(R^{MKT})}{\sqrt{1-\rho_l + a^2(1-\rho_d \rho_l) + \rho_d \rho_l - \rho_d^2 \rho_l}}
$$

$$
\hat{\rho} = \frac{\sqrt{\rho_d \rho_l (1-\rho_d)}}{\sqrt{1-\rho_l + a^2(1-\rho_d \rho_l) + \rho_d \rho_l - \rho_d^2 \rho_l}}
$$

The variance of conditional loss up to time t is

$$
\begin{aligned}
V_t^i(z) &= \int_0^t \left(\int_0^1 (1-r)^2 \cdot d_r P(R_i \le r \mid \tau_i = s, Z = z) \right) \cdot dp_i(s,z) - L_t^i(z)^2 \\
&= \int_0^t \left(\int_0^1 P(R_i \le r \mid \tau_i = s, Z = z) \cdot 2(1-r) \cdot dr \right) \cdot dp_i(s,z) - L_t^i(z)^2 \\
&= 2\int_0^t \Phi_2 \left(e(s,z), 0; -\frac{a\sqrt{1-\rho_d \rho_l}}{\sqrt{2(1-\rho_l + a^2(1-\rho_d \rho_l))}} \right) \cdot dp_i(s,z) - L_t^i(z)^2 \\
&= 2\Phi_3 \left(c(t,z), b(z), 0; \begin{pmatrix} 1 & -\hat{\rho} & 0 \\ -\hat{\rho} & 1 & -\breve{\rho} \\ 0 & -\breve{\rho} & 1 \end{pmatrix} \right) - L_t^i(z)^2
\end{aligned}
\tag{46}
$$

where

$$
e(t,z) = \frac{-\sqrt{\rho_l}\, z - \sqrt{1-\rho_d \rho_l}\sqrt{1+a^2}\,\Phi^{-1}(R^{MKT}) + \sqrt{\rho_d \rho_l}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_l + a^2(1-\rho_d \rho_l)}}
$$

$$
\breve{\rho} = \frac{a\sqrt{1-\rho_d \rho_l}}{\sqrt{2(1-\rho_l + a^2(1-\rho_d \rho_l) + \rho_d \rho_l - \rho_d^2 \rho_l)}}
$$

If we set $\rho_l = 1$, then recovery rate is only driven by the systematic factor and the model reduces to that of Bennani and Maetz [6] with $\gamma_\rho = \frac{1}{a}$.

Another way is to define a new model with the expected conditional spot loss of the above model as the deterministic conditional spot loss, and we will again have a model similar to Bennani and Maetz [6] with (see equation (44))

$$
\gamma_\rho = \sqrt{\frac{\rho_l(1-\rho_d)}{1-\rho_l + a^2(1-\rho_d \rho_l)}} \text{ and } \delta = \sqrt{\frac{(1-\rho_d \rho_l)(1+a^2)}{1-\rho_l + a^2(1-\rho_d \rho_l)}}\Phi^{-1}(R^{MKT}) \tag{47}
$$

The expected conditional loss will be the same. The variance of conditional loss is

$$
\begin{aligned}
V_t^i(z) &= \int_0^t (1 - r_i(s,z))^2 \cdot dp_i(s,z) - L_t^i(z)^2 \\
&= \int_0^t \left(1 - \Phi\left(\frac{\sqrt{\rho_l}\, z + \sqrt{1-\rho_d \rho_l} \cdot \sqrt{1+a^2}\, \Phi^{-1}(R^{MKT}) - \sqrt{\rho_d \rho_l}\, \Phi^{-1}(p_i(s))}{\sqrt{1-\rho_l + a^2(1-\rho_d \rho_l)}} \right) \right)^2 \cdot dp_i(s,z) - L_t^i(z)^2 \\
&= \Phi_3\left(c(t,z), b(z), b(z); \begin{pmatrix} 1 & -\hat{\rho} & -\hat{\rho} \\ -\hat{\rho} & 1 & \hat{\rho}^2 \\ -\hat{\rho} & \hat{\rho}^2 & 1 \end{pmatrix} \right) - L_t^i(z)^2
\end{aligned}
\tag{48}
$$

which will be smaller than the original variance in equation (46).

