Amplifying Business Cycles through Credit Constraints

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Abstract

Theory suggests that endogenous borrowing constraints amplify the impact of external shocks on the economy. How big is the amplification? In this paper, we quantitatively investigate this question in the context of a dynamic general equilibrium model with borrowing constraints under two alternatives: (1) borrowing constraint endogenously depends on the borrowers' net worth (2) borrowing constraint is exogenous. Calibrating our model to the Japanese economy, we find evidence of significant amplification. Next, we apply our model to Japan and find that TFP fluctuations can well account for the boom and the bust of the Japanese economy during 1980 to 2000, and the impact is much more significant when borrowing constraint is endogenous.

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1 Introduction

"A banker is a fellow who lends you his umbrella when the sun is shining, but wants it back the minute it begins to rain" : Mark Twain (1835-1910)

In the wake of the Japanese and the East Asian crisis of the nineties, there has been a resurgence of interest in the link between the asset market and the aggregate economy and the role played by endogenous borrowing constraints. The economic experience of Japan and East Asia in the periods just before and during the financial crisis is characterized by two distinct facts: (1) a sharp increase in real estate and asset prices in the years preceding the crisis followed by a significant and prolonged decline during the crisis years and (2) a booming economy just before the crisis characterized by a growth rate exceeding the long run growth rate of the economy that suddenly gave way to a marked economic recession. Are these observations linked? Empirical studies have shown that in Japan and East Asia real estate comprises an important collateral asset so theory tells us to expect the financial accelerator mechanism to play a significant role. Given the endogeneity of borrowing constraints, external shocks would have an amplified effect on the economy through fluctuations in asset values that translate to fluctuations in a firm's ability to borrow which in turn would affect business investment and output. The question is how big is the amplification?

In this paper we quantitatively test the significance of the financial accelerator in neoclassical dynamic general equilibrium model.

We consider two cases: one where borrowing constraint is endogenous and real estate comprises a part of the collateral asset and the second case is one where borrowing constraint is exogenously determined and does not depend on the wealth holding of the borrower. According to the financial accelerator theory, the effect of a shock is much more amplified in the former case as compared to the latter. We apply the models to data and quantitatively search for evidence of such amplification.

The study of the financial accelerator in macro literature can be traced back to Irving Fisher (1933)[1] who stressed the importance of financial factors in the Great Depression. The formal dynamic general equilibrium modelling of the concept of financial accelerator began with Ben Bernanke and Mark Gertler(1989) and has been extended by Nobuhiro Kiyotaki and John Moore (1997) and Charles Carlstrom and Timothy Fuerst (1997). Since then a host of business cycle researchers have shown the importance of asset price fluctuations in the real economy. Ben Bernanke, Mark Gertler and Simon Gilchrist

[1] See Matteo Iacoviello (2005) for a discussion of the historical evolution of financial accelerator theories in macroeconomics.

(1997) studied the role played by financial accelerator under alternative situations, including sticky prices and in a monetary model. Glenn Hubbard (1998) summarizes the empirical evidence gathered from various studies that establish the sensitivity of business investment to a firm's net worth. Matteo Iacoviello (2005) and Matteo Iacoviello and Raoul Minetti (2006) study the link between fluctuations in house values and the aggregate economy.

The real challenge to the importance of financial accelerator came with Narayana Kocherlakota's(2000) study where he questions the quantitative magnitude of amplification and shows that the amplification caused by the financial accelerator is particularly sensitive to the factor shares. Our study is motivated by this challenge. We construct a fully calibrated dynamic general equilibrium model with variable capital and land, allowing for heterogeneity amongst borrowers in terms of their borrowing constraint. We calibrate our model to the Japanese case and test for the degree of amplification under alternative borrowing constraints. Next, we study the sensitivity of the magnitude of amplification to alternative factor shares. Finally, we feed in exogenous fluctuations in TFP to our model and test to what extent our model can quantitatively replicate the Japanese experience during 1980 to 2000.

Given the economic experiences of Japan in the last two decades of the twentieth century, Japan looks like an ideal case for testing the financial accelerator theory . After a decade of boom in the eighties characterized by an average per capita GDP growth rate of 3.6% and a doubling of real estate prices between 1985 and 1991, the average growth rate of per capita GDP fell to .5% in the nineties and real estate prices declined to their pre-1980 level. This overall economic recession characterizing the nineties in Japan has been called "the lost decade" by Edward C. Prescott and Fumio Hayashi(2002). Currently, there are two strands of literature on business cycle fluctuations in Japan. Most studies till date have concentrated on explaining the fluctuations in real economic aggregates. Explanations forwarded include fluctuations in financial factors including credit market fluctuations (see for example Takeo Hoshi and Anil Kashyap (2004) or Robert Dekle and Kenneth Kletzer(2003) and Levon Barseghyan (2006)) and fluctuations in total factor productivity (Hayashi and Prescott (2002). The study that comes closest to ours is by Kenneth Kasa (1998) who studies the impact of falling asset and land prices on welfare in Japan, Korea and Hong Kong but he takes the fluctuations in real estate prices as exogenous to the model. Quantitative studies on fluctuations in real estate prices are limited. Sami Alpanda (2006) looks at fluctuations in tax on land as a explanation for fluctuations in land prices. Tomoyuki Nakajima (2003) analyzes the importance of expectations in explaining the fluctuations in real estate prices.

The model we consider is a real business cycle model with heterogenous agents comprising of workers and entrepreneurs, and a government. The tax policy is such that the entrepreneurs can claim the interest paid on borrowed funds as a deduction from the taxable income for calculation of corporate income tax. Such a tax deduction cannot be claimed for funds collected through sale of equity hence entrepreneurs have a bias in favor of debt financing as opposed to equity financing. However, the firms face an endogenous collateral constraint on borrowings which limits their ability to borrow. We consider a borrowing constraint of the "margin call" form where entrepreneurs can only borrow upto a fraction of their wealth where the fraction is commonly referred to as the loan-to-value ratio. This type of borrowing constraint specification has gained ground in international macroeconomics literature (see Enrique Mendoza and Katherine Smith (2006) for a discussion of the "margin call" form). There are three exogenous shocks that affect the economy at any period: shocks to productivity, shocks to loan to value ratio and exogenous changes in government expenditure. Given our model, any external shock has an amplified and persistent impact on aggregate output through changes in wealth of an entrepreneur which affects the maximum amount that an entrepreneur can borrow. Our model is similar to the credit cycle model of Kiyotaki and Moore (1997), henceforth referred to as KM. However, there are two important distinctions: we consider a "margin call" form of the borrowing constraint which is an ex-ante borrowing constraint as opposed to the ex-post borrowing constraint considered in KM; more importantly, KM assumes that entrepreneurs are more impatient than workers which leads to binding borrowing constraints. In our model, the binding borrowing constraints are a result of a tax policy rather than differences in discount rate[2].

Calibrating our model to the Japanese economy we find that the model with endogenous borrowing constraints result in significant amplification of TFP shocks. However, the degree of amplification is sensitive to factor shares as suggested by Kocherlakota(2000). The rate of amplification decreases as the share of capital in output goes down.

We next calculate the time series of TFP fluctuations from the Japanese economy. Feeding in the TFP fluctuations in our model, we find that the model with endogenous borrowing constraints can explain a significant portion of the fluctuation in output though the performance of the model with respect to fluctuations in land prices fall short when compared to the data.

We also feed in the TFP fluctuations in the model with exogenous borrowing constraints. The results reinforces the fact that the endogeneity of borrowing constraints lead to significant amplification of external shocks.

The rest of the paper is organized as follows. The stylized facts from the Japanese economy during 1980 to 2000 are discussed in Section 2. We present the theoretical model in Section 3 and the quantitative results in Section 4. Section 5 concludes the paper.

[2] Alpanda (2006) also uses the same technique.

2 Stylized facts from Japanese economy

We begin by tracing the fluctuations in per capita output in Japan during the period 1980 to 2000. The national income accounts data is collected from Hayashi and Prescott (2002). In our analysis, the population comprises of working age population i.e. population in the age groups 20 to 69. The average growth rate of population is .78% and the average growth rate of per capita output is 2.2% which is slightly higher than the long term average growth rate of per capita output in United States which stands at 2%.

In Figure 1-a we plot the per capita output detrended at 2.2%. After years of stable growth during 1980 to 1985, Japanese economy experienced an economic boom during 1986 to 1991 when output grew at a rate of 1.39% above trend. However, since 1991 the growth stalls and except for a brief recovery during 1994 to 1996, output continues to decline. During 1991 to 2000 the average growth rate of per capita output is 1.68% below trend.

In Figure 1-b we plot the capital output ratio that registers a continuous increase throughout the eighties and the nineties. Capital output ratio increases from 1.74 in 1980 to 2.36 in 2000 exhibiting significant capital deepening.

In Figure 1-c we plot the share of non-residential land in total land where we normalize the total land holdings in the economy to 1. Non-residential land in our setup comprises of land used for cultivation and land underlying the non-residential buildings. Residential land is the land underlying residential buildings. We assume that the sum of non-residential and residential land in the economy is 1. The data is collected from Japan Statistical Yearbook . The share on non-residential land increases by .91% during 1986 to 1991 and eventually falls by 16.7% by 2000.

In Figure 1-d we plot the index of urban land prices as collected from Japan Statistical Yearbook. In the yearbook, the urban land price index in year 2000 is taken as 100. For our analysis, we take the land price in year 1980 as 100. During the eighties the average growth rate of urban land prices is 7.6%. The trend continues till 1991 when land prices start falling. During the nineties urban land prices *fall* at an average rate of 4.24%.

3 Theoretical model

We model the Japanese economy as a heterogenous-agent economy model with two groups of infinitely lived agents: entrepreneurs and workers. Entrepreneurs are of two types: type 1 produces residential services (or housing) and type 2 produces non-residential goods. To keep matters simple, we assume that a person is either born as an entrepreneur (either type 1 or type 2) or as a worker and it is not possible to switch types. We further assume that the fraction of workers and entrepreneurs of each type remain constant every period. In addition to the workers and the entrepreneurs, there exists an infinite number of perfectly competitive firms that use the residential services and non-residential goods as inputs to produce the aggregate consumption good y that is used for final consumption and investment. Lastly, there is a government that collects tax revenue, spends a part of it for government consumption and disburses the rest as transfers to the economic agents.

Workers are endowed every period with one unit of time. They spend a part of their time working for the entrepreneurs and the rest of the time is spent in leisure. There are two sources of income for the worker: wage income and interest income. After paying labor income taxes and taxes on interest income, the workers use the rest of their income to buy consumption goods and the remainder is saved.

The type 1 entrepreneurs begin any period t with certain amount of land holding. They own the technology to produce residential services denoted by y_1 which is produced using land and labor. For simplicity we do not introduce capital in production of residential services. The revenue earned from residential services is used by the entrepreneur to pay wages to the labor and the remainder is spent on consumption and investment in land. The entrepreneur can also borrow but they face a borrowing constraint which allows them to borrow only a fraction of their wealth.

The second group of agents are the entrepreneurs who own the production process to produce the non-residential goods denoted by y_2 . The type 2 entrepreneurs begin every period with a certain amount of capital and land..The entrepreneurs combine capital, land and labor to produce the non-residential goods, the returns to which are used for consumption and investment in capital and land. The type 2 entrepreneurs can borrow upto a certain fraction of their wealth every period.

Both groups of entrepreneurs have to pay corporate income taxes but can claim the interest paid on borrowed funds as a deductible from their taxable income. This "tax shelter" creates an incentive for the entrepreneurs to raise funds through debt financing as opposed to equity financing.

We further assume that entrepreneurs do not value leisure. This assumption is common in literature and has been used by Iacoviello (2005), Iacoviello and Minetti (2006) amongst others.

The amount of land in the economy is constant and is normalized to one. The final good is treated as the numeraire so that wages, interest rate and prices are in terms of the final consumption good.

Let us begin with the economic problem of the representative worker.

3.1 Workers

The representative worker's (denoted by a superscript w) problem is a standard one. Workers maximize the presented discounted value of a lifetime utility function given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u_t^w(c_t^w, h_t^w)$$

where E_0 is the expectations operator, $\beta \in (0,1)$ is the discount factor, c_t^w denotes consumption of the workers at time t, and h_t^w denotes the hours worked by the workers. The workers maximize the lifetime utility subject to a budget constraint:

$$c_t^w + s_{t+1} - s_t \leq w_t h_t^w (1 - \tau_{ht}) + r_t(1 - \tau_{st}) s_t + Tr_t^w$$

where w_t is the wage rate and r_t is the interest rate on savings in units of final consumption good y. The tax on their labor income is τ_{ht}, and tax on their interest earnings is denoted by τ_{st}. In addition, workers also get transfers Tr_t^w from the government every period.

