Private Debt with Pervasive Default Risk

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Abstract

This paper studies the effects of private debts on risk sharing and welfare, in which I assume individual residents have access to both international and domestic capital markets. Like Jeske (2006), I make the assumption that domestic residents cannot commit to repay their debts across border. The previous literature assumes contracts are perfectly enforceable within border, and hence the marginal rate of substitution must be equal among all residents in any one country. The novel feature in this model is to bring limited commitment into debt contracts signed between domestic residents. The pervasive risk of repudiation creates different domestic asset pricing rules for countries that are constrained in international financial market. Constrained country's domestic interest rate is equal to the reciprocal of the lowest marginal rate of substitution within that country. However, non-constrained countries still have equalized marginal rate of substitution which determines the international interest rate. A wider gap between domestic and international financing cost emerges in this model and leads to harsher punishment for international debt defaulters. Although limited domestic risk sharing hinders aggregate welfare reaching an even higher standard, it has no negative effect on the original level in Jeske's setup. As a result, my model allows more international risk sharing and higher welfare. I show how this improvement depends on the interaction between preventing within and across border default in equilibrium. I also explore the role of endogenous borrowing constraints, international borrowing by using other domestic residents as intermediaries and the specification of deviation penalty.

Keywords: Default risk, private debt, limited commitment.

JEL Classification: F34, F41.

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1 Introduction

In the presence of limited commitment, regardless of complete financial markets, international loans are available only to the extent that their repayments can be enforced by the threat of reversion to autarky. Frictions of this kind result in limited risk sharing between countries across the world. The size of risk sharing is determined by the specification of the outside option. Jeske (2006) predicts that a centralized arrangement, where only government borrows internationally and redistributes domestically, allows more international risk sharing and higher aggregate welfare than a decentralized arrangement, where individual residents have access to capital markets. An intuitive proof is as follow. One can think of the decentralized arrangement as a centralized setup only this imaginary government assumes that it can ignore the resource constraint in autarky after default and instead keep borrowing at domestic interest rate just like residents, which is a better outside option than pure autarky in the original centralized arrangement. Higher post-default value leads to tighter participation constraint in international financial market and accordingly smaller capital flow. A crucial assumption in this proof is that domestic debts are perfectly enforceable. For this very reason, the decentralized arrangement is equivalent to an corresponding imaginary planner's problem, in which the marginal rate of substitution (MRS, from now on) is equalized across all residents within any country.

In this paper, I add limited commitment problem in debt contracts signed between domestic residents. This revision introduces a new way of deviation from risk sharing agreement: domestic debt default. The penalty for domestic debt defaulters is harsher than international debt defaulters. While international debt defaulters are excluded from international financial market but retain access to domestic financial market, domestic debt defaulters are prohibited to participate in all financial markets. This assumption of some discrimination against foreign creditors seems to me more realistic than the previous one of infinite discrimination in which domestic creditors are fully protected. This paper shows that limited commitment of domestic debts has opposite effects on country's aggregate welfare after globalization. Pervasive risk of default may hamper domestic exchange and worsen welfare, whereas it can raise the volume of international capital flow thus improve welfare. The reason for the latter statement is as follow. Thanks to participation constraints in domestic financial market, international debt defaulters now confront more severe punishment than before. After a default, the channel of using other non-defaulted domestic residents as intermediaries to access international financial market is restricted. In contrast, international debt defaulters in Jeske's setup can lend and borrow *freely* and indirectly in international financial market afterwards. The question then is which effect dominates. I prove that when residents are heterogeneous and there exist some residents who are sometimes on the edge of reneging on domestic debt, the positive effect enables these residents to enjoy welfare gains from an increment in foreign capital flow while the welfare levels for others stay unaffected. Although the negative effect hinders aggregate welfare reaching an even higher standard when comparing with a centralized distribution mechanism, it is not functioning as by reference to Jeske's decentralized arrangement. Therefore, more international risk sharing and higher (at least the same if there are no indifferent residents) aggregate welfare can be supported in my amended model when comparing with Jeske's decentralized arrangement. The policy implication is that perfect domestic risk sharing and infinite discrimination against foreign creditors together lower the level of international risk sharing if pervasive risk of default is a fact of life. There is a rationale for government in decentralized countries to exchange perfect domestic enforcement for better international connections.

Now that aggregate welfare has been improved by not enforcing domestic debt contracts, does governments still want to control private capital and carry out the centralized arrangement? The answer depends on the trade-off that centralization wins by gains from domestic exchange but may lose by few gains from international exchange. In a numerical example, I show that capital control is always the best disregarding domestic limited commitment if the endowment structure is such that international income fluctuation is large relative to domestic income fluctuation.

The novel commitment problem within border also overturns the domestic bond pricing rule for countries that are participation constrained in equilibrium. In closed economy models with heterogeneous residents, domestic interest rate is determined by the highest MRS possible in that economy to ensure repayment.[1] In open economy models with perfect domestic enforcement, international interest rate is determined by the highest MRS in all countries across the world, whereas domestic interest rate is determined by the equalized MRS within any country.[2] In this model, international interest rate is, as usual, determined by highest MRS, however, domestic interest rate is equalized to the reciprocal of the lowest MRS among all residents. This result is consistent with the assumption that international debt defaulters are penalized harsher in my model. When international debt defaulters come down to seeking help in domestic financial market, they face a wider gap between domestic and international financing cost since domestic bond price is lower than ever. This cruel situation raises international borrowing quota for all residents and pays off by higher welfare

[1] Bond price is negatively related to corresponding interest rate. When price is determined by the highest MRS, interest rate is the loweset possible. See, for example, Alvarez and Jermann (2000, 2001) and Azariadis and Lambertini (2002).

[2] See, for example, Jeske (2006) and Wright (2006).

level.

