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Properties of distributions with increasing failure rate

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Abstract

This paper solves the search for interior solutions to optimization problems using stochastic variables. This is done by way of some new properties of distribution functions with increasing failure rates as characterized in Barlow and Proschan (1965). Building upon Lariviere (2006), we show that an objective function of the type $R(x) = F(x) + x\overline{F}(x)$, where $\overline{F}(x) = 1 - F(x)$, can also admit one interior maximal solution when the distribution function F has an increasing failure rate (IFR).

Keywords: optimization problem, probability distribution, increasing failure rate

1. Introduction

The problem solved here emerges from a mechanism design approach where a principal wishes to offer a price for some product or service to an agent without being aware of the agent's willingness to pay for such product

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or service. If the agent rejects the offer, the principal is left with his reservation utility. The seller is in a Bayesian setting of incomplete information and must form a belief about the agent's willingness to pay.

We now consider the case of a principal who wishes to sell to a agent but is not informed of the agent's willingness to pay. The principal has to form a Bayesian belief about this willingness to pay X. This Bayesian belief can be modeled as a random variable which follows a distribution F supported on $[\underline{X}, \overline{X}]$. If the principal fails to estimate correctly the customer's willingness to pay, his revenue is reduced to 1. The principal wishes to determine the point x which maximizes his revenue within the range $[\underline{X}, \overline{X}]$.

Let X be a non-negative random variable with distribution F and let $\overline{F}(X) = 1 - F(X)$. Let f be its probability density. We define a failure rate function as in Barlow and Proschan (1965): $r(X) = f(X)/\overline{F}(X)$. The generalized failure rate is defined in Lariviere and Porteus (2001) as $g(X) = Xf(X)/\overline{F}(X)$. X has an increasing failure rate (IFR) or, equivalently, F is an IFR distribution if $r(X)$ is weakly increasing for all X such that $F(X) < 1$. The IFR distributions are of interest in operations and supply chain management research because of the implications in the evaluation of some types of objective functions which model stochastic events. In Lariviere (2006), the example cited is of a service's pricing. One customer arrives per period, and service takes one period. The cost of service is zero. Customers privately observe their valuations, which are independent and identically distributed according to $F(X)$. A firm posting price p then faces demand $D(p) = \overline{F}(p)$ and sets p to maximize revenue, $R(p) = pD(p)$.

An optimal price p^* must solve

$$R'(p^*) = D(p^*) + p^* D'(p^*) = \overline{F}(p^*)(1 - g(p^*)) = 0, \tag{1}$$

when $R''(p^*) < 0$.

The uniqueness of p^* depends on the generalized failure rate $g(X)$. In particular, as stated in Lariviere (2006), if $g(X)$ is increasing, it can equal one at only a single point, and the unique p^* must solve $g(p^*) = 1$.

2. Problem

In the case of the principal, he must maximize his revenue function. This revenue function can be spelled out as

$$\begin{aligned} R(x) &= F(x) + x\overline{F}(x), \\ &= F(x)(1-x) + x. \end{aligned} \tag{2}$$

Let us demonstrate that when the distribution F is IFR, the revenue function admits an interior unique point x which maximizes it.

We first mention a corollary of the increasing failure rate of interest in what follows.

An increasing failure rate has equivalently the following property

$$r'(x) \geq 0. \tag{3}$$

This means that

$$f'(x)(1 - F(x)) + f(x)^2 \geq 0. \tag{4}$$

The F.O.C. requires that

$$R'(x) = f(x)(1-x) + \overline{F}(x) = 0. \tag{5}$$

If f is such that $F(1) = 1$, then $x = 1$ is solution and is also a maximum. This case is when the distribution ranges over $[0, 1]$.