Our description of the workers problem is different from other financial accelerator models like the ones in Iacoviello(2005), Minetti and Iacoviello (2006) in that we do not have land in the workers' utility function. In earlier studies the description of the economy was such that the workers valued land for consumption and entrepreneurs value land only as an input. In our model we assume that the workers as well as the entrepreneurs both value residential services. The way we model this is by assuming that the final consumption good y in the economy that is consumed by workers and entrepreneurs is made up of both non-residential goods as well as residential services. In addition to the budget constraint, the workers also face non-negativity constraints every period.

Given the information listed above, we can summarize the representative worker's problem by the following set of first-order conditions, where $\beta^t \lambda_t^w$ is the lagrange multiplier attached to the workers' budget constraint at time t. We denote the derivative of an utility function $u_t^w(.)$ with respect to a variable z with $u_{zt}^w(.)$.

$$c_t^w : \beta^t u_{ct}^w(c_t^w, h_t^w) = \beta^t \lambda_t^w \tag{1}$$

$$l_t^w : \beta^t u_{ht}^w(c_t^w, h_t^w) = \beta^t \lambda_t^w w_t (1 - \tau_{ht}) \tag{2}$$

$$a_{t+1} : \beta^{t+1} E_t \lambda_{t+1}^w (1 + r_{t+1}(1 - \tau_{st+1})) = \beta^t \lambda_t^w \tag{3}$$

$$\lambda_t^w : c_t^w + s_{t+1} - s_t = w_t h_t^w (1 - \tau_{ht}) + r_t(1 - \tau_{st}) s_t + Tr_t^w \tag{4}$$

Combining the equations we get:

$$
\begin{aligned}
u^w_{lt}(c^w_t, h^w_t) - u^w_{ct}(c^w_t, h^w_t) w_t (1 - \tau_{ht}) &= 0 \qquad (5)\\
\beta E_t u^w_{ct+1}(c^w_{t+1}, h^w_{t+1})(1 + r_{t+1}(1 - \tau_{st+1})) - u^w_{ct}(c^w_t, h^w_t) &= 0 \qquad (6)\\
c^w_t + s_{t+1} - s_t - w_t h^w_t (1 - \tau_{ht}) - r_t (1 - \tau_{st}) s_t - Tr^w_t &= 0 \qquad (7)
\end{aligned}
$$

Equation (5) tells us that in equilibrium, the marginal rate of substitution between consumption and leisure of the worker equals the after-tax marginal product of labor. Equation (6) tells us that in equilibrium, the present discounted value of lifetime utility that would ensue from withholding one unit of consumption today and saving it instead for consumption in future (after deducting for the tax on interest income earned) is equal to the marginal disutility that stems today from the lost consumption. Equation (7) is the workers' budget constraint.

3.2 Type 1 Entrepreneurs

The representative type one entrepreneur (denoted by a superscript 1) maximizes the presented discounted value of a lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t u^1_t(c^1_t)$$

where E_0 is the expectations operator, $\beta \in (0, 1)$ is the discount factor, c^1_tdenotes consumption of the workers at time t. The entrepreneur's maximize the lifetime utility subject to a budget constraint:

$$c^1_t + q_t(l^1_{t+1} - l^1_t) \leq (1 - \tau_{yt})(p_{1t} y_{1t} - w_t h^1_t - r_t b^1_t) + b^1_{t+1} - b^1_t + Tr^1_t$$

where y_{1t} is the amount of residential services produced at time t and p_{1t} is the price of residential services in terms of final consumption good. The entrepreneurs are assessed a tax on their corporate income at the rate τ_{yt}. The entrepreneurs also get transfers Tr^1_t from the government every period. The entrepreneurs can claim the interest paid on borrowed funds as a deductible from the corporate income tax calculation as given by the expression $(1-\tau_{yt})(p_{1t} y_{1t} - w_t h^1_t - r_t b^1_t)$, where h^1_t is the demand of worker's labor by the representative entrepreneur. The entrepreneurs also borrow an amount $b^1_{t+1} - b^1_t$ every period. The after-tax earnings of the entrepreneur are used to finance consumption c^1_t and invest in land for the next period (denoted by $l^1_{t+1} - l^1_t$). In addition to the budget constraint, the entrepreneurs also face the collateral constraint:

$$b^1_{t+1} \le B^1_{t+1}$$

$$B^1_{t+1} = \phi_{1t}(q_t l^1_{t+1})$$

when borrowing constraint is endogenously determined by the net tangible asset holdings of the entrepreneur and

$$B^1_{t+1} = \overline{B}_{t+1}$$

when borrowing constraint is exogenous

This collateral constraint is of the ex-ante form and is distinct from the ex-post borrowing constraint considered in literature where the borrowing limit is determined as a fraction of the present value of future wealth holdings.

Given the setup of our problem, we can show that the borrowing constraint holds with equality in the steady state (please find the proof in Appendix 1).

The entrepreneur also faces a constraint in the form of the production technology that can be summarized by:

$$y_{1t} \le A_{1t}F_t(h^1_t,\ l^1_t)$$

where A_{1t} is time varying productivity.

We can summarize the representative type 1 entrepreneur's problem by the following set of first-order conditions, where $\beta^t\lambda^1_t$ is the lagrange multiplier attached to the entrepreneur's budget constraint at time t, and $\beta^t\mu_{1t}$ denotes the lagrange multiplier attached to the collateral constraint at time t. We denote the derivative of an utility function $u^1_t(.)$ with respect to a variable z with $u^1_{zt}(.)$ and the derivative of the production function $F_t(.)$ with respect to a variable z by $F_{tz}(.)$. Assuming entrepreneurs face an endogenous borrowing constraint, the first order conditions that define the entrepreneurs problem are:

$$c^1_t \ : \ \beta^t u^1_{ct}(c^1_t) - \beta^t\lambda^1_t = 0 \tag{8}$$

$$h^1_t \ : \ \beta^t(1-\tau_{yt})\lambda^1_t p_{1t}A_{1t}F_{ht}(h^1_t,\ l^1_t) - \beta^t\lambda^1_t w_t(1-\tau_{yt}) = 0 \tag{9}$$

$$l^1_{t+1} \ : \ \begin{array}{c} \beta^{t+1}E_t\{(1-\tau_{yt+1})\lambda^1_{t+1}p_{1t+1}A_{1t+1}F_{lt+1}(h^1_{t+1},\ l^1_{t+1}) \\ q_{t+1}\lambda^1_{t+1}\} - \beta^t q_t\lambda^1_t + \beta^t\mu_{1t}\phi_{1t}q_t = 0 \end{array} \tag{10}$$

$$b_{1t+1} \ : \ \left(-\beta^{t+1}E_t\lambda^1_{t+1}(1+r_{t+1}(1-\tau_{yt+1})) + \beta^t\lambda^1_t - \beta^t\mu_{1t}\right)\mu_{1t} = 0 \tag{11}$$

Note that when borrowing constraint binds with equality, $\mu_{1t} > 0$ and the set of equations (8) to (11) reduces to:

$$p_{1t}A_{1t}F_{ht}(h^1_t,\ l^1_t) - w_t = 0 \tag{12}$$

$$\begin{array}{c} \beta^{t+1}E_t u^1_{ct+1}(c^1_{t+1})\{(1-\tau_{yt+1})(p_{1t+1}A_{1t+1}F_{lt+1}(h^1_{t+1},\ l^1_{t+1}) \\ -r_{t+1}\phi_{1t}q_t + (q_{t+1} - q_t\phi_{1t})\} - \beta^t q_t u^1_{ct}(c^1_t)(1-\phi_{1t}) = 0 \end{array} \tag{13}$$

Equation (12) reflects the fact that in equilibrium, wage rate is equal to the after-tax marginal productivity of labor. Equation (13) gives us the equilibrium condition for investment in land by the entrepreneur, which essentially tells us that in equilibrium, the present discounted value of lifetime utility obtained by withholding one unit of consumption and instead investing it in land and deriving future consumption from it (after deducting the appropriate taxes) is equal to the loss in utility that ensues today due to consumption withheld. This is a standard condition except there are two sources of deriving benefits from investment in land: firstly, the entrepreneur gets the benefit of increased production when she uses investment today as an input in the production process tomorrow (this is reflected in the marginal product of land), secondly, the entrepreneur can use the investment in land as a collateral asset that helps the entrepreneur in boosting her borrowing capability. This second source of benefit that the entrepreneur derives by additional investment results from the collateral constraint being a function of the net worth of the entrepreneur and is unique to models with endogenous collateral constraints.

3.3 Type 2 Entrepreneurs

The economic problem of the type 2 entrepreneur is very similar to that of type 1.

The representative type two entrepreneur (denoted by a superscript 2) maximizes the presented discounted value of a lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t u_t^2(c_t^2)$$

where E_0 is the expectations operator, $\beta \in (0,1)$ is the discount factor, c_t^2 denotes consumption of the entrepreneur at time t.

The entrepreneur's maximize the lifetime utility subject to a budget constraint:

$$c_t^2 + q_t(l_{t+1}^2 - l_t^2) + k_{t+1} - k_t \leq (1 - \tau_{yt})(p_{2t} y_{2t} - w_t h_t^2 - \delta k_t - r_t b_t^2) + b_{t+1}^2 - b_t^2 + Tr_t^2$$

where y_{2t} is non-residential good produced at time t and p_{2t} is the price of the non-residential good in terms of final consumption good. The entrepreneurs are assessed a tax on their corporate income at the rate τ_{yt}. The entrepreneurs also get transfers Tr_t^2 from the government every period. The entrepreneurs can claim the depreciated capital and the interest paid on borrowed funds as a deductible from the corporate income tax calculation as given by the expression $(1 - \tau_{yt})(p_{2t} y_{2t} - w_t h_t^2 - \delta k_t - r_t b_t^2)$, where h_t^2 is the demand of worker's labor by the representative type two entrepreneur and δ is the depreciation rate. The entrepreneurs also borrows an amount $b_{t+1}^2 - b_t^2$ every period. The after-tax earnings of the entrepreneur are used to finance consumption c_t^2 and investment in land and capital for the next period (denoted by $l_{t+1}^2 - l_t^2$ and $k_{t+1} - k_t$ respectively). In addition to the budget constraint, the entrepreneurs also face the collateral constraint:

$$b_{t+1}^{21} \leq B_{t+1}^2$$

$$B_{t+1}^2 = \phi_{2t}(k_{t+1} + q_t l_{t+1}^2)$$

when borrowing constraint is endogenously determined by the net tangible asset holdings of the entrepreneur and

$$B_{t+1}^2 = \overline{B}_{t+1}$$

when borrowing constraint is exogenous

The proof that the borrowing constraint holds with equality in the steady state is same as for type one entrepreneur and is summarized in Appendix 1.

The production technology is given by:

$$y_{2t} \leq A_{2t} F_t(k_t,\ h_t^2,\ l_t^2)$$

where A_{2t} is time varying productivity.