Of critical importance in this strand of literature is the specification of what defaulters may be entitled. In the remainder of this section, I review several closely related work which try to relax the assumption of compelet exclusion from future trading in early theories.[3]

A growing research body uses the assumption of partial exclusion, under which defaulters retain some access to financial markets or have alternative ways to smooth consumption. International risk sharing diminishes in size since life after default is less painful than it would be otherwise. Partial exclusion arises if defaulters can reenter foreign capital market indirectly through intermediaries as in Jeske (2006). Wright (2006) builds on Jeske's model and argues that international borrowing subsidies can also lead to constrained efficient allocations instead of Jeske's radical way of centralization. Defaulters continuing to have access to international savings may also cause partial exclusion. Bulow and Rogoff (1989) first use this idea and prove that international borrowing cannot be supported in a small open economy that takes the international interest rate as given (partial equilibrium). Hellwig and Lorenzoni (2007) carry their work further to a multi-country (general equilibrium) setup in which they show that international risk sharing can exist with low interest rate. Then Wright establishes an equivalence between the above two modeling methods if the extra dimension of heterogeneity among residents in Jeske's model is taken care of. Reduced penalty can be due to other internal opportunities as well. For example, Kehoe and Perri (2002, 2004) study international risk sharing in a real business cycle model under productivity shock, in which autarky value depends on the quantity of capital the country has accumulated up to defaults. In their paper, defaulters can continue to produce and consume capital in *autarky*, but they may not buy or sell capital and other financial assets to foreigners.

Broner and Ventura (2009) assume that countries cannot discriminate against foreign creditors. Thus, international risk sharing is obtained even in the absence of international debt default penalties. Unlike my model in which residents make default decisions, in their setup government endogenously chooses whether to enforce all debt contracts or none. They show that decrease in trade barriers in goods market facilitates international trade and raises the costs of enforcement. As a result, globalization might hamper domestic trade and lower aggregate welfare. In this paper, government can identify the citizenship but still chooses to enforce none because more international risk sharing can be supported.

[3] See, for example, Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jermann (2000) and Kehoe and Perri (2002, 2004)

The remainder of the paper is organized as follows. In section 2, I present the model of private international lending and borrowing with limited commitment problem within and across border and derive equilibrium results. Section 3 compares aggregate welfare level in different setups, thus generates some policy implications. Section 4 introduces a crude numerical example which illustrates the essence of this paper. Section 5 concludes and finally a technical appendix contains all proofs.

2 The Model

The model considers a world that consists of a finite number of countries denoted as $m \in \{1, 2, ..., M\}$ and each country m is populated by N types of residents with a continuum of them in each type $n \in \{1, 2, ..., N\}$.[4] Residents live forever so that time is infinite and discrete, denoted by $t = 0, 1, 2, ..., \infty$. Information about current and future endowments is indexed by the state $\theta_t \in \Theta$. History is summarized in $\theta^t \equiv \{\theta^0, \theta_1, \theta_2, ..., \theta_t\} \in \Theta^t$ with θ^0 given. Transition probability from history θ^t to next period's state θ_{t+1} is known as $\pi(\theta_{t+1}|\theta^t)$ with θ^t given. $\pi(\theta^t)$ is the unconditional probability of observing history θ^t and $\pi(\theta^r|\theta^t)$ is the probability of observing history θ^r conditional on having been in θ^t. There is only one non-storable consumption good which can be exchanged within and across border. I denote by $e_n^m(\theta^t)$ the endowment of type n in country m after history θ^t and by $c_n^m(\theta^t)$ the corresponding consumption. There are M domestic bonds for each country m and only one international bond traded across the world. Let $b_n^m(\theta^t, \theta_{t+1})$ and $f_n^m(\theta^t, \theta_{t+1})$ respectively be the amounts of domestic and foreign state-contingent securities held by agents of type n living in country m, which are purchased at θ^t and for payment next period in state θ_{t+1}; $p^m(\theta^t, \theta_{t+1})$ and $q(\theta^t, \theta_{t+1})$ are their respective prices.

For all n and m, I use $\beta \in (0, 1)$ as the discount factor and denote by $U(\cdot)$ the period utility function which is strictly increasing, strictly concave, and twice continuously differentiable. Type n residents in country m have preferences

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)),$$

[4] Unlike Jeske, I assume that in any country m the mass of type n residents λ_n^m is normalized to 1 for all $n \in \{1, 2, ..., N\}$. Note that in Jeske's model one's endowment only depends on type. Although living in different countries, same types receive the same endowment each period. As a result, having $\lambda_n^m = 1$ for all n and m in Jeske would imply that countries are identical ex-ante, thus there may not be any role for international capital flow. However, in this paper one's endowment vary upon both type and country, which will be clear after I introduce history and endowment structure. Assuming $\lambda_n^m = 1$ simplifies notation, whereas still justifies international borrowing and lending.

after θ^t with $t \in [0, \infty)$.

I take the existence of limited commitment problem as a fact of life. In particular, debt contracts between any two parties are not enforced. Previous theories have instead assumed that debt contracts between domestic residents and foreigners are not enforced while all domestic payments are perfectly enforced. Border still matters here because default within border leads to different penalty from default across border. International debt defaulters can still trade internationally indirectly, through borrowing from other non-defaulted domestic residents in the same country. I refer this situation as resident's international autarky henceforth. However, domestic debt defaulters would be denied from all financial markets, which I call resident's autarky hereafter.