For all cases such that $f(x) > 0$, we can write

$$1 - x = -\frac{\overline{F}(x)}{f(x)}. \tag{6}$$

The S.O.C. for a maximum requires that

$$R''(x) = -2f(x) + (1 - x)f'(x) < 0. \tag{7}$$

In the case when $f(x) \neq 0$ When we replace $(1 - x)$ from (6) in (7), we obtain

$$-2f(x) - f'(x)\frac{\overline{F}(x)}{f(x)} < 0. \tag{8}$$

Let us define

$$\forall x, \mid f(x) > 0,\ Q(x) = f'(x)\frac{F(x)}{f(x)} - 2f(x) \tag{9}$$

We wish to demonstrate that

$$\exists\, x_0 \,|\, Q(x_0) < 0. \tag{10}$$

From (4), we can write

$$f'(x)F(x) \leq f'(x) + f(x)^2. \tag{11}$$

When replacing $f'(x)F(x)$ in (10), it is equivalent to demonstrate that

$$f'(x) - f(x)^2 < 0, \tag{12}$$

since $f(x) > 0$. Let us call

$$H(x) = f(x)^2 - f'(x). \tag{13}$$

Demonstrating (12) is equivalent to verifying the conditions for which

$$H(x) > 0. \tag{14}$$

From Barlow and Proschan (1965), we know that the distributions which enjoy increasing failure rates include the gamma, Weibull, modified extreme value and the truncated normal distribution when their parameters are the commonly accepted ones.

Let us see if the most popular distributions who have increasing failure rates comply.

[The poisson distribution also has an increasing failure rate, but the exponential, which has a constant failure rate, is not studied here.]

2.1. Gamma distribution

The parameters of the gamma distribution which allow for an IFR are $\alpha > 1$ and $\lambda > 0$.

$$f(x) = \frac{e^{-x\lambda}\lambda(x\lambda)^{\alpha-1}}{\Gamma(\alpha)} \tag{15}$$

and we obtain

$$H(x) = \frac{e^{-2x\lambda}(x\lambda)^{\alpha}\left((x\lambda)^{\alpha} + e^{x\lambda}(-\alpha + x\lambda + 1)\Gamma(\alpha)\right)}{x^2\Gamma(\alpha)^2} \tag{16}$$

which is strictly positive for the domain of the parameters $\alpha > 1$ and $\lambda > 0$.

2.2. Weibull distribution

The parameters of the Weibull distribution are $\lambda > 0, \alpha > 1$ and $x > 0$. We have

$$f(x) = e^{-x^{\alpha}\lambda}x^{\alpha-1}\alpha\lambda. \tag{17}$$

In this case,

$$H(x) = \alpha\lambda e^{-2\lambda x^{\alpha}} x^{\alpha-2} \left(\alpha\lambda x^{\alpha} + e^{\lambda x^{\alpha}} \left(\alpha\left(\lambda x^{\alpha} - 1\right) + 1\right)\right). \tag{18}$$

Given the domain of the parameters, $H(x) > 0$ when $x > 0$.

2.3. Modified extreme value distribution

This distribution function has as pdf

$$f(x) = \frac{e^{(\alpha-x)/\beta - \frac{x-\alpha}{\beta}}}{\beta} \tag{19}$$

and as failure rate function

$$r(x) = \frac{e^{-\frac{x-\alpha}{\beta}}}{\beta(e^{e^{-\frac{x-\alpha}{\beta}}} - 1)} \tag{20}$$

for this distribution, we have

$$Q(x) = \frac{1}{\beta}\left(e^{-e^{-\frac{x-\alpha}{\beta}}}(e^{-\frac{x-\alpha)}{\beta}} - 1) - 2e^{-\frac{x-\alpha}{\beta} - e^{-\frac{x-\alpha}{\beta}}}\right) \tag{21}$$

We observe that, when $\beta > 0$

$$\lim_{x\to\infty} Q(x) = -\frac{1}{\beta}. \tag{22}$$

Further, for $\beta > 0$

$$\begin{aligned} \lim_{x\to 0} Q(x) &= -\frac{(1+e^{\frac{\alpha}{\beta}})e^{-e^{\alpha/\beta}}}{\beta}, \\ \lim_{\alpha\to\infty} Q(x) &= 0. \end{aligned} \tag{23}$$