We can summarize the representative type 2 entrepreneur's problem by the following set of first-order conditions, where $\beta^t \lambda_t^2$ is the lagrange multiplier attached to the entrepreneur's budget constraint at time t, and $\beta^t \mu_{2t}$ denotes the lagrange multiplier attached to the collateral constraint at time t. We denote the derivative of an utility function $u_t^2(.)$ with respect to a variable z with $u_{zt}^2(.)$ and the derivative of the production function $F_t(.)$ with respect to a variable z by $F_{tz}(.)$. Assuming entrepreneurs face an endogenous borrowing constraint, the first order conditions that define the entrepreneurs' problem are:

$$c_t^2 : \beta^t u_{ct}^2(c_t^2) - \beta^t \lambda_t^2 = 0 \tag{14}$$

$$h_t^2 : \beta^t (1 - \tau_{yt}) \lambda_t^2 p_{2t} A_{2t} F_{ht}(k_t,\ h_t^2,\ l_t^2) - \beta^t \lambda_t^{12} w_t (1 - \tau_{yt}) = 0 \tag{15}$$

$$l_{t+1}^2 : \begin{array}{c} \beta^{t+1} E_t \{(1 - \tau_{yt+1}) \lambda_{t+1}^2 p_{2t+1} A_{2t+1} F_{lt+1}(k_{t+1},\ h_{t+1}^2,\ l_{t+1}^2) \\ + q_{t+1} \lambda_{t+1}^2\} - \beta^t q_t \lambda_t^2 + \beta^t \mu_{2t} \phi_{2t} q_t = 0 \end{array} \tag{16}$$

$$k_{t+1} : \begin{array}{c} \beta^{t+1} E_t \{+(1 - \tau_{yt+1}) \lambda_{t+1}^2 p_{2t+1} A_{2t+1} F_{kt+1}(k_{t+1},\ h_{t+1}^2,\ l_{t+1}^2) \\ + \lambda_{t+1}^2 (1 - \delta)\} - \beta^t \lambda_t^2 + \beta^t \mu_{2t} \phi_{2t} = 0 \end{array} \tag{17}$$

$$b_{2t+1} : \left(-\beta^{t+1} E_t \lambda_{t+1}^2 (1 + r_{t+1}(1 - \tau_{yt+1})) + \beta^t \lambda_t^2 - \beta^t \mu_{2t}\right) \mu_{2t} = 0 \tag{18}$$

Note that when borrowing constraint binds with equality, $\mu_{2t} > 0$ and the set of equations (14) to (18) reduces to:

$$p_{2t}A_{2t}F_{ht}(k_t,\ h_t^2,\ l_t^2) - w_t = 0 \tag{19}$$

$$\begin{aligned}\beta^{t+1}E_t u_{ct+1}^2(c_{t+1}^2)\{(1-\tau_{yt+1})(p_{2t+1}A_{2t+1}F_{lt+1}(k_{t+1},\ h_{t+1}^2,\ l_{t+1}^2) \\ +(q_{t+1}-q_t\phi_{2t}) - r_{t+1}\phi_{2t}q_t\} - \beta^t q_t u_{ct}^2(c_t^2)(1-\phi_{2t}) = 0\end{aligned} \tag{20}$$

$$\begin{aligned}\beta^{t+1}E_t u_{ct+1}^2(c_{t+1}^2)\{(1-\tau_{yt+1})(p_{2t+1}A_{2t+1}F_{kt+1}(k_{t+1},\ h_{t+1}^2,\ l_{t+1}^2) \\ +1-\phi_{2t} - r_{t+1}\phi_{2t}\} - \beta^t u_{ct}^2(c_t^2)(1-\phi_{2t}) = 0\end{aligned} \tag{21}$$

The equations are similar to the ones described in Equations (12) and (13). The only additional equation is Equation (21) that gives us the equilibrium condition for investment in capital by the entrepreneur, which essentially tells us that in equilibrium, the present discounted value of lifetime utility obtained by withholding one unit of consumption and instead investing it in capital and deriving future consumption from it (after deducting the appropriate taxes) is equal to the loss in utility that ensues today due to consumption withheld.

3.4 Firms producing final goods

The profit maximizing problem of a representative firm that produces the final good y is summarized by:

$$Max\ y_t - \sum_{i=1}^{2} p_{it}y_{it}$$

subject to the production constraint:

$$y_t \leq v(y_{1t}, y_{2t})$$

The necessary first order condition reduces to:

$$(v_1(y_{1t}, y_{2t}) - p_{1t})\,\lambda_{1yt} = 0 \tag{22}$$

$$(v_2(y_{1t}, y_{2t}) - p_{2t})\,\lambda_{2yt} = 0 \tag{23}$$

where $\lambda_{iyt}, i \in (1, 2)$ is the lagrange multiplier on the production technology constraint.

3.5 Government and Resource Constraints

The role of the government in my model is a passive one. Government collects tax revenues and spends it on expenditure and transfers back to the entrepreneurs and workers so as to balance the budget every period. Assuming that $N_{it}, i \in (1, 2)$ denotes the number of entrepreneurs of type i and N_{wt} denotes the total number of workers in period t, the budget constraint of the government can be summarized as:

$$\begin{array}{c} N_{wt}\left(w_t h_t^w \tau_{ht} + r_t \tau_{st} s_t\right) + N_{1t}\tau_{yt}(p_{1t}y_{1t} - w_t h_t^1 - r_t b_t^1) \\ +N_{2t}\tau_{yt}(p_{2t}y_{2t} - w_t h_t^2 - \delta k_t - r_t b_t^2) \leq g_t + T_t \end{array}$$

where T_t denotes the aggregate transfer. We further assume that the share of transfers to workers and entrepreneurs in total transfers is fixed and is given by:

$$\begin{aligned} N_{wt}Tr_t^w &= t_w T_t && (24) \\ N_{1t}Tr_t^1 &= t_1 T_t && (25) \\ N_{2t}Tr_t^2 &= (1 - t_w - t_1)T_t && (26) \end{aligned}$$

We close our model by summarizing the resource constraints every period:

$$\begin{aligned} N_{wt}c_t^w + \sum_{i=1}^{2} N_{it}c_t^i + N_{2t+1}k_{t+1} - (1-\delta)N_{2t}k_t + g_t - y_t &\leq 0 && (27) \\ \sum_{i=1}^{2} N_{it}b_t^i - N_{wt}s_t &\leq 0 && (28) \\ \sum_{i=0}^{2} N_{it}h_t^i - N_{wt}h_t^w &\leq 0 && (29) \\ \sum_{i=0}^{2} N_{it}l_t^i - 1 &\leq 0 && (30) \end{aligned}$$

Equation (27) summarizes the goods market clearing condition. Equation (28) summarizes the loan-market clearing condition and states that borrowings must be less than or equal to savings every period. Equation (29) is the labor-market clearing condition that says demand for labor by the entrepreneurs every period is less than or equal to supply of labor by workers. Equation (30) tells us that the sum of land held by the entrepreneurs must be less than the total endowment of land in the economy, which we assume is one unit every period.

3.6 Equilibrium

Given that the utility function of the workers and the entrepreneurs is strictly quasi-concave and the constraints are linear, the model has a unique equilibrium that is characterized by a set of allocation functions summarized by the vector of allocations$\{c_t^w, c_t^1, c_t^2, h_t^w, h_t^1, h_t^2, l_{t+1}^1, l_{t+1}^2, k_{t+1}, y_t, y_{1t,} y_{2t}, b_{t+1}^1, b_{t+1}^2, s_{t+1},$

$Tr_t^w, Tr_t^1, Tr_t^2, T_t\}_{t=0}^{\infty}$,and a sequence of prices $\{w_t,\ r_t,\ q_t, p_{1t,}\ p_{2t}\}_{t=0}^{\infty}$ such that given the state of the economy at any period t summarized by the state vector, $\{k_t,\ l_t^1,\ l_t^2,\ b_t,\ s_t\}$,and a sequence of exogenous variables summarized by the vector $\{A_{1t},\ A_{2t},\ g_t,\ \phi_{1t},\ \phi_{2t}\}_{t=0}^{\infty}$ the set of allocation functions and the sequence of price functions satisfy the workers utility maximization problem, the entrepreneurs utility maximization problem and the final goods producing firms profit maximization problem summarized in the previous subsections, the government budget is balanced every period, and the resource constraints are satisfied along with the the relevant transversality conditions.

A balanced growth in this economy occurs when per capita consumption, outputs, investment in capital, borrowings and transfers grow at a constant rate $(1 + g_z)$ where g_z is the long run rate of technical progress and all other variables remains constant. For our analysis we also assume that the growth rate of population η is constant and same for all groups.

4 Quantitative Results

In this section, we calibrate our model to match the moments of the Japanese economy. Next, we solve for the decision rules and feed in exogenous shocks in our model to account for the observed economic fluctuations in Japan during 1980 to 2000

4.1 Model Propositions

We begin with a proposition that establishes the condition for borrowing constraints to bind in equilibrium. Note that we can only formally prove the proposition for the steady state.

Proposition 2 *In the steady state, the borrowing constraint of the entrepreneur will hold with equality if and only if the tax charged on the interest earnings of the workers is less than the corporate income tax rate, i.e.*

$$\begin{aligned}
b_{t+1} &= B_{t+1},\ iff\ \tau_y > \tau_s \\
b_{t+1} &\in [0,\ B_{t+1}],\ iff\ \tau_y = \tau_s \\
b_{t+1} &= 0,\ iff\ \tau_y < \tau_s
\end{aligned}$$

Proof. We provide a formal proof of the proposition in Appendix 1, but the idea is a simple one. The entrepreneurs have two ways to finance their investment: equity-financing and debt-financing. Given all other things remain the same, entrepreneurs will prefer one mode of financing over the other if there are some tax advantages attached to one mode. Typically, government allows entrepreneurs to claim the interest paid on their borrowed funds as a deductible from their corporate income in calculation of the corporate income tax payment. Such an option is not allowed for equity-financed investment, which creates a bias in favor of debt-financing. The government, on the other hand, makes up partly for revenue loss due to this tax-shelter by charging a tax on the interest earnings of the workers who lend to the entrepreneurs. As we can show in the proof, the workers can pass on this tax to the entrepreneurs through the market interest rate which partly depends on the tax on interest earnings. So, essentially the entrepreneurs have to weigh the two taxes: the savings generated by the ability to deduct interest payments from corporate tax calculations and the expenses of tax on interest income that is often passed on to the entrepreneurs through the market interest rate. A rational entrepreneur will therefore favor debt financing as long as the savings of the tax shelter scheme outweigh the cost of having to bear the tax on interest earnings of the lenders. This would happen if the corporate income tax rate is higher than the tax rate on interest earnings, which is often the case.

When the two tax rates are equal, the entrepreneur is essentially indifferent between the two modes of financing, and when the corporate income tax rate is lower than the tax charged on interest earnings of the lenders, the entrepreneurs favor equity-financing. This last option is hard to come by in the real world but is definitely a theoretical possibility in which case the borrowing constraint does not hold for the entrepreneur and the model is no longer a relevant one.

Given that the borrowing constraint binds in the steady-state, we assume that it also binds in the neighborhood of the steady state and apply usual log-linearization techniques to solve for the decision rules. ■

4.2 Steady state and Calibration

At this point it is useful to look at the steady state equations that summarize the model.

For our future analysis we assume the following functional forms of our utility and production functions:

$$\begin{aligned} u_t^w(c_t^w, h_t^w) &= \frac{\left((c_t^w)^\varsigma (1-h_t^w)^{1-\varsigma}\right)^{1-\sigma} - 1}{1-\sigma}, \sigma > 1 \\ &= \varsigma \log\ c_t^w + (1-\varsigma) \log\ (1-l_t^w), \sigma = 1 \end{aligned}$$

$$
\begin{aligned}
u_t^i(c_t^i) &= \frac{c_t^{i1-\sigma}-1}{1-\sigma}, \sigma > 1 \\
&= \log\ c_t^i, \sigma = 1 \text{ where } i \in (1,2)
\end{aligned}
$$

$$
\begin{aligned}
A_{1t}F_t(h_t^1,\ l_t^1) &= A_{1t}l_t^{1\theta_{l1}}h_t^{1(1-\theta_{l1})} \\
A_{2t}F_t(k_t,\ h_t^1,\ l_t^1) &= A_{2t}k_t^{\theta_k}l_t^{2\theta_{l2}}h_t^{2(1-\theta_k-\theta_{l2})} \\
f(y_{1t},\ y_{2t}) &= y_{1t}^{\alpha}y_{2t}^{1-\alpha}
\end{aligned}
$$

ς is the elasticity of substitution between consumption and leisure of the worker. α is the share of residential services in output. θ_{l1} is the share of land in residential services, θ_k and θ_{l2} are the shares of capital and land in the non-residential goods. For our analysis, we assume that the size of workers and both types of entrepreneurs is same and we normalize the size to one.

Given the functional forms and the long term growth rate of output and population, the steady state equations are summarized in Appendix 2.

The objective of the calibration exercise is to estimate the parameters of our model such that the moments of our model match the moments from data. We take the moments from Japanese data during the period 1980 to 1984 when Japan was growing at a steady pace and by many accounts (including Hayashi and Prescott (2002)) had reached a steady state. The data is from Hayashi and Prescott (2002) dataset which in turn is collected from Japan Statistical Yearbooks and the Economic and Social Research Institute of Japan. The dataset has become the standard in studies about Japanese economy and has been used by Imrohoroglu, Imrohoroglu and Chen(2006) amongst others. The average share of investment in output is .25 and the average share of government consumption in output is .14 during this period. The average capital output ratio is 1.8 which yields a depreciation rate $\delta = .1$. The share of housing which we use as a proxy for residential services in output is .04 which yields $\alpha = .04$. The data for taxes is collected from Hiromitshu Ishi(1999). The average labor income tax rate in Japan $\tau_h = 33\%$, the average corporate tax rate $\tau_y = 49.5\%$ and the tax rate on return to savings $\tau_s = 19\%$. Note that the corporate income tax rate in Japan is much higher than the tax rate on interest on savings which makes borrowing constraint hold with equality in the steady state. For our analysis, we abstract from any changes in taxes, so taxes in my model are time-invariant.