Despite the fact whether residents have defaulted on international debt or not, the continuation value for type n residents in country m who renege on domestic debt at θ^r is

$$A_n^m(\theta^r) \equiv \sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(e_n^m(\theta^s)). \tag{RA}$$

Definition 1 *Type n residents in country m live in* ***resident's autarky*** *after any history θ^r if their period consumption $c_n^{m,A}(\theta^s) = e_n^m(\theta^s)$ for all histories θ^s with $s \in [r, \infty)$.*

Since all residents are small relative to the market, individual defaulter on international debt does so at θ^t by assuming that the sequence of all future domestic bond prices $\{p^m(\theta^r, \theta_{r+1})\}_{r \in [t,\infty)}$ stays unchanged. Consider type n residents in country m who renege on international debt alone at θ^t, their continuation value in resident's international autarky can be represented as

$$V_n^m(\theta^t, b_n^m(\theta^t)) \equiv \max_{\{c_n^m(\theta^r), b_n^m(\theta^r, \theta_{r+1})\}_{r \in [t,\infty)}} \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \tag{RIA}$$

subject to the budget constraint

$$e_n^m(\theta^r) + b_n^m(\theta^r) \geqslant c_n^m(\theta^r) + \sum_{\theta_{r+1}} p^m(\theta^r, \theta_{r+1}) b_n^m(\theta^r, \theta_{r+1}), \tag{1}$$

the participation constraint in domestic financial markets

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^m(\theta^s)) \geqslant A_n^m(\theta^r), \tag{2}$$

the no-Ponzi condition

$$b_n^m(\theta^r, \theta_{r+1}) \geqslant -\overline{B},$$

with

$$b_n^m(\theta^t) \text{ and } p^m(\theta^r, \theta_{r+1}) \text{ given,}$$

for all histories θ^r and states (θ^r, θ_{r+1}) with $r \in [t, \infty)$. $b_n^m(\theta^t)$ denotes the domestic bond holdings residents inherit when entering period t. $\overline{B} > 0$ is large enough such that no-Ponzi conditions never bind in equilibrium, which ensures compactness of the budget set. For the problem to be interesting, I assume for some n, m and θ^t there exist future histories θ^r at which domestic participation constraints (2) are binding.

Unlike Jeske's model, in which domestic debt default is not allowed in nature though it might be a better choice in some future histories θ^r if

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^m(\theta^s)) < A_n^m(\theta^r) \text{ with } r > t,$$

this model prevents domestic debt defaults at all future histories θ^r with $r \in [t, \infty)$ by a set of constraints (2).

Definition 2 *Given a price sequence* $\{p^m(\theta^r, \theta_{r+1}), q(\theta^r, \theta_{r+1})\}_{r\in[t,\infty)}$ *and domestic bond holdings* $b_n^m(\theta^t)$*, type* n *residents in country* m *live in* ***resident's international autarky*** *after any history* θ^t *if their consumption allocation* $\{c_n^{m,D}(\theta^r)\}_{r\in[t,\infty)}$ *solves resident's international autarky problem (RIA).*

Up to now, I have defined the outside options for domestic debt defaulters in (RA) and international debt defaulters in (RIA). We are ready to write down resident's problem before any default. Residents choose sequences for consumption and for both domestic and international bond holdings to maximize life time preference

$$\max_{\{c_n^m(\theta^t), b_n^m(\theta^t, \theta_{t+1}), f_n^m(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)}} \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)), \tag{RP}$$

subject to the budget constraint

$$\begin{aligned} & e_n^m(\theta^t) + b_n^m(\theta^t) + f_n^m(\theta^t) \\ \geqslant \quad & c_n^m(\theta^t) + \sum_{\theta_{t+1}} p^m(\theta^t, \theta_{t+1}) b_n^m(\theta^t, \theta_{t+1}) + \sum_{\theta_{t+1}} q(\theta^t, \theta_{t+1}) f_n^m(\theta^t, \theta_{t+1}), \end{aligned} \tag{3}$$

the participation constraint in international financial markets

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant V_n^m(\theta^t, b_n^m(\theta^t)), \tag{4}$$

the participation constraint in domestic financial markets

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant A_n^m(\theta^t), \tag{5}$$

the no-Ponzi conditions

$$b_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{B}, f_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{F},$$

with initial bond holdings

$$b_n^m(\theta^0), f_n^m(\theta^0) \text{ given}$$

and bond price sequences

$$p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1}) \text{ given,}$$

for all histories θ^t and states $\left(\theta^t, \theta_{t+1}\right)$ with $t \in [0, \infty)$. No-Ponzi conditions again require $\overline{B}$ and $\overline{F} > 0$ to be large enough to ensure that these two constraints never bind.

To solve the problem, notice that international participation constraint (4) implies the domestic participation constraint (5) for all θ^t with $t \in [0, \infty)$. By definition,

$$V_n^m(\theta^t, b_n^m(\theta^t)) \geqslant A_n^m(\theta^t) \text{ for all } \theta^t.$$

The reason is that problem (RA)'s consumption allocation $\left\{c_n^{m,A}(\theta^r)\right\}_{r\in[t,\infty)}$ is always affordable and individual rational in problem (RIA). Intuitively, no one reneges on domestic debt before a default on international debt in equilibrium. This is a direct result from the assumption of harsher penalty for domestic debt defaulters than international debt defaulters. Hence, constraint (5) is redundant in finding optimal solutions as can be seen in Figure 1. This cut simplifies life since domestic enforcement problem will only affect optimal allocation indirectly through the utility level of resident's international autarky, i.e., $V_n^m(\theta^t, b_n^m(\theta^t))$. Rest of this section first defines, then characterizes the equilibrium results.

Definition 3 *A* ***trade equilibrium*** *is an allocation* $\{c_n^m(\theta^t), b_n^m(\theta^t, \theta_{t+1}), f_n^m(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)}$ *and a price sequence* $\{p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)}$ *such that each type solves resident's problem (RP) given prices and initial bond holdings, while resource feasibility*

$$\sum_{m=1}^{M}\sum_{n=1}^{N} c_n^m(\theta^t) \leq \sum_{m=1}^{M}\sum_{n=1}^{N} e_n^m(\theta^t),$$

domestic financial markets clearing condition

$$\sum_{n=1}^{N} b_n^m(\theta^t, \theta_{t+1}) = 0, \text{ for all } m,$$

and international financial market clearing condition

$$\sum_{m=1}^{M}\sum_{n=1}^{N} f_n^m(\theta^t, \theta_{t+1}) = 0,$$

are satisfied for all histories θ^t *and states* $\left(\theta^t, \theta_{t+1}\right)$ *with* $t \in [0, \infty)$.