5. Numerical Results

We use a simple model specification to calibrate CDX S9 as of Norvember 18, 2008. The marginal spot recovery distribution will be the same as the first example with $\tilde{R} = 0$, where recovery can only take the value of 0 or 1 with probabilities $1 - R^{MKT}$ and R^{MKT}. We will use the parameter specification in Bennani and Maetz [6] (see equation (38))

$$
\alpha_\rho = \frac{\rho_d}{1-\rho_d} \qquad \text{or} \qquad \rho_l = \frac{{\rho_d}^2}{(1-\rho_d)^2 + {\rho_d}^2}
$$

Using equations (30), (31), we can calculate the conditional expected loss and conditional loss variance, and then use the conditional normal approximation to calibrate the base correlations of the tranches.

The table below shows the market quotes for CDX S9 on Norvember 18, 2008.

CDX S9	Upfront	Running	Upfront	Running	Upfront	Running
Index Maturity/Spread	5Y	2.48%	7Y	2.26%	10Y	2.07%
0%-3%	77.40%	5.00%	79.90%	5.00%	81.18%	5.00%
3%-7%	45.64%	5.00%	51.91%	5.00%	55.25%	5.00%
7%-10%	0.00%	9.65%	0.00%	10.06%	0.00%	10.13%
10%-15%	0.00%	4.64%	0.00%	4.78%	0.00%	4.82%
15%-30%	0.00%	1.35%	0.00%	1.40%	0.00%	1.44%

The next table shows the calibration results for base correlation.

CDX S9	Fixed Recovery			Stochastic Recovery		
Index Maturity	5Y	7Y	10Y	5Y	7Y	10Y
3%	52.81%	55.09%	56.36%	32.12%	32.47%	31.90%
7%	50.75%	51.87%	53.63%	31.62%	31.40%	31.31%
10%	57.10%	56.95%	57.51%	35.17%	34.06%	33.17%
15%	69.92%	70.49%	71.68%	42.36%	41.32%	40.56%
30%	-	-	-	67.46%	68.73%	70.23%

As expected, the standard Gaussian Copula model with fixed recovery rate can not calibrate the senior most tranche, but the stochastic spot recovery model can. Besides, the base correlation curve is lower and less steep.

It should be emphasized that this example is for illustration purpose only. The recovery distribution used is not realistic, but is close to the recovery markdown to zero. It has maximum variance, which should help with calibration range. A more detailed and systematic study of calibration ranges, loss distributions and hedging properties is needed to justify any choice of marginal spot recovery distribution, the correlation parameters and the possible relationship between the default correlation and recovery correlation. However, as observed by Andensen and Sidenius [5], stochastic recovery alone can not produce the strong correlation skew observed in the market.

6. Conclusion

In this paper, we present a general model of stochastic spot recovery rate as an extension to the Gaussian Copula framework. The model has several distinct features. First, the marginal recovery distribution is a free parameter, which can be chosen based on either historical data or market view. Second, spot recovery conditional on the systematic factor is not deterministic, which better reflects the empirical evidence. Third, the model separates the correlation between default latent variables (ρ_d), the correlation between default latent variable and recovery rate latent variable ($\sqrt{\rho_d \rho_l}$) and the correlation between recovery rate latent variables (ρ_l). So it is not driven by a single correlation parameter. It is straight forward to extend the model to include a third independent parameter to further control the correlation between recovery rate latent variables by introducing a second independent systematic factor to the latent variable that drives the recovery rate as in Andersen and Sidenius [5].

Further analysis is needed to understand how to choose the marginal recovery distribution, how to calibrate the extra correlation parameter and the impact of stochastic recovery on pricing and hedging CDOs. The recent research work of Amraoui et al [2] should be extended to include spot recovery models. This may help us determine what kind of model will better describe the CDO market.

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