The data on borrowings is collected from the Japan Statistical Yearbook. The net borrowings of the consumers for housing purposes is taken as a proxy for the share of borrowings of entrepreneur 1 in total borrowing so that $\frac{b_1}{y} = .45$. Given the net borrowings of the non-corporate sector, we take the share of borrowings in output $\frac{b_2}{y} = 1.75$. Note that Alpanda (2006) takes the share of aggregate borrowing in GDP in Japan to be 2.2 which matches the share of aggregate borrowing in our model which is the ratio of sum of borrowings of entrepreneur 1 and 2 to output. The value of residential land (taken as the value of land underlying residential buildings) as a ratio of output or $\frac{ql_1}{y} = 1.55$ and the sum of the value of land underlying non-residential buildings and cultivated land is taken as the value of land held by type 2 such that $\frac{ql_2}{y} = 1.08$. In our analysis, we assume that the economy is closed and that all entrepreneurs face borrowing constraints. In Japan, it is well documented that the small and the medium firms face borrowings constraints and use their land holdings as a collateral. As for the bigger firms, we assume that the ability to raise capital depends on their wealth holdings. This is not a very unrealistic assumption given the fact that the *keiretsu* or the firms that had a preferred position in the Japanese loan markets were bigger firms that had a good network based on reputation.

Given the value of land and share of borrowings of entrepreneur 1, $\phi_1 = .299$ and given the capital output ratio, value of land held by entrepreneur 2 and $\frac{b_2}{y}$, $\phi_2 = .6143$ (a value similar to Alpanda (2006)).

We take the interest rate to be 5% and the rate of relative risk aversion $\sigma = 1.5$ that is common in literature. Further, the long term growth rate of per capita output $g_z = 2.2\%$ and the long term growth rate of working population $\eta = .78\%$ which yields $\beta = .99$.

Labor $h = .33$ which yields $\frac{1-\xi}{\xi} = \psi = 1.513$. The corresponding measure for Hayashi and Prescott (2002) is 1.37.

In our model, as opposed to the common symbol in macro literature, θ_k denotes the share of capital in non-residential output, θ_{1l} denotes the share of land in housing or residential output, θ_{2l} denotes the share of land in non-residential output and θ_h denotes the share of labor in non-residential output.

The calibration exercise yields $\theta_k = .3198$, $\theta_{1l} = .4446$, $\theta_{2l} = .0025$ and $\theta_h = .6777$.

Note that given the share of residential output in total output or $\alpha = .04$, the share of capital in total output $y = (1-\alpha)\theta_k = .3070$,share of residential land in total output $\alpha * \theta_{1l} = .0178$, share of non-residential land in total output $(1-\alpha) * \theta_{2l} = .0024$ and the share of labor in total output $= \alpha * (1-\theta_{1l}) + (1-\alpha) * \theta_h = .6143$ which adds up to 1.

The data for transfers is not available by employment or wealth cohorts. We assume that 80% of the total transfers go to the workers and the remaining goes to the entrepreneurs. We justify our findings by the fact that a larger portion of transfers in developed countries typically go to the workers and medium businesses as compared to big businesses. Literature on transfer payments by employment or wealth cohorts in general is limited. Studies on US economy suggests that about 75% to 90% of social security transfers go to the poorer households. We check the robustness of our results for alternative share of transfers and find our results are not sensitive to alternative shares. The steady state values and the calibrated parameters are summarized in Table1 and Table 2.

4.3 Impulse responses

Given the theoretical proposition, borrowing constraint will bind in the steady state as long as $\tau_y > \tau_s$.

We assume that the borrowing constraint binds in the neighborhood of the steady state and solve for the decision rules using the technique of log-linearization outlined in Robert King, Charles Plosser and Sergio Rebelo (1988). The set of log-linearized equations that we solve to derive our decision rules are summarized in Appendix 2.

We denote the deviations of the variables from their steady state by a tilda where:

$$\begin{aligned}
\widetilde{A}_{1t} &= \log A_{1t} - \log A_1 \\
\widetilde{A}_{2t} &= \log A_{2t} - \log A_2 \\
\widetilde{\phi}_{1t} &= \phi_{1t} - \phi_1 \\
\widetilde{\phi}_{2t} &= \phi_{2t} - \phi_2 \\
\widetilde{g}_t &= \log g_t - \log g
\end{aligned}$$

We assume that the deviations in exogenous variables follow a vector autoregressive process of order one.

$$\begin{aligned}
\widetilde{A}_{1t+1} &= \rho_{a1}\widetilde{A}_{1t} + e_{a1t} \\
\widetilde{A}_{2t+1} &= \rho_{a2}\widetilde{A}_{2t} + e_{a2t} \\
\widetilde{\phi}_{1t+1} &= \rho_{\phi 1}\widetilde{\phi}_{1t} + e_{\phi 1t} \\
\widetilde{\phi}_{2t+1} &= \rho_{\phi 2}\widetilde{\phi}_{2t} + e_{\phi 2t} \\
\widetilde{g}_{t+1} &= \rho_g \widetilde{g}_t + e_{gt}
\end{aligned}$$

Given the time series of estimated productivities, the loan to value ratios and government consumption, we estimate:

$\rho_{a1} = .72,\ \rho_{a2} = .78,\ \rho_{\phi 1} = .84,\ \rho_{\phi 2} = .96$ and $\rho_g = .95$.

The vector of standard deviations of the errors :

$$\sigma = \begin{bmatrix} .04 & 0 & 0 & 0 & 0 \\ 0 & .04 & 0 & 0 & 0 \\ 0 & 0 & .06 & 0 & 0 \\ 0 & 0 & 0 & .09 & 0 \\ 0 & 0 & 0 & 0 & .15 \end{bmatrix}$$

We next plot the impulse response of output, land price, labor and capital to a 1% positive shock to productivities A_1 and A_2 and a 1% positive shock to the loan to value ratios ϕ_1 and ϕ_2.

4.3.1 Impulse response : Amplification and endogeneity of borrowing constraint

As Figure 2-a depicts, a 1% positive shock to A_1 increases output by .065% and land prices by .005% at impact. The effect is much smaller if we consider an exogenous borrowing constraint. As depicted in Figure 2-b, a 1% positive shock to A_1 increases output by .055% . However, land prices fall by .038% on impact.

The result is similar when we consider a 1% positive shock to A_2. As Figure 3-a depicts output increases by 1.7% and land price increases by .2% on impact before the effects start diminishing. When we consider borrowing constraint to be exogenous, output increases by 1.4% and land price decreases by 1.2%. This is depicted in Figure 3-b.

Given our model parameters as calibrated to the Japanese economy we thus find evidence of amplification when borrowing constraint is endogenous.

Apart from A_{1t} and A_{2t}, fluctuations in the loan to value ratio ϕ_{1t} and ϕ_{2t} also affect real macroeconomic variables. In Figures 4-a and 4-b, we plot the impulse responses of a 1% positive shock to the loan to value ratio Due to a 1% positive shock to ϕ_1 output falls by 4.2% and land price falls by 3.8%. At the same time a 1% positive shock to ϕ_2 results in a .8% fall in output and a .5% decline in land price.

The response to a positive shock to loan to value ratio generates a decline in labor and output as opposed to an increase. The result follows from the fact that an increase in the loan to value ratio has a dominant wealth effect on the workers which results in an increase in consumption of leisure , a normal good. This results in reduced labor supply that given predetermined capital stock and land for production, results in a decrease in output.

4.3.2 Impulse response: Sensitivity of amplification to factor shares

One of the main arguments forwarded by Kocherlakota(2000) is that the neoclassical general equilibrium models with credit constraints lose their power of amplification as factor shares change. More specifically, the decline in share of capital in output results in a decline in amplification. We test this finding in our model. We keep the factor share of land the same as before and reduce the share of capital θ_k to .2 (a measure similar to Kocherlakota(2000)).

Hence the new set of factor shares are : $\{\theta_k = .2, \theta_{l1} = .4446, \theta_{l2} = .0025, \theta_h = .8\}$.

In Figures 5-a and 5-b, we plot the impulse response of a 1% positive shock to A_1 in a model with endogenous and a model with exogenous borrowing constraint respectively. In the former case, output increases by .064% and land price increases by .002%. In the latter case when borrowing constraint is exogenous, output increases by .061% and land price registers a fall by .008%.

As for a 1% positive shock to A_2, output in a model with endogenous borrowing constraint increases by 1.7% and land price increases by .2%. In comparison, in a model with exogenous borrowing constraint, output increases by 1.6% and land price falls by .025% as depicted in Figure 6-a and 6-b.

Hence as compared to the results when $\theta_k = .3198$, reducing θ_k to .2 reduces the degree of amplification generated by an endogenous borrowing constraint.

One possible explanation of this sensitivity might be that the share of capital in output comprises a part of the entrepreneurial return. In Kocherlakota(2000), share of capital was the only entrepreneurial return. In our model, entrepreneurial return comprises of share of land and share of capital in output. Financial accelerator works through its impact on entrepreneurial net worth and consequently its impact on business investment. When the return to business investment in total output is lower we would expect the financial accelerator mechanism to lose some of its amplifying power.

This is what we see when we reduce θ_k.

4.4 Application to Japan

The results from plotting impulse responses indicate that a model with endogenous borrowing constraints amplify the impact of external shocks on the economy.

We next apply the general equilibrium model with endogenous borrowing constraints discussed above to the Japanese economy during 1980 to 2000. Our objective is to see to what extent endogeneity of the borrowing constraint aggravated the economic situation in Japan. For our analysis, we concentrate on fluctuations in total factor productivity which by many accounts was one of the primary reasons behind the boom and bust of the Japanese economy.

For our analysis we assume that $A_{1t} = A_{2t}$ for every t which is the aggregate total factor productivity. The intuition is that production technology for producing residential services and non-residential goods are both subject to the same technology shock, which macro literature often refers to as a country-wide technology shock (as opposed to sector-specific technology shock).

Given the calibrated parameter values and data on capital output ratio, share of land in non-residential sector, share of land in residential sector and labor we calculate TFP or the Solow Residual.

Figure 7 shows the fluctuation in detrended productivity over 1980 to 2000. Total factor productivity continuously grows above the long term growth rate of 2.2% between 1986 and 1991. In 1991, the growth rate of total factor productivity starts declining and except for a brief respite during 1994 to 1996, growth rate of TFP continues to decline till 2000.

We feed in the time series of TFP thus estimated in a model with endogenous borrowing constraints and compare the results with the case when borrowing constraint is exogenous.

In Figures 8-a to 8-d, we plot the predictions of feeding in time series of TFP in our model. We summarize our findings in Table 3-a and 3-b.

We divide 1980 to 2000 in two sub-periods: the boom years of 1986 to 1991 and the recession years of 1991 to 2000.

The boom years of the Japanese economy are characterized by an 8.4% increase in per capita output accompanied by a 35.5% increase in land price between 1986 to 1991. The period also witnessed an increase in non-residential land holding by .91% and a capital deepening such that capital output ratio increased from 1.8 in 1986 to 1.9 in 1991.

How well does TFP fluctuations account for these facts?

TFP fluctuations in a model with endogenous borrowing constraint results in an increase in output per capita by 5.7% as compared to 4.9% in a model with exogenous borrowing constraints. Land prices increase by 1.46% in the former case. However, in the latter case it registers a decline by 1.8%. In case of non-residential land, we find that a model with endogenous borrowing constraint results in an increase by 11.33% while a model with exogenous borrowing constraint predicts an increase by 8.65%. With respect to non-residential land holdings, the model predictions are much higher as compared to data. We suspect that in the real world, land is not very easily tradable as there are zoning laws and restrictions that prevent residential land from being costlessly converted to non-residential land. Our model did not have any such restrictions and consequently the shift in land holdings between residential and non-residential use is much more easy. In this paper for the sake of brevity we abstract from introducing any zoning laws but it can be easily incorporated in our model by including an adjustment cost of transfer of land between residential and non-residential uses with a high adjustment cost reflecting steeper zoning laws.

Finally, as far as the capital output ratio is concerned, the ratio increases from 1.72 to 1.77 when the borrowing constraint is endogenous but remains constant at 1.78 when the borrowing constraint is exogenous.

Next we look at the recession years which are characterized by a 14.3% drop in per capita output accompanied by a 48% drop in land prices and a 16.7% decline in non-residential land. Capital output ratio on the other hand increases from 1.9 to 2.36.