The Lagrangian of the resident's problem (RP) is (drop the superscript and subscript for simplicity)

$$\begin{aligned} L_{RP} &= \sum_{t=0}^{\infty}\beta^t \sum_{\theta^t}\pi(\theta^t)U(c(\theta^t)) + \\ &+\sum_{t=0}^{\infty}\sum_{\theta^t}\kappa(\theta^t)\left[e(\theta^t)+b(\theta^t)+f(\theta^t)-c(\theta^t)\right] \\ &-\sum_{t=0}^{\infty}\sum_{\theta^t}\kappa(\theta^t)\left[p(\theta^t,\theta_{t+1})b(\theta^t,\theta_{t+1})+q(\theta^t,\theta_{t+1})f(\theta^t,\theta_{t+1})\right] \\ &+\sum_{t=0}^{\infty}\sum_{\theta^t}\mu(\theta^t)\left[\sum_{s=t}^{\infty}\beta^{s-t}\sum_{\theta^s|\theta^t}\pi(\theta^s|\theta^t)U(c(\theta^s))-V(\theta^t,b_n^m(\theta^t))\right], \end{aligned}$$

where $\kappa_n^m(\theta^t)$ and $\mu_n^m(\theta^t)$ denote respectively the Lagrange multipliers on the budget constraint (3) and international participation constraint (4) if θ^t occurs.

First order conditions are: with respect to $c(\theta^t)$,

$$\beta^t\pi(\theta^t)U'(c(\theta^t)) - \kappa(\theta^t) + \sum_{s=0}^{t}\mu(\theta^s)\beta^{t-s}\sum_{\theta^t|\theta^s}\pi(\theta^t|\theta^s)U'(c(\theta^t)) = 0; \tag{6}$$

with respect to $b(\theta^t,\theta_{t+1})$,

$$-p(\theta^t,\theta_{t+1})\kappa(\theta^t) + \kappa(\theta^t,\theta_{t+1}) - \mu(\theta^t,\theta_{t+1})\frac{dV(\theta^t,\theta_{t+1},b(\theta^t,\theta_{t+1}))}{db(\theta^t,\theta_{t+1})} = 0; \tag{7}$$

and with respect to $f(\theta^t,\theta_{t+1})$,

$$-q(\theta^t,\theta_{t+1})\kappa(\theta^t) + \kappa(\theta^t,\theta_{t+1}) = 0. \tag{8}$$

Rearrange (6) to get

$$\kappa(\theta^t) = \beta^{r-t}\pi(\theta^t)U'(c(\theta^t))\left[1+\sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}\right]. \tag{9}$$

Before using (7), I went back and solved the post-default optimization problem (RIA) in order to get a closed form of its envelope condition, $\frac{dV(\theta^t,\theta_{t+1},b(\theta^t,\theta_{t+1}))}{db(\theta^t,\theta_{t+1})}$.

To solve (RIA) with initial history θ^t, first write down the Lagrangian.

$$\begin{aligned} L_{RIA} &= \sum_{r=t}^{\infty}\beta^{r-t}\sum_{\theta^r|\theta^t}\pi(\theta^r|\theta^t)U(c(\theta^r)) \\ &+\sum_{r=t}^{\infty}\sum_{\theta^r|\theta^t}\lambda(\theta^r)\left[e(\theta^r)+b(\theta^r)-c(\theta^r)-\sum_{\theta_{r+1}}p(\theta^r,\theta_{r+1})b(\theta^r,\theta_{r+1})\right] \\ &+\sum_{r=t}^{\infty}\sum_{\theta^r|\theta^t}\nu(\theta^r)\left[\sum_{s=r}^{\infty}\beta^{s-r}\sum_{\theta^s|\theta^r}\pi(\theta^s|\theta^r)U(c(\theta^s))-A_n^m(\theta^r)\right]. \end{aligned}$$

Let $\lambda_n^m(\theta^r)$ and $\nu_n^m(\theta^r)$ be the multipliers on the budget constraint (1) and domestic participation constraint (2) if θ^r occurs. First order condition with respect to $c(\theta^r)$ is

$$\beta^{r-t}\pi(\theta^r|\theta^t)U'(c(\theta^r)) - \lambda(\theta^r) + \sum_{s=t}^{r}\nu(\theta^s)\beta^{r-s}\sum_{\theta^r|\theta^s}\pi(\theta^r|\theta^s)U'(c(\theta^r)) = 0.$$

Rewrite it to get

$$\lambda(\theta^r) = \beta^{r-t}\pi(\theta^r|\theta^t)U'(c(\theta^r))\left[1 + \sum_{s=t}^{r}\sum_{\theta^r|\theta^s}\nu(\theta^s)\beta^{t-s}\frac{\pi(\theta^r|\theta^s)}{\pi(\theta^r|\theta^t)}\right]. \tag{10}$$

Given initial domestic bond holdings $b_n^m(\theta^t)$, envelope theorem yields

$$\frac{dV(\theta^t, b(\theta^t))}{db(\theta^t)} = \frac{\partial L_{RIA}}{\partial b(\theta^t)} = \lambda(\theta^t). \tag{11}$$

Combine (10) and (11) together.

$$\begin{aligned}\frac{dV(\theta^t, b(\theta^t))}{db(\theta^t)} &= \beta^{t-t}\pi(\theta^t|\theta^t)U'(c^D(\theta^t))\left[1 + \sum_{s=t}^{t}\sum_{\theta^t|\theta^s}\nu(\theta^s)\beta^{t-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t|\theta^t)}\right] \\ &= \left[1 + \nu(\theta^t)\right]U'(c^D(\theta^t)),\end{aligned}$$

where $c^D(\theta^t)$ is the first element in the optimal consumption allocation $\left\{c^D(\theta^r)\right\}_{r\in[t,\infty)}$ that solves (RIA) with initial history θ^t. Iterating $\frac{dV(\theta^t,b(\theta^t))}{db(\theta^t)}$ one period forward generates

$$\frac{dV(\theta^t, \theta_{t+1}, b(\theta^t, \theta_{t+1}))}{db(\theta^t, \theta_{t+1})} = \left[1 + \nu(\theta^t, \theta_{t+1})\right]U'(c^D(\theta^t, \theta_{t+1})). \tag{12}$$