And as

$$Q'(x) = -\frac{e^{e^{-\frac{x-\alpha}{\beta}} - \frac{2(x-\alpha)}{\beta}}}{\beta^2} < 0, \tag{24}$$

we conclude that

$$\forall x \geq 0, \forall\alpha \in \mathbb{R},\, \beta \in \mathbb{R}^{*+},\, Q(x) \leq 0. \tag{25}$$

When the parameter α is not large, which would be the general case as this parameter represents a shifting parameter, we have $Q(x) < 0$.

2.4. Truncated normal distribution

The parameters of the Normal distribution which enable to have an IFR are $\mu > 0$, $\sigma > 0$, for which we have

$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \tag{26}$$

but the ensuing

$$H(x) = \frac{\sqrt{2\pi}(x-\mu) + \sigma e^{-\frac{(x-\mu)^2}{\sigma^2}}}{2\pi e^{-\frac{(x-\mu)^2}{\sigma^2}}\sigma^3} \tag{27}$$

is complicated to study.

Instead, we have to look at the initial inequality in (10) to see when it is satisfied.

In fact, since μ is only a translation factor and the factor σ is a scale factor, this function is always negative for $\mu > 0$ and $\sigma > 0$. Figure 1 represents this function with $\mu = 0.5$ and $\sigma = 1$. We obtain

$$Q(x) = -\frac{(x-\mu)\left(\operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)+1\right)}{2\sigma^2} - \frac{\sqrt{\frac{2}{\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma}, \tag{28}$$

with erf(x) as the integral of the gaussian distribution $F(x)$ with $\mu = 0$ and $\sigma = 1$.

3. Conclusion

This problem was numerically solved in Brusset (2010) as a particular instance of the much broader problem presented here. We prove here that for the parameters which enable the above distributions to be IFR distributions, the objective function of the form $R(x) = F(x) + x\overline{F}(x)$, different from the ones contemplated in Lariviere (2006), admit one single interior point over the range of the decision variable x which maximizes it.

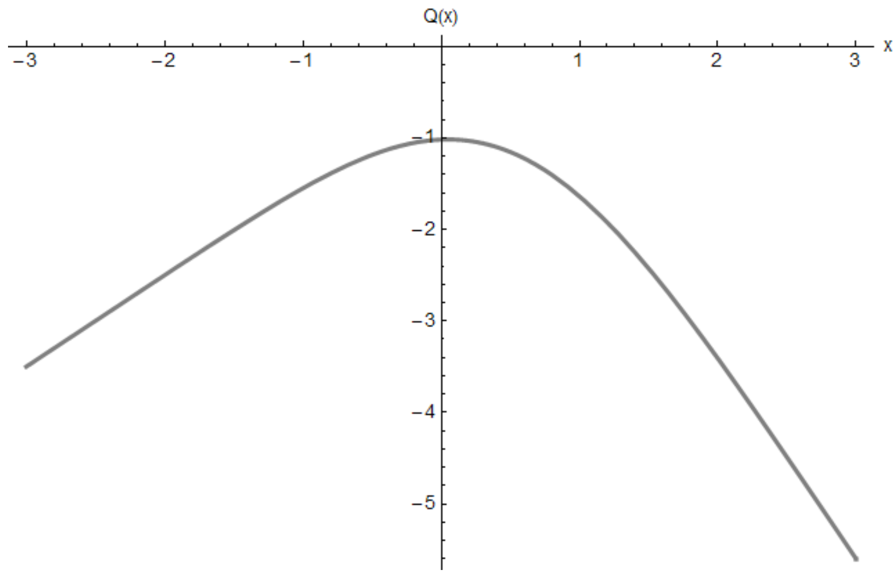

Figure 1: Evolution of function $Q(x)$ when $\mu = 0.5$ and $\sigma = 1$.

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