In a model with endogenous borrowing constraints, TFP fluctuations predict a 20.8% drop in output accompanied by a 4.8% drop in land prices. Non-residential land falls by 37.1% and contrary to data, capital output ratio falls from 1.77 to 1.59. In our model with endogenous borrowing constraints, the effect of TFP on business investment is greater than the effect of TFP on output. This explains why we get a decline in capital output ratio in response to a decline in TFP in contrast to data where there is a capital deepening.

When borrowing constraint is exogenous, output drops by 18.1%. However, land price increases by 8.89%, a prediction opposite to that of data. Non-residential land falls by 28.9% while capital deepens such that the capital output ratio increases from 1.78 to 1.8.

Our results indicate that the impact of fluctuations in TFP get amplified when borrowing constraints are endogenous. Increases in TFP in the presence of endogenous borrowing constraints resulted in an increased boom of the economy which would have been much lower had the borrowing constraint been exogenous. This blessing became a curse during the nineties when endogeneity aggravated the negative impact of falling TFP. Our model falls short in explaining fluctuations in land prices. The magnitude of change in price of land in our model is much smaller than that witnessed in data and even increasing the persistence of the TFP process does not improve the model predictions with regard to land prices. This result is consistent with the works of Nakajima(2003) who finds that a model with rational expectations cannot account for fluctuations in real estate prices in Japan.

In our model, TFP fluctuations alone have not been able to generate significant fluctuations in land prices. We do have rational expectations in our model. That begs the question: is fluctuations in land price in Japan really a bubble phenomenon which cannot be explained by economic fundamentals (in our case productivity shocks)? Or is it sensitive to expectations specifications and if we had adaptive expectations instead of rational expectations framework, we would have better results?

This analysis is outside the scope of this paper and we leave it for the future.

5 Conclusion

A wide spectrum of theoretical studies suggest that in the presence of endogenous borrowing constraints small shocks can have a much amplified and persistent impact on the economy. The mechanism is a simple one: exogenous shocks lead to changes in the borrower's net worth. Given endogenous borrowing constraint, fluctuations in a borrowers net worth translates to reduced funds for business investment which magnifies the impact of a small shock. The popular name of this mechanism is : financial accelerator.

However quantitative studies have posed quite a challenge for the financial accelerator mechanism. Kocherlakota (2000) showed that the amplification might be quite small and is very sensitive to factor shares.

That leaves us with a question: quantitatively how important is the financial accelerator mechanism? The question has assumed particular significance in the aftermath of the Japanese and the East Asian crisis which were characterized by significant fluctuations in the real estate and asset prices along with fluctuations in real economic aggregates. Given the financial environment where debt financing is a common way of raising funds for investment and firms often borrow against a collateral, the two phenomenon might be intrinsically linked. We expect external shocks to cause fluctuations in real estate and asset prices which would then get transmitted to the real economy resulting in amplified fluctuations of output.

In this paper, we build a dynamic general equilibrium model with borrowing constraints to quantitatively test the importance of the financial accelerator mechanism. We allow entrepreneurs to borrow funds for investment. However there is an upper limit on the amount the entrepreneurs can borrow that is determined by the net worth of the entrepreneurs. Calibrating our model to the Japanese economy we find that the magnitude of fluctuations in real estate prices and per capita output in response to a shock to total factor productivity is much more amplified when borrowing constraint is endogenously determined by the borrowers' net worth as opposed to an exogenously determined cap.

We do find that the degree of amplification is sensitive to factor shares. We tested a case where the share of capital in output was reduced as compared to our benchmark case. We find that the degree of amplification goes down in the former case as compared to the latter.

Next, we applied our model to Japan. Our objective was to see to what extent endogeneity of borrowing constraint was instrumental in magnifying the boom and the bust of the Japanese economy during 1986 to 2000. We find that our model with endogeneity registers greater increase and decline in output per capita in response to TFP fluctuations as compared to a model where borrowing constraints are exogenous. However, the model falls short of explaining the land price fluctuations of the same magnitude as witnessed in data, though endogeneity does amplify the impact of TFP on land prices. This suggests the possibility of a bubble which renders land price not so dependent on economic fundamentals, or it opens up the possibility of a need to modify the model, with one possible modification being in the structure of expectations as suggested by Nakajima (2003).

One assumption in our model was that all entrepreneurs are subject to a binding borrowing constraint. An area for future research might be to relax this assumption and test a case where only a fraction of the entrepreneurs at any time face binding borrowing constraint. It would be interesting to see to what extent the effect of the external shock would get amplified in such a case. We suspect the effect to be smaller than what we find and we also suspect the effect would be sensitive to the degree of interaction between the constrained and the unconstrained entrepreneur. For example, if the output of the credit constrained entrepreneur is an important input used by the unconstrained entrepreneur, we might expect the effect of TFP shocks to be bigger. A detailed general equilibrium model in this area would help us find answers.

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Table 1: Moments from Japanese data (average 1980 to 1984)

Data moments	Model moments	Steady state values(1980 to 1984)
Interest rate	r	5%
Capital output ratio	$\frac{k}{y}$	1.8
Residential land to output ratio	$\frac{ql_1}{y}$	1.55
Non-residential land to output ratio	$\frac{ql_2}{y}$	1.08
Residential borrowing to output ratio	$\frac{b_1}{y}$	.45
Nonresidential borrowing to output ratio	$\frac{b_2}{y}$	1.75
Labor	l	.33
Per capita output growth rate	g_z	2.2%
Working population growth rate	η	.78%
Labor income tax rate	τ_h	33%
Tax rate on interest earnings	τ_s	19%
Corporate income tax rate	τ_y	49.5%

Table 2: Calibrated value of parameters

Parameter	Description	Values
ψ	Elasticity of substitution between consumption and leisure	1.513
β	Rate of time preference	.99
δ	Depreciation rate	.1
$(1-\alpha)\theta_k$	Share of capital in final output	.3070
$\alpha\theta_{1l}$	Share of residential land in final output	.0178
$(1-\alpha)\theta_{2l}$	Share of non-residential land in final output	.0024
$\theta_h(1-\alpha)+\alpha(1-\theta_{1l})$	Share of labor in final output	.6728
α	Share of housing in total output	4%
ϕ_1	Loan to value ratio of entrepreneur 1	.299
ϕ_2	Loan to value ratio of entrepreneur 2	.6143

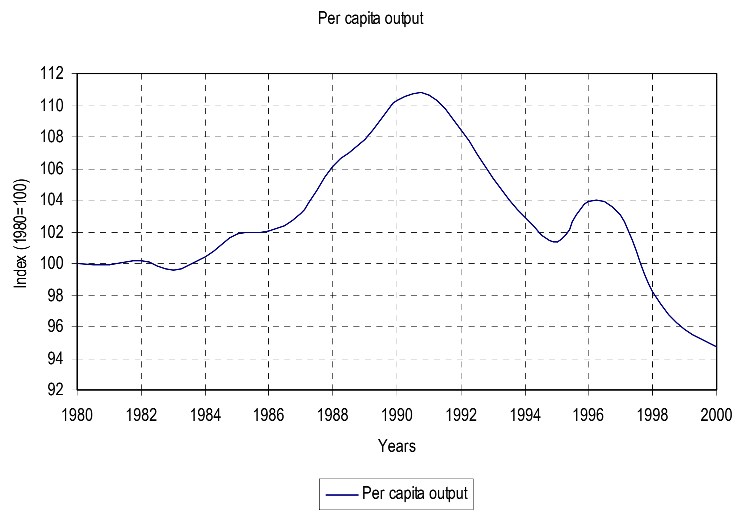

Figure 1-a: Per capita output with respect to the long tern trend

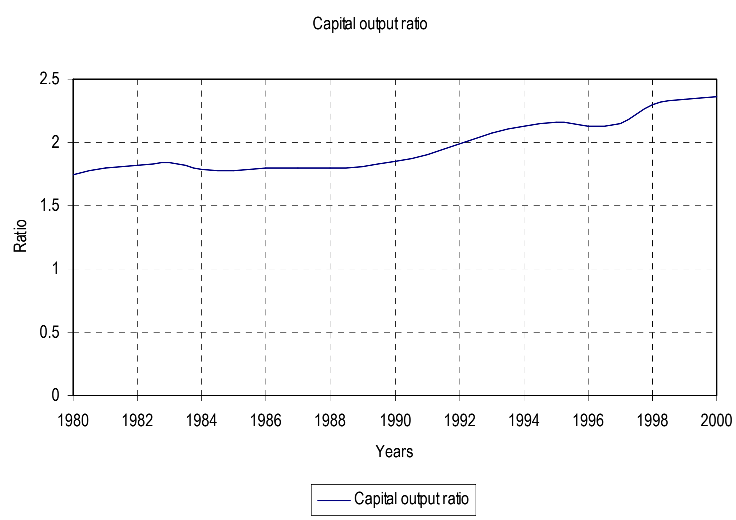

Figure 1-b: Capital output ratio

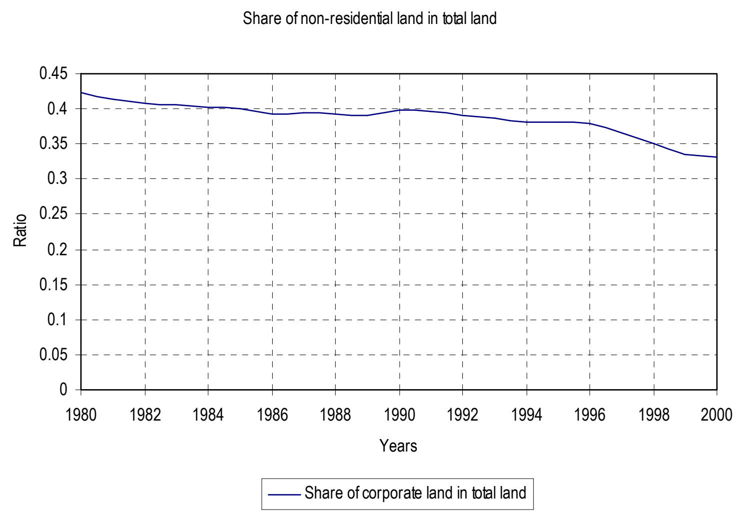

Figure 1-c: Share of non-residential land in total land

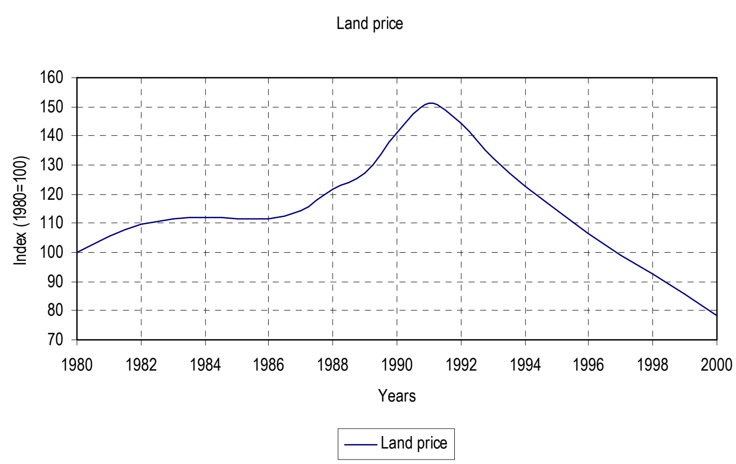

Figure 1-d: Land price with respect to the long term trend

Impulse response to a 1% shock to $A_1(\theta_k = .32, \theta_{l1} = .45, \theta_{l2} = .0025, \theta_h = .68, \alpha = .04)$