Now continue with problem (RP). Substitute (12) into (7) and solve for the domestic bond price together with (9).

$$\begin{aligned}p(\theta^t, \theta_{t+1}) &= \frac{\kappa(\theta^t, \theta_{t+1}) - \mu(\theta^t, \theta_{t+1})\frac{dV(\theta^t, \theta_{t+1}, b(\theta^t, \theta_{t+1}))}{db(\theta^t, \theta_{t+1})}}{\kappa(\theta^t)} \\ &= \frac{\kappa(\theta^t, \theta_{t+1}) - \mu(\theta^t, \theta_{t+1})\left[1 + \nu(\theta^t, \theta_{t+1})\right]U'(c^D(\theta^t, \theta_{t+1}))}{\kappa(\theta^t)} \\ &= \beta\frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1 + A_2 - \left(1 + \nu(\theta^t, \theta_{t+1})\right)A_1}{1 + A_3},\end{aligned} \tag{13}$$

where

$$\begin{aligned}A_1 &= \mu(\theta^t, \theta_{t+1})\beta^{-t-1}\frac{U'(c^D(\theta^t, \theta_{t+1}))}{U'(c(\theta^t, \theta_{t+1}))}\frac{1}{\pi(\theta^t, \theta_{t+1})}; \\ A_2 &= \sum_{s=0}^{t+1}\sum_{\theta^t, \theta_{t+1}|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t, \theta_{t+1}|\theta^s)}{\pi(\theta^t, \theta_{t+1})}; \\ A_3 &= \sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}.\end{aligned}$$

Finally, solve for international bond price with (8) and (9).

$$
\begin{aligned}
q(\theta^t,\theta_{t+1}) &= \frac{\kappa(\theta^t,\theta_{t+1})}{\kappa(\theta^t)} \\
&= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1+A_2}{1+A_3}.
\end{aligned} \tag{14}
$$

Most propositions in this section closely follow Jeske (2006) with slight change to accommodate the enforcement problem within border. In country m, consider residents of some types $n \subseteq \{1,2,...,N\}$ for whom $\mu_n^m(\theta^t) > 0$ with $t \in [0,\infty)$, i.e., they are internationally participation constrained at θ^t in trade equilibrium. In contrast to the work by Jeske, these constrained types might be different from each other because of domestic enforcement problem. Specifically, type n residents fall into two categories: type n_A and n_B with $n_A \cup n_B = n$. In resident's international autarky, type n_A with $v_{n_A}^m(\theta^r) = 0$ for all $r \in [t,\infty)$ are *never* domestically participation constrained while type n_B with $v_{n_B}^m(\theta^r) > 0$ for some $r \in [t,\infty)$ are *sometimes* domestically participation constrained at θ^r.

Type n residents at the brink of default attain the same continuation value by staying with trade equilibrium and reversing to resident's international autarky. Proposition 1 states that types n also consume the same amount of goods at every future history from θ^t on irrespective of reneging on international debt.

Proposition 1 *In trade equilibrium, if $\mu_n^m(\theta^t) > 0$ for some n, m and θ^t with $t \in [0,\infty)$, then $c_n^{m,D}(\theta^r)$ and $c_n^m(\theta^r)$ are identical for all θ^r happening with positive probability and $r \in [t,\infty)$, where $\left\{c_n^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ and $\{c_n^m(\theta^r)\}_{r\in[0,\infty)}$ respectively denote the optimal consumption allocations in resident's international autarky problem (RIA) started at θ^r and in resident's problem (RP).*

Proof. *See Appendix A.1.* ■

I get the same result above as in Jeske's (2006) proposition 1, which has been extremely useful in proving all following propositions. Proposition 2 states that, for any country and history, either every type is internationally participation constrained or no type is, even if types are heterogeneous.

Proposition 2 *For all m and (θ^t,θ_{t+1}) with $t \in [0,\infty)$, either $q(\theta^t,\theta_{t+1}) > p^m(\theta^t,\theta_{t+1})$ and $\mu_n^m(\theta^t,\theta_{t+1}) > 0$ for all n, or $q(\theta^t,\theta_{t+1}) = p^m(\theta^t,\theta_{t+1})$ and $\mu_n^m(\theta^t,\theta_{t+1}) = 0$ for all n.*

Proof. *See Appendix A.2.* ■

All types in country m must be participation constrained in international financial market at the same time. Otherwise, it would be profitable for non-constrained types to borrow internationally up to their constraints and relend at a higher interest rate to constrained ones. No arbitrage in equilbrium allows us to replace internationally constrained residents with the notion of internationally constrained country. This result is again the same as Jeske's (2006) proposition 4. Removing the assumption of perfectly enforceable contracts within border, therfore, does not alter the basic characteristics of countries that are internationally participation constrained in trade equilibrium. However, within the border of any constrained country, domestic enforcement problem affects the restrictions imposed on type n_B when participating in international financial market.

If type n_B residents from one internationally constrained country indeed choose to default on international debt at θ^t and instead live in resident's international autarky thereafter, then they will again find themselves indifferent between reneging on and repaying domestic debt at some future history θ^r. Proposition 3 says that type n_B residents can neither borrow nor lend domestically beyond some constant domestic debt holdings at θ^{t-1} in trade equilibrium.

Proposition 3 *In trade equilibrium, for some* n, m *and* θ^t *with* $t \in (0, \infty)$, *if* $\nu_n^m(\theta^r) > 0$ *at* θ^r *with* $r \in [t, \infty)$ *in addition to* $\mu_n^m(\theta^t) > 0$, *then*

$$b_n^m(\theta^t) = \overline{B}_n^m(\theta^t),$$

where $\overline{B}_n^m(\theta^t)$ *is some constant determined by*

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^{m,D}(\theta^s, \theta^t, \overline{B}_n^m(\theta^t))) = A_n^m(\theta^r).$$

Proof. *See Appendix A.3.* ■

when it comes to decide the domestic bond holdings $b_{n_B}^m(\theta^t)$ at θ^{t-1}, type n_B residents in internationally constrained country must buy a specified amount $\overline{B}_{n_B}^m(\theta^r)$ even if they want to borrow or lend more. This observation is critical to prove the next two propositions. Meanwhile, type n_A residents in the same country can freely choose their domestic bond holdings, but they do not want to deviate whenever the optimum is arrivied.