$\rho_{a1} = .72$

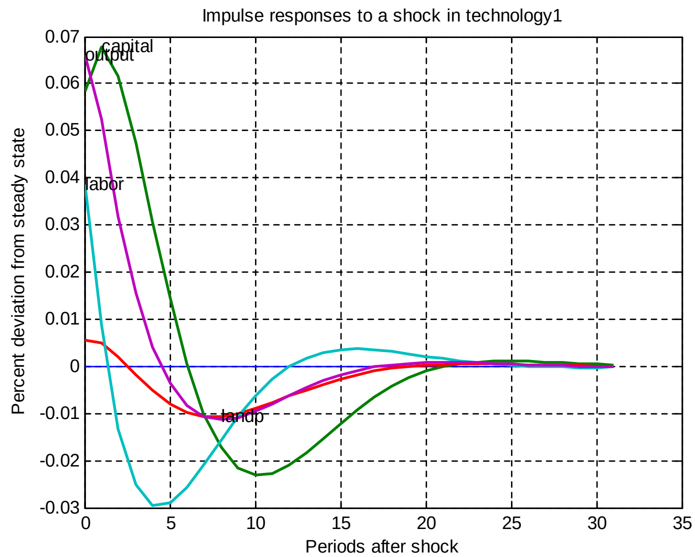

Figure 2-a: Endogenous borrowing constraint

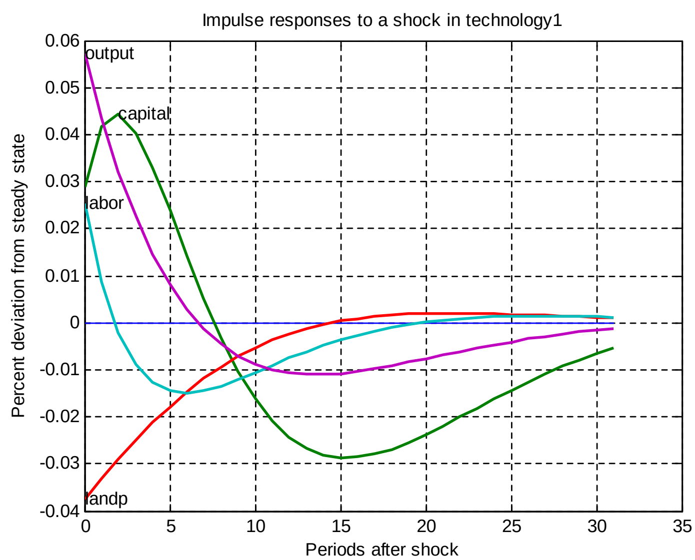

Figure 2-b: Exogenous borrowing constraint

Impulse response to a 1% shock to $A_2(\theta_k = .32, \theta_{l1} = .45, \theta_{l2} = .0025, \theta_h = .68, \alpha = .04)$

$\rho_{a1} = .78$

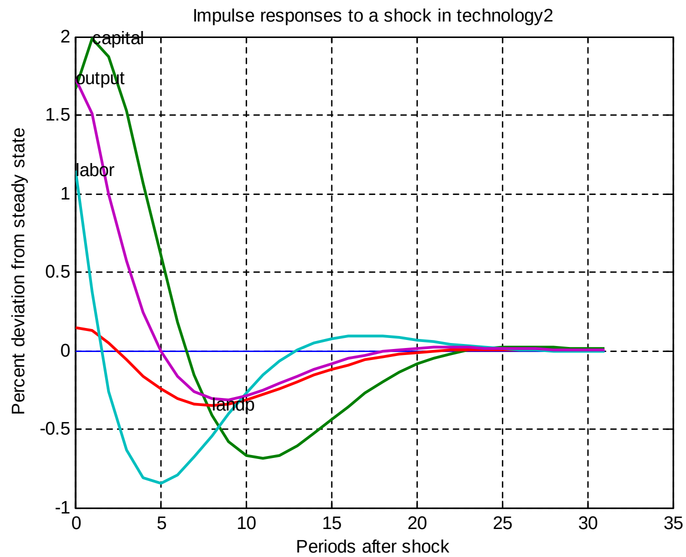

Figure 3-a : Endogenous borrowing constraint

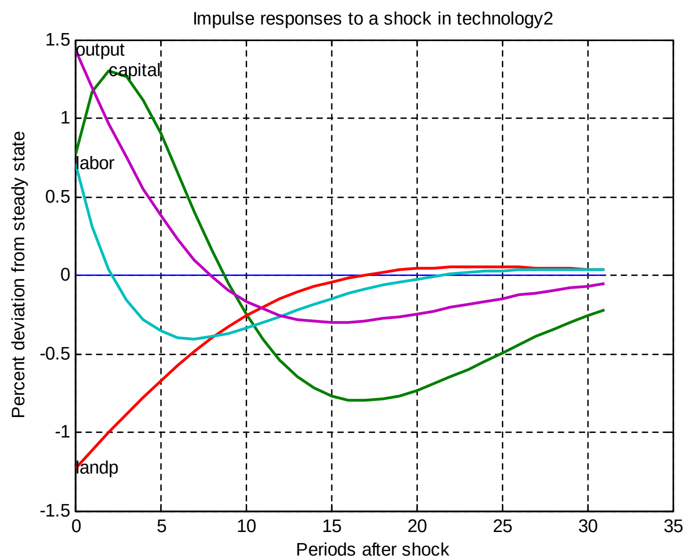

Figure 3-b : Exogenous borrowing constraint

Impulse response to 1% shock to loan-to-value ratio
$(\theta_k = .32, \theta_{l1} = .45, \theta_{l2} = .0025, \theta_h = .68, \alpha = .04)$

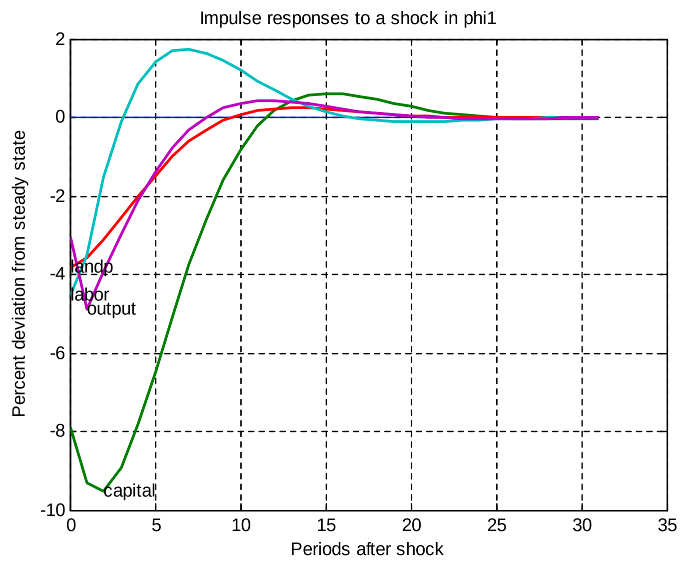

Figure 4-a : 1% positive response to $\phi_1(\rho_{\phi 1} = .84)$

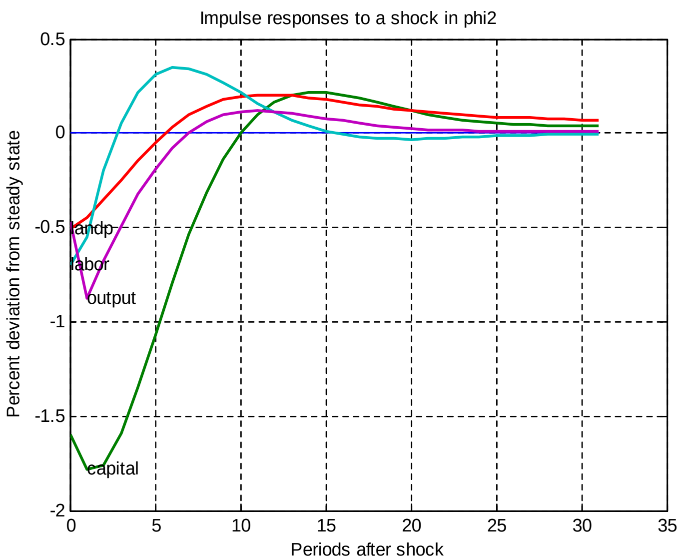

Figure 4-b: 1% positive shock to $\phi_2(\rho_{\phi 2} = .96)$

Impulse response to a 1% shock to $A_1(\theta_k = .2, \theta_{l1} = .45, \theta_{l2} = .0025, \theta_h = .8, \alpha = .04)$

$\rho_{a1} = .72$

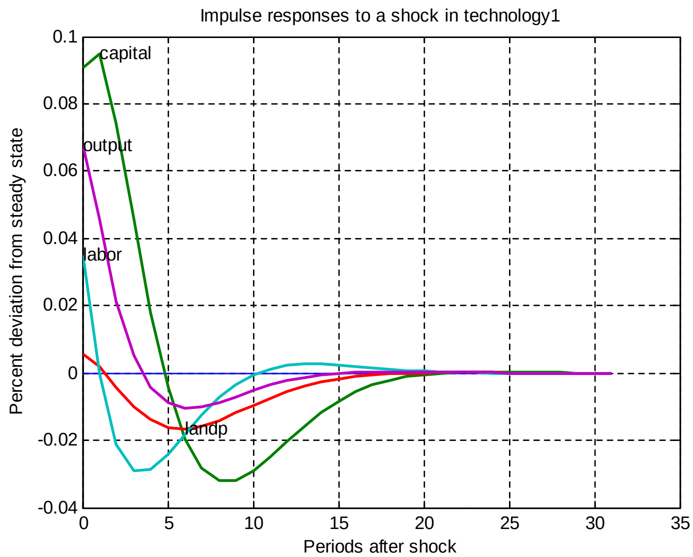

Figure 5-a: Endogenous borrowing constraint

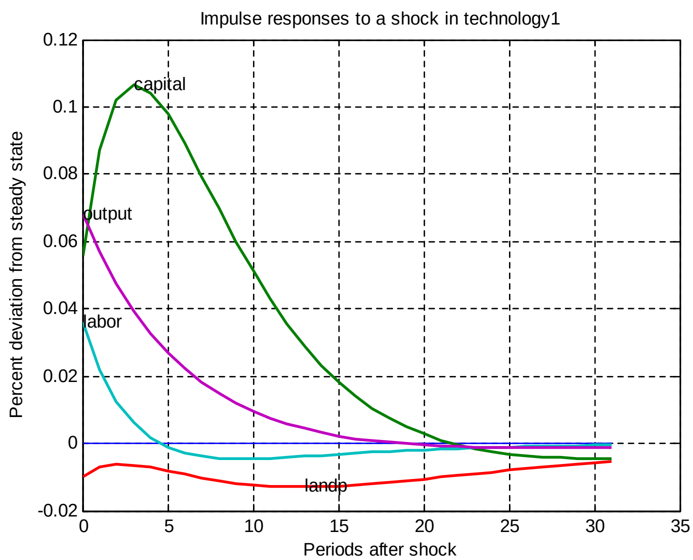

Figure 5-b: Exogenous borrowing constraint

Impulse response to a 1% shock to $A_2(\theta_k = .2, \theta_{l1} = .45, \theta_{l2} = .0025, \theta_h = .8, \alpha = .04)$

$\rho_{a2} = .78$

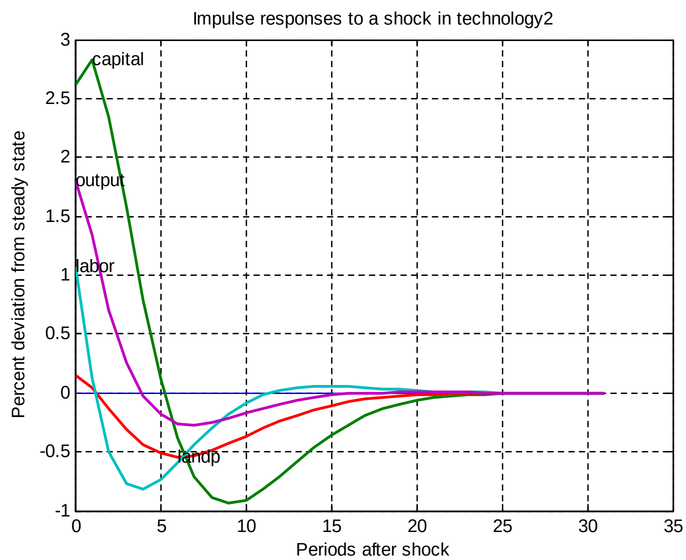

Figure 6-a: Endogenous borrowing constraint

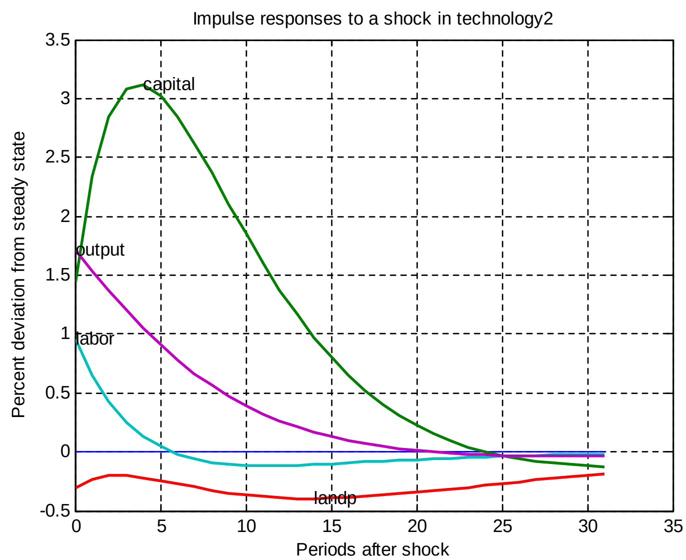

Figure 6-b: Exogenous borrowing constraint

Application to Japan

Evolution of total factor productivity in Japan during 1980 to 2000

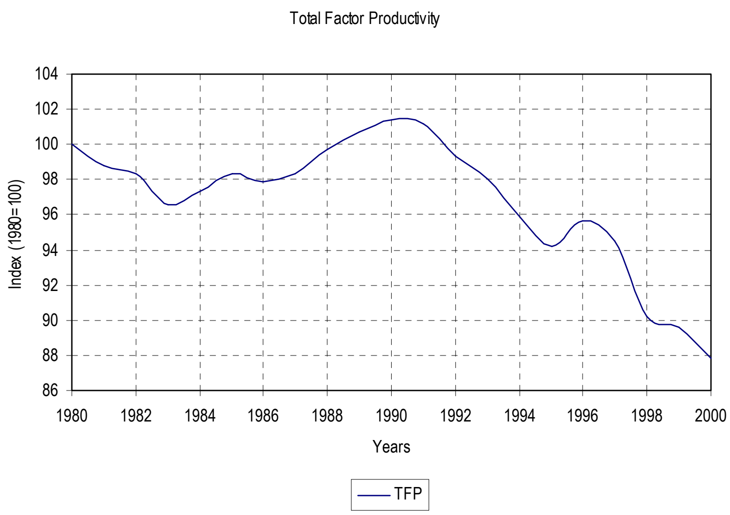

Figure 7: Total Factor Productivity (Solow Residual)