In general, the international participation constraint (4) makes resident's problem (RP) non-convex. Proposition 4 justifies the sufficiency of first-order-condition approach for a maximum. A similar method has been used in the proof of Jeske's (2006) proposition 2. First of all, I define an alternative maximization problem with the same objective function and a convex constraint set that is a superset of the non-convex constraint set in the original non-convex problem. Then I show that a solution to the original problem is also affordable and individually rational in the alternative convex problem. It turns out that both problems have identical first order conditions, thus the same optimal solutions. Therefore, the first order conditions for the alternative problem also characterize the global maximum of the original problem.

Proposition 4 *For all n and m, together with a transversality condition*

$$\lim_{T\to\infty}\beta^T\sum_{\theta^T}U'(c_n^m(\theta^T))\pi(\theta^T)\left[b_n^m(\theta^T)+f_n^m(\theta^T)\right]=0,$$

first order conditions (7), (8) and (9) are sufficient to characterize the maximum of (RP).

Proof. *See Appendix A.4.* ■

Proposition 5 shows how domestic and international bond prices are determined in trade equilibrium. The domestic bond pricing rule differs from the one in Jeske's (2006) proposition 3 since MRS's across different types in any internationally constrained country are no longer equalized in general.

Proposition 5 *In trade equilibrium, for all n, m and $\left(\theta^t,\theta_{t+1}\right)$ with $t\in[0,\infty)$,*

(I) the international bond price in the world is

$$q(\theta^t,\theta_{t+1})=\max_{m=1,\ldots,M;n=1,\ldots,N}\left\{\beta\frac{U'(c_n^m(\theta^t,\theta_{t+1}))}{U'(c_n^m(\theta^t))}\pi(\theta_{t+1}|\theta^t)\right\},$$

(II) the domestic bond price in country $m\in\{1,2,...,M\}$ is

$$p^m(\theta^t,\theta_{t+1})=\min_{n=1,\ldots,N}\left\{\beta\frac{U'(c_n^m(\theta^t,\theta_{t+1}))}{U'(c_n^m(\theta^t))}\pi(\theta_{t+1}|\theta^t)\right\},$$

and finally

(III) the relationship between international bond price and all M domestic bond prices is

$$q(\theta^t,\theta_{t+1})=\max_{m=1,\ldots,M}\left\{p^m(\theta^t,\theta_{t+1})\right\}.$$

Proof. *See Appendix A.5.* ■

In the first part, the international bond price is equal to the maximum MRS among all N types and across all M countries. Intuitively, international interest rate has to be to the lowest so that repaying international debt would not hurt debtors as much as living isolated from the world.

The second part says that the domestic bond price in m is equal to the minimum MRS among all N types in m. This rule overturns the result from closed economy models in which price must be the highest to guarantee the incentive of fulfilling debtor's obligation. The reason is that domestic debt default can never occur without an international debt default in this paper**.** In contrast to closed economy models, domestic interest rate as a device to ensure repayment is no longer needed. Instead, domestic interest rate plays aother role of punishing international debt defaulters harsher and raising the ceiling of foreign capital inflow.

Consider a country with $\mu_n^m(\theta^t, \theta_{t+1}) > 0$ for example. When its residents are contemplating international debt default, they find themselves more miserable in resident's international autarky since the domestic interest rate is higher than the level they would have accepted. Country with $\mu_n^m(\theta^t, \theta_{t+1}) = 0$ is a special case in part two since the MRS's are equalized among all N types in this unconstrained country. Moreover, its domestic bond price must equal to the international bond price in order to rule out the possibilities of arbitrage. All the above results combining together reveals the relationship between international and domestic bond prices in the thrid part.

3 Welfare Analysis

In this section, I rank the aggregate welfare level in two scenarios for any country $m \in \{1, 2, ..., M\}$. The first scenario is Jeske's private international borrowing setup in which only debt contracts across border are subject to the risk of repudiation. The second scenario is my decentralized model presented in section 2, in which both within and across border contracts are not enforced. I can show that country m's aggregate welfare in my model is higher than Jeske's setup given some exogenous welfare-weighted index.

Recall Jeske (2001, 2006), after type n residents in country m renege on international debt at θ^t, their value can be represented as

$$V_n^{m,J}(\theta^t, b_n^m(\theta^t)) \equiv \max_{\{c_n^m(\theta^r), b_n^m(\theta^r, \theta_{r+1})\}_{r\in[t,\infty)}} \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \qquad (\mathrm{RIA^J})$$

subject to the budget constraint

$$e_n^m(\theta^r) + b_n^m(\theta^r) \geqslant c_n^m(\theta^r) + \sum_{\theta_{r+1}} p^m(\theta^r, \theta_{r+1}) b_n^m(\theta^r, \theta_{r+1}),$$

the no-Ponzi condition

$$b_n^m(\theta^r, \theta_{r+1}) \geqslant -\overline{B},$$

with

$$b_n^m(\theta^t) \text{ and } p^m(\theta^r, \theta_{r+1}) \text{ given,}$$

for all histories θ^r and all states (θ^r, θ_{r+1}) with $r \in [t, \infty)$.

Again at date 0 before any default could happen, resident's problem is to choose sequences for consumption and for holdings of both domestic and international bonds to maximize life time utility

$$\max_{\{c_n^m(\theta^t), b_n^m(\theta^t, \theta_{t+1}), f_n^m(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)}} \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)), \qquad (\mathrm{RP^J})$$

subject to the budget constraint

$$
\begin{aligned}
& e_n^m(\theta^t) + b_n^m(\theta^t) + f_n^m(\theta^t) \\
\geqslant \quad & c_n^m(\theta^t) + \sum_{\theta_{t+1}} p^m(\theta^t, \theta_{t+1}) b_n^m(\theta^t, \theta_{t+1}) + \sum_{\theta_{t+1}} q(\theta^t, \theta_{t+1}) f_n^m(\theta^t, \theta_{t+1}),
\end{aligned}
$$

the participation constraint in international financial markets

$$
\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant V_n^{m,J}(\theta^t, b_n^m(\theta^t)), \tag{15}
$$

the no-Ponzi conditions

$$
b_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{B}, f_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{F},
$$

with initial bond holdings

$$
b_n^m(\theta^0), f_n^m(\theta^0) \text{ given}
$$

and bond price sequences

$$
p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1}) \text{ given},
$$

for all histories θ^t and all states $\left(\theta^t, \theta_{t+1}\right)$ with $t \in [0, \infty)$.

For any m, all types in my model achieve higher or at least the same welfare level than in Jeske's setup above. One intuitive explanation is that type n_B residents now confront with more severe penalty after international debt default (lower continuation value in resident's international autarky). Assume a small open economy m stops enforcing domestic contracts after a bad shock. Type n_A's optimization problem can still be defined by ($\mathrm{RP^J}$), thus their welfare level stays the same. However, type n_B's optimization problem shifts from ($\mathrm{RP^J}$) to (RP). Types n_B residents' original optimal allocations in ($\mathrm{RP^J}$) are both affordable in (RP), since bond prices determined by type n_A stay unchanged, and individual rational in (RP), since international participation constraints (4) are less tighter than (15) and the newly added domestic participation constraints (5) are superfluous. If there do exist type n_B residents whose domestic participation constraints (2) bind after some future histories in (RIA) when their country is internationally constrained after some present histories in (RP), and if there are positive international capital inflows at these present histories in (RP), then type n_B residents after the shock can do strictly better by relaxing their international participation constraints (4) at present histories in (RP). Adding up welfare levels of all types with exogenously given weights assigned to each type, the country's aggregate welfare is improved since type n_A residents stay the same and type n_B residents are strictly better off. The following proposition 6 formalizes this argument.

Proposition 6 *Assume the welfare-weighted index is given by* $\{\varphi_n^m\}_{n=1}^N$ *with* $\varphi_n^m \in \mathbb{R}_{++}$ *for all* n *in any* m*. Let* $\left\{c_n^m\left(\theta^t\right), b_n^m(\theta^t, \theta_{t+1}), f_n^m(\theta^t, \theta_{t+1})\right\}_{t\in[0,\infty)}$ *solves resident's problem (RP) in section 2 and* $\left\{c_n^{m,J}\left(\theta^t\right), b_n^{m,J}(\theta^t, \theta_{t+1}), f_n^{m,J}(\theta^t, \theta_{t+1})\right\}_{t\in[0,\infty)}$ *solves resident's problem (RP*^{J}*) in Jeske (2006). Then*

$$\sum_{n=1}^{N} \varphi_n^m \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)) \geqslant \sum_{n=1}^{N} \varphi_n^m \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^{m,J}(\theta^t))$$

with strict inequality if there is type n *at history* (θ^t, θ_{t+1}) *with* $f_n^{m,J}(\theta^t, \theta_{t+1}) < 0$ *(non-autarkic allocation) and* $\frac{U'\left(c_n^m(\theta^t,\theta_{t+1})\right)}{U'(c_n^m(\theta^t))} \neq \frac{U'\left(c_1^m(\theta^t,\theta_{t+1})\right)}{U'\left(c_1^m(\theta^t)\right)}$ *for some* n *(imperfect sharing domestically).*

Proof. *See Appendix A.6.* ■

4 Numerical Example

I illustrate my results through a simple example in this section. Consider a world of country 1 and 2, each of them is populated by a unit mass of residents with static preference $U(c) = \log(c)$. Residents born at period 0 live forever and time is discrete. A sort of non-storable goods is traded every period.

The initial endowment structure in the world could be $1 + y$ in 1 and $1 - y$ in 2 or $1 - y$ in 1 and $1 + y$ in 2. But after the initial endowment in each country is known, aggregate endowment alternates between high state and low state deterministically. Residents in each country could be type A, who faces idiosyncratic endowment shock with negative ε in low state and positive ε in high state, or type B who faces just the opposite shock as type A. It is assumed that y is large relative to ε, capturing the idea that income fluctuations across countries are much volatile than income fluctuations within a country.

The timeline of contracting is as follow before any transaction. First of all, domestic debt contracts are signed between residents within the same country. Secondly, a coin flip determines the type of half random residents in both countries, and then domestic contracts are either enforced under the assumption of full domestic commitment or subject to default risk under the assumption of no domestic commitment. Thirdly, international debt contracts are signed between domestic residents and foreigners. Eventually, another independent coin flip determines the initial endowment structure in the world, and agents will not deviate from ex-ante international agreement if they are as better off as autarky.

Without loss of generality, suppose country 1 has initial aggregate endowment $1 + y$. Then, the endowment structure at period 0 can be summarized in Table 1, where superscript $m = \{1, 2\}$ denotes country and subscript $n = \{A, B\}$ denotes type. Table 2 summarizes the endowment structure at period 1 and the endowment structure in either country repeats itself every two periods. Therefore, country 1 is internationally participation constrained at all even numbered periods $0, 2, 4, ..., \infty$, even type B with a relatively lower endowment. At all odd numbered periods $1, 3, 5, ..., \infty$, country 1 as a borrower is unconstrained in international financial markets.

After the type is known to residents and before the realization of initial aggregate endowment for countries, a representative resident of any type has expected life time preference

$$E\left[u(z)\right] = \frac{1}{2}\left[\log(1+z) + \beta\log(1-z)\right] + \frac{1}{2}\left[\log(1-z) + \beta\log(1+z)\right],$$

where z represents consumption deviation and β denotes discount factor. $E\left[u(z)\right]$ is strictly decreasing in z as depicted in Figure 2. More international risk sharing means higher ex-ante welfare.