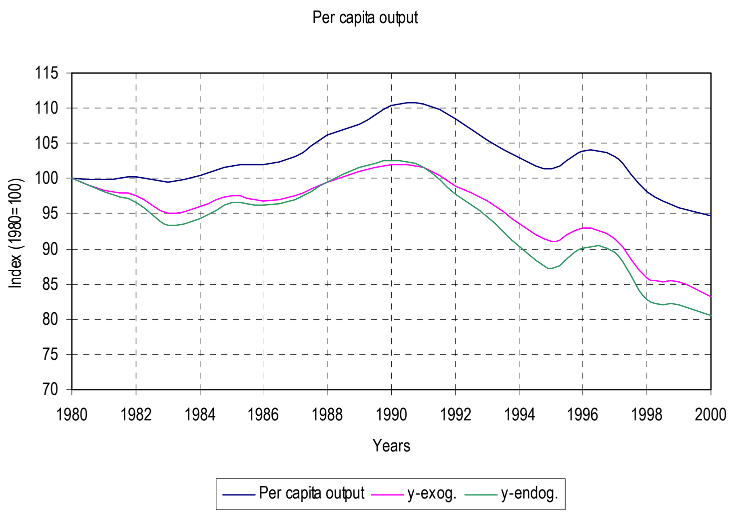

Figure 8-a: Per capita output ($\rho_{a1} = .72, \rho_{a2} = .78$)

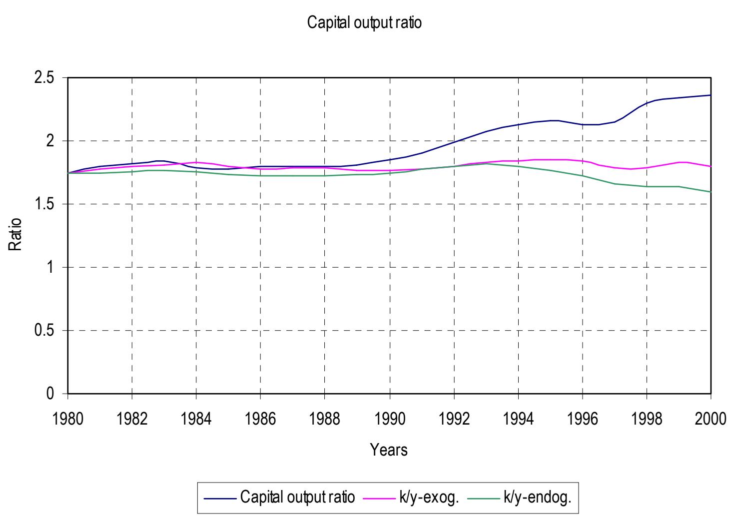

Figure 8-b:Capital output ratio ($\rho_{a1} = .72, \rho_{a2} = .78$)

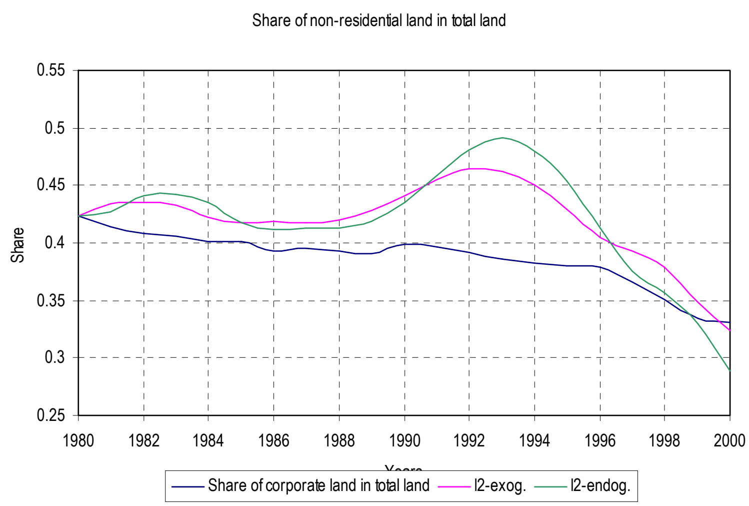

Figure 8-c: Non-residential land ($\rho_{a1} = .72, \rho_{a2} = .78$)

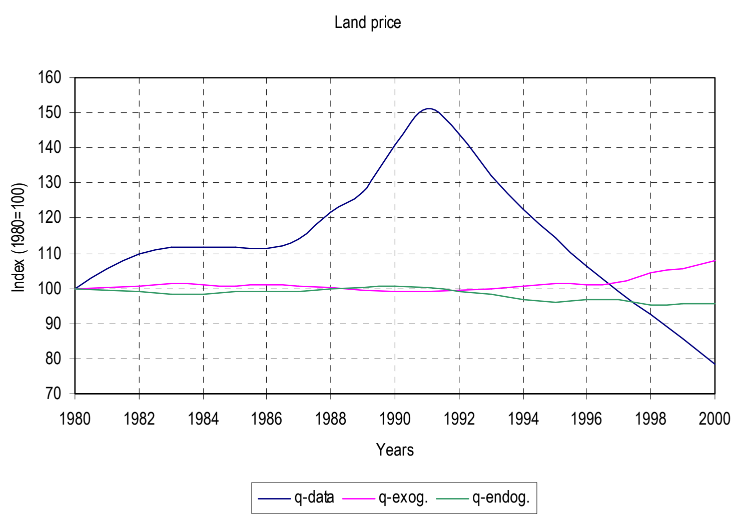

Figure 8-d: Land price ($\rho_{a1} = .72, \rho_{a2} = .78$)

Model Performance : Exogenous vs. Endogenous borrowing constraint

Table 3-a: Pre-crisis years (1986:1991)

		Borrowing constraint	
	Data	Endogenous	Exogenous
Output per capita (% change)	8.4%	5.7%	4.9%
Land price(% change)	35.5%	1.46%	-1.8%
Share of non-residential land (%change)	.91%	11.3%	8.65%
Capital output ratio(level-1986 to 1991)	1.8 to 1.9	1.72 to 1.77	1.78 to 1.78

Table 3-b: Crisis years (1991:2000)

		Borrowing constraint	
	Data	Endogenous	Exogenous
Output per capita (% change)	-14.3%	-20.8%	-18.1%
Land price(% change)	-48%	-4.8%	8.89%
Share of non-residential land (%change)	-16.7%	-37.1%	-28.9%
Capital output ratio(level-1986 to 1991)	1.9 to 2.36	1.77 to 1.59	1.78 to 1.8

Appendix 1: Proof of Proposition 1

The first order condition of the worker with respect to savings s_{t+1} can be summarized by:

$$a_{t+1} : \beta^{t+1} E_t \lambda^w_{t+1}(1 + r_{t+1}(1 - \tau_{st+1}) = \beta^t \lambda^w_t \tag{31}$$

Allowing for long term trend in output and population the equation reduces to:

$$\begin{aligned} a_{t+1} \quad &: \quad \left((1+g_z)^{1-\sigma}(\beta\eta)\right)^{t+1} E_t \lambda^w_{t+1}(1 + r_{t+1}(1 - \tau_{st+1})) && (32) \\ &= \quad \left((1+g_z)^{1-\sigma}(\beta\eta)\right)^{t} \eta (1+g_z) \lambda^w_t && (33) \end{aligned}$$

In the steady state, the equation reduces to:

$$(1 + r(1 - \tau_s)) = \frac{(1+g_z)^\sigma}{\beta} \tag{34}$$

The first order condition of the entrepreneur i with respect to borrowings b^i_{t+1} can be summarized by:

$$\left(-\beta^{t+1} E_t \lambda^i_{t+1}(1 + r_{t+1}(1 - \tau_{yt+1}) + \beta^t \lambda^i_t - \beta^t \mu_{it}\right) \mu_{it} = 0 \tag{35}$$

Allowing for long term trend in output and population the equation reduces to:

$$\begin{pmatrix} -\left((1+g_z)^{1-\sigma}(\beta\eta)\right)^{t+1} E_t \lambda^i_{t+1}(1 + r_{t+1}(1 - \tau_{yt+1})) \\ +\eta(1+g_z)\left((1+g_z)^{1-\sigma}(\beta\eta)\right)^{t} \lambda^i_t - \mu_{it}\left((1+g_z)^{1-\sigma}(\beta\eta)\right)^{t} \end{pmatrix} \mu_{it} = 0 \tag{36}$$

Note that if borrowing constraint binds with equality then $\mu_{it} > 0$ which implies that

$$\begin{pmatrix} -\left((1+g_z)^{1-\sigma}(\beta\eta)\right)^{t+1} E_t \lambda^i_{t+1}(1 + r_{t+1}(1 - \tau_{yt+1})) \\ +\eta(1+g_z)\left((1+g_z)^{1-\sigma}(\beta\eta)\right)^{t} \lambda^i_t - \mu_{it}\left((1+g_z)^{1-\sigma}(\beta\eta)\right)^{t} \end{pmatrix} = 0 \tag{37}$$

In the steady-state the equation reduces to:

$$\begin{aligned} -(1 + r(1 - \tau_y)) + \frac{(1+g_z)^\sigma}{\beta} - \frac{\mu_i}{\beta\eta(1+g_z)^{1-\sigma}} &= 0 && (38) \\ \Rightarrow \frac{(1+g_z)^\sigma}{\beta} - (1 + r(1 - \tau_y)) &= \frac{\mu_i}{\beta\eta(1+g_z)^{1-\sigma}} && (39) \end{aligned}$$

Substituting for $\frac{(1+g_z)^\sigma}{\beta}$ from Equation : we get

$$(1 + r(1 - \tau_s)) - (1 + r(1 - \tau_y)) = \frac{\mu_i}{\beta\eta\,(1+g_z)^{1-\sigma}} \tag{40}$$

$$\Rightarrow \quad r(\tau_y - \tau_s) = \frac{\mu_i}{\beta\eta\,(1+g_z)^{1-\sigma}} \tag{41}$$

Hence the sufficient condition for $\mu_i > 0$ is that $(\tau_y - \tau_s) > 0$ or $\tau_y > \tau_s$.

Appendix 2: Technical Appendix

The set of steady state equations of the model is summarized by:

$$\frac{1-\varsigma}{\varsigma} - \frac{w}{c^w}(1-\tau_h)(1-h^w) = 0 \quad (42)$$

$$\alpha - \frac{p_1 y_1}{y} = 0 \quad (43)$$

$$1-\alpha - \frac{p_2 y_2}{y} = 0 \quad (44)$$

$$\beta - \frac{(1+g_z)^{\sigma}}{1+(1-\tau_s)r} = 0 \quad (45)$$

$$\delta - \frac{x}{k} - 1 + \eta(1+g_z) = 0 \quad (46)$$

$$\theta_k(1-\alpha) - \left[\frac{1-\phi_2}{1-\tau_y}\left(\frac{(1+g_z)^{\sigma}}{\beta} - 1\right) + \delta + \phi_2 r\right]\frac{k}{y} = 0 \quad (47)$$

$$\theta_{1l}\alpha - \left[\frac{1-\phi_1}{1-\tau_y}\left(\frac{(1+g_z)^{\sigma-1}}{\beta\eta}\right) - \frac{1}{1-\tau_y} + \frac{\phi_1}{\eta(1+\gamma)(1-\tau_y)} + \frac{\phi_1 r}{\eta(1+\gamma)}\right]\frac{ql_1}{y} = 0 \quad (48)$$

$$\theta_{2l}(1-\alpha) - \left[\frac{1-\phi_2}{1-\tau_y}\left(\frac{(1+g_z)^{\sigma-1}}{\beta\eta}\right) - \frac{1}{1-\tau_y} + \frac{\phi_2}{\eta(1+\gamma)(1-\tau_y)} + \frac{\phi_2 r}{\eta(1+\gamma)}\right]\frac{ql_2}{y} = 0 \quad (49)$$

$$w - \alpha(1-\theta_{l1})\frac{y}{h_1} = 0 \quad (50)$$

$$w - (1-\alpha)(1-\theta_k-\theta_{l2})\frac{y}{h2} = 0 \quad (51)$$

$$\frac{b_1}{y} - \phi_1\frac{ql_1}{y\eta(1+\gamma)} = 0 \quad (52)$$

$$\frac{b_2}{y} - \phi_2\left(\frac{k}{y} + \frac{ql_1}{y\eta(1+\gamma)}\right) = 0 \quad (53)$$

$$y_1 - A_1 l_1^{\theta_{l1}} h_1^{1-\theta_{l1}} = 0 \quad (54)$$

$$y_2 - A_2 k^{\theta_k} l_2^{\theta_{l1}} h_2^{1-\theta_{l1}} = 0 \quad (55)$$

$$y - y_1^{\alpha} y_2^{1-\alpha} = 0 \quad (56)$$

$$l_1 + l_2 - 1 = 0 \quad (57)$$

$$b_1 + b_2 - s = 0 \quad (58)$$

$$c^w + (1+\gamma)\eta s - s - wh^w(1-\tau_h) - r(1-\tau_s)s - Tr^w = 0 \quad (59)$$

$$c^1 - (1-\tau_y)(p_1 y_1 - wh^1 - rb^1) - (1+\gamma)\eta b^1 + b^1 - Tr^1 = 0 \quad (60)$$

$$c^w + c^1 + c^2 + (1+\gamma)\eta k - (1-\delta)k + g - y = 0 \quad (61)$$

$$wh^w\tau_h + r\tau_s s + \tau_y(p_1 y_1 - wh^1 - rb^1) + \tau_y(p_2 y_2 - wh^2 - \delta k - rb^2) - g - T = 0 \quad (62)$$

$$Tr^w - t_w T = 0 \quad (63)$$

$$Tr^1 - t_1 T = 0 \quad (64)$$

$$Tr^2 - (1-t_w-t_1)T = 0 \quad (65)$$

Note that a variable in the steady state is denoted without a time subscript.

We have 24 unknowns summarized by the vector of allocations $\{c^{w}, c^{1}, c^{2}, h^{w}, h^{1}, h^{2}, l^{1}, l^{2}, k, y, y_{1,} y_{2}, b^{1}, b^{2}, s_{,} Tr^{w}, Tr^{1}, Tr^{2}, T\}$,and a vector of prices

$\{w,\ r,\ q, p_{1,}\ p_{2}\}_{t=0}^{\infty}$ and 24 equations,.so we can solve for the steady state model variables uniquely.

Next we summarize the log-linearized form of our first order conditions.

Using the terminology of King, Plosser and Rebelo (1988) we get two sets of equations: deterministic equations and expectations equations.

Any variable with a tilda denotes the log-deviation of the variable from its steady state except for $\widetilde{r}_t = r_t - r$

The set of deterministic equations can be summarized by:

$$\widetilde{\lambda}_t^w + \sigma \widetilde{c}_t^w + \psi (1-\sigma) \frac{h^w}{1-h^w} \widetilde{h}_t^w = 0 \quad (66)$$

$$\widetilde{y}_t - \widetilde{c}_t^w - \frac{1}{1-h^w} \widetilde{h}_t^w = 0 \quad (67)$$

$$\begin{array}{c} \frac{c^w}{y} \widetilde{c}_t^w + \eta (1+g_z) \frac{s}{y} \widetilde{s}_{t+1} - [1 + r*(1-\tau_s)] \frac{s}{y} \widetilde{s}_t - (1-\tau_s) \frac{s}{y} \widetilde{r}_t \\ -(1-\tau_h) [(1-\alpha)*(1-\theta_k - \theta_{l2}) + \alpha * (1-\theta_{l1})] \widetilde{y}_t - \frac{t^w}{y} \widetilde{T}_t = 0 \end{array} \quad (68)$$

$$\widetilde{c}_{1t} + \frac{1}{\sigma} \widetilde{\lambda}_{1t} = 0 \quad (69)$$

$$\begin{array}{c} \frac{c_1}{y} \widetilde{c}_{1t} - \eta (1+g_z) \frac{b_1}{y} \widetilde{b}_{1t+1} + [1 + r*(1-\tau_y)] \frac{b_1}{y} \widetilde{b}_{1t} + (1-\tau_y) \frac{b_1}{y} \widetilde{r}_t \\ -(1-\tau_y) * \alpha * \theta_{l1} \widetilde{y}_t - \frac{t^1}{y} \widetilde{T}_t + \frac{ql_1}{y} \widetilde{l}_{1t+1} - \frac{ql_1}{y} \widetilde{l}_{1t} = 0 \end{array} \quad (70)$$

$$\widetilde{c}_{2t} + \frac{1}{\sigma} \widetilde{\lambda}_{2t} = 0 \quad (71)$$

$$\frac{c^w}{y} \widetilde{c}_t^w + \frac{c^1}{y} \widetilde{c}_{1t} + \frac{c^2}{y} \widetilde{c}_{2t} + \eta (1+g_z) \frac{k}{y} \widetilde{k}_{t+1} - (1-\delta) \frac{k}{y} \widetilde{k}_t + \frac{g}{y} \widetilde{g}_t - \widetilde{y}_t = 0 \quad (72)$$

$$\frac{g}{y} \widetilde{g}_t - \tau_h * [(1-\alpha)*(1-\theta_k - \theta_{l2}) + \alpha * (1-\theta_{l1})] \widetilde{y}_t - (\tau_s - \tau_y) \frac{s}{y} \widetilde{r}_t - (\tau_s - \tau_y) \frac{s}{y} \widetilde{s}_t = 0 \quad (73)$$

$$\begin{array}{c} \widetilde{y}_t - \alpha \widetilde{A}_{1t} - (1-\alpha) \widetilde{A}_{2t} - \alpha * \theta_{l1} * \widetilde{l}_{1t} - \alpha * (1-\theta_{l1}) * \widetilde{h}_{1t} \\ -(1-\alpha) \theta_k \widetilde{k}_t - (1-\alpha) \theta_{2l} \widetilde{l}_{2t} - (1-\alpha) * (1-\theta_k - \theta_{l2}) * \widetilde{h}_{2t} = 0 \end{array} \quad (74)$$

$$\widetilde{h}_{1t} - \widetilde{h}_{2t} = 0 \quad (75)$$

$$h_1 \widetilde{h}_{1t} + h_2 \widetilde{h}_{2t} - h \widetilde{h}_t^w = 0 \quad (76)$$

The set of expectational equations are summarized by:

$$\frac{(1+g_z)^\sigma}{\beta}\widetilde{\lambda}^w_{t+1} + (1-\tau_s)\,\widetilde{r}_{t+1} - \frac{(1+g_z)^\sigma}{\beta}\widetilde{\lambda}^w_t = 0 \quad (77)$$

$$\begin{aligned} &\tfrac{b_1}{y}\widetilde{b}_{1t+1} - \tfrac{ql_1}{y\eta(1+g_z)}\widetilde{\phi}_{1t} - \tfrac{\phi_1 ql_1}{y\eta(1+g_z)}\widetilde{q}_t \\ &+\tfrac{\phi_1 ql_1}{y\eta(1+g_z)}\widetilde{l}_{1t+1} = 0 \end{aligned} \quad (78)$$

$$\begin{aligned} &\tfrac{b_2}{y}\widetilde{b}_{2t+1} - \tfrac{\phi_2 k}{y}\widetilde{k}_{t+1} - \left(\tfrac{k+}{y}\tfrac{ql_2}{y\eta(1+g_z)}\right)\widetilde{\phi}_{2t} \\ &-\tfrac{\phi_2 ql_2}{y\eta(1+g_z)}\widetilde{q}_t + \tfrac{\phi_2 ql_2}{y\eta(1+g_z)}\widetilde{l}_{2t+1} = 0 \end{aligned} \quad (79)$$

$$l_1\widetilde{l}_{1t+1} + l_2\widetilde{l}_{2t+1} = 0 \quad (80)$$

$$\frac{b_1}{y}\widetilde{b}_{1t+1} + \frac{b_2}{y}\widetilde{b}_{2t+1} - \frac{s}{y}\widetilde{s}_{t+1} = 0 \quad (81)$$

$$\begin{aligned} &\tfrac{(1+g_z)^\sigma}{\beta}(1-\phi_2)\,\widetilde{\lambda}^2_{t+1} + \theta_k(1-\alpha)(1-\tau_y)\,\tfrac{y}{k}E_t\left(\widetilde{y}_{t+1} - \widetilde{k}_{t+1}\right) \\ &-(1-\tau_y)\,\phi_2\widetilde{r}_{t+1} - \left(1 + r(1-\tau_y) - \tfrac{(1+g_z)^\sigma}{\beta}\right)\widetilde{\phi}_{2t} \\ &-\tfrac{(1+g_z)^\sigma}{\beta}(1-\phi_2)\,\widetilde{\lambda}^2_t = 0 \end{aligned} \quad (82)$$

$$\begin{aligned} &\tfrac{(1+g_z)^{\sigma-1}}{\beta\eta}(1-\phi_1)\,\widetilde{\lambda}^1_{t+1} + \theta_{1l}\alpha(1-\tau_y)\,\tfrac{y}{ql_1}E_t\left(\widetilde{y}_{t+1} - \widetilde{l}_{1t+1}\right) - (1-\tau_y)\,\tfrac{\phi_1}{\eta(1+g_z)}\widetilde{r}_{t+1} \\ &-\widetilde{q}_t\left(1 + (1-\tau_y)\,\theta_{l1}\alpha\tfrac{y}{ql_1}\right) - \left(\tfrac{1}{\eta(1+g_z)} + \tfrac{r(1-\tau_y)}{\eta(1+g_z)} - \tfrac{(1+g_z)^{\sigma-1}}{\beta\eta}\right)\widetilde{\phi}_{1t} \\ &-\tfrac{(1+g_z)^{\sigma-1}}{\beta\eta}(1-\phi_1)\,\widetilde{\lambda}^1_t = 0 \end{aligned} \quad (83)$$

$$\begin{aligned} &\tfrac{(1+g_z)^{\sigma-1}}{\beta\eta}(1-\phi_2)\,\widetilde{\lambda}^2_{t+1} + \theta_{2l}(1-\alpha)(1-\tau_y)\,\tfrac{y}{ql_2}E_t\left(\widetilde{y}_{t+1} - \widetilde{l}_{2t+1}\right) - (1-\tau_y)\,\tfrac{\phi_2}{\eta(1+g_z)}\widetilde{r}_{t+1} \\ &-\widetilde{q}_t\left(1 + (1-\tau_y)\,\theta_{l2}(1-\alpha)\tfrac{y}{ql_2}\right) - \left(\tfrac{1}{\eta(1+g_z)} + \tfrac{r(1-\tau_y)}{\eta(1+g_z)} - \tfrac{(1+g_z)^{\sigma-1}}{\beta\eta}\right)\widetilde{\phi}_{2t} \\ &-\tfrac{(1+g_z)^{\sigma-1}}{\beta\eta}(1-\phi_1)\,\widetilde{\lambda}^2_t = 0 \end{aligned} \quad (84)$$

Given the set of deterministic equations, and the set of expectational equations, the set of unknowns is summarized by the vector:
$\{\widetilde{\lambda}^w_t, \widetilde{c}^w_t, \widetilde{h}^w_t, \widetilde{y}_t, \widetilde{T}_t, \widetilde{c}_{1t}, \widetilde{c}_{2t}, \widetilde{\lambda}_{1t}, \widetilde{\lambda}_{2t}, \widetilde{h}_{1t}, \widetilde{h}_{2t}, \widetilde{k}_{t+1}, \widetilde{s}_{t+1}, \widetilde{b}_{1t+1}, \widetilde{b}_{2t+1}, \widetilde{l}_{1t+1}, \widetilde{l}_{2t+1}, \widetilde{r}_t, \widetilde{q}_t\}$.

We have a set of 19 equations and 19 unknowns so we can uniquely determine the decision rules of the model.