4.1 Private Borrowing with Full Commitment Domestically

Because domestic debt contracts are perfectly enforced, different types in the same country consume the same amount of goods every period. The consumption pattern in the world is as follow

$$\begin{aligned} c_t^1 &= 1 + x^J, c_{t+1}^1 = 1 - x^J, \text{ for country 1;} \\ c_t^2 &= 1 - x^J, c_{t+1}^2 = 1 + x^J, \text{ for country 2.} \end{aligned}$$

Country 1 is participation constrained in international financial market at period 0, which implies that the present value of all future net payments to foreigners when discounted by Arrow-Debreu domestic bond prices is zero.[5]

$$\frac{x^J - y + q\left(y - x^J\right)}{1 - pq} = 0. \tag{16}$$

The price for international bonds today denoted by q is determined by the MRS in countries that are unconstrained next period. In other words, international bond price can be found in countries whose residents consume $1 + x^J$ today and $1 - x^J$ tomorrow.

$$q = \beta\frac{1 + x^J}{1 - x^J}. \tag{17}$$

And p denotes the price for domestically traded bonds when the country is participation constrained next period

$$p = \beta\frac{1 - x^J}{1 + x^J}.$$

[5] See the derivation in Appendix A.7.

The price sequences for domestic bonds traded in country 1 and country 2 are

$$p^1 = \begin{cases} p, \text{ for period } 0,2,4,... \\ q, \text{ for period } 1,3,5,... \end{cases}; \tag{18}$$

$$p^2 = \begin{cases} q, \text{ for period } 0,2,4,... \\ p, \text{ for period } 1,3,5,... \end{cases}.$$

There are two solutions to equation (16). The first is autarky, or $x^J = y$, while the second requires $q = 1$, which further implies $x^J = \frac{1-\beta}{1+\beta}$ using equation (17).

4.2 Centralized Borrowing

Consider a world of two centralized economies where government lends and borrows internationally, decides whether or not to renege on debt owned by the country, and apportions total endowment plus net foreign capital inflow among residents. This government regulation is introduced to help the welfare comparison between my model and Jeske's setup. Because of government's intervention in domestic allocation, one can aggregate each country into a representative agent with the following consumption pattern

$$\begin{aligned} c_t^1 &= 1 + x^c, c_{t+1}^1 = 1 - x^c, \text{ for country 1;} \\ c_t^2 &= 1 - x^c, c_{t+1}^2 = 1 + x^c, \text{ for country 2,} \end{aligned}$$

where x^c is the minimum deviation satisfying country 1's international participation constraint at time t

$$x^c = \min_{z \geqslant 0} \left\{ z : \log(1+z) + \beta \log(1-z) \geq \log(1+y) + \beta \log(1-y) \right\}.$$

For the problem to be interesting, I am looking for the situation in which some (not full) risk sharing can be supported across centralized economy. This is only possible given two restrictions on endowment are satisfied. The first one is

$$\beta < -\frac{\log(1+y)}{\log(1-y)}, \tag{19}$$

otherwise countries can fully smooth consumption, $x^c = 0$ as in Figure 3. The second one is

$$\beta > \frac{1-y}{1+y},$$

or equivalently

$$y > \frac{1-\beta}{1+\beta}, \tag{20}$$

otherwise autarky is the highest utility one can achieve and there is no trade in equilibrium, $x^c = y$ as in Figure 4.

If (19) and (20) are both satisfied, some international risk sharing can be supported across border. As can be seen in Figure 5, the risk sharing level in centralized economy turns out to be better than the private international borrowing with full domestic commitment in section 4.1,

$$0 < x^c < \frac{1-\beta}{1+\beta} < y.$$

Given that $x^J = \frac{1-\beta}{1+\beta}$ or y, I know the following relationship

$$0 < x^c < x^J \leqslant y.$$

4.3 Private Borrowing with Limited Commitment Domestically

Next I remove the assumption of perfect enforcement on domestic debts. Therefore, type A and type B agents in the same country might have different consumption allocations. As a result, the allocations in trade equilibrium alternate not only between different types within a country but also across countries. By symmetry, I have

$$\begin{aligned}
c^1_{A,t} &= 1 + x + \varepsilon^p, c^1_{A,t+1} = 1 - x - \varepsilon^p, \text{ for type A in country 1;} \\
c^1_{B,t} &= 1 + x - \varepsilon^p, c^1_{B,t+1} = 1 - x + \varepsilon^p, \text{ for type B in country 1;} \\
c^2_{A,t} &= 1 - x - \varepsilon^p, c^2_{A,t+1} = 1 + x + \varepsilon^p, \text{ for type A in country 2;} \\
c^1_{B,t} &= 1 - x + \varepsilon^p, c^1_{B,t+1} = 1 + x - \varepsilon^p, \text{ for type B in country 2.}
\end{aligned}$$

The consumption structure implies two possibilities: $\varepsilon^p > 0$ and $\varepsilon^p = 0$.

$\varepsilon^p > 0$ means different types within the same country can not match consumption even though international financial markets are accessible, thus Type B agents with a less volatile consumption deviation must be domestically participation constrained if the country is internationally participation constrained in trade equilibrium. x and ε^p are jointly determined by three conditions. The first two conditions are derived from the fact that type A and B in country 1 are internationally participation constrained at the same time. For both types in country 1, the present value of all future foreign net payments from t on being zero when discounted by domestic bond prices.

$$\begin{aligned} \frac{(x+\varepsilon^p)-(y+\varepsilon)+q\left[(y+\varepsilon)-(x+\varepsilon^p)\right]}{1-pq} &= 0 \text{ for type A;} \\ \frac{(x-\varepsilon^p)-(y-\varepsilon)+q\left[(y-\varepsilon)-(x-\varepsilon^p)\right]}{1-pq} &= 0 \text{ for type B.} \end{aligned} \tag{21}$$

The third condition says that if type B residents are participation constrained in domestic and international financial markets at the same time, then their continuation values in trade equilibrium and resident's autarky must be equalized at some even numbered period.

$$\log(1+x-\varepsilon^p)+\beta\log(1-x+\varepsilon^p)=\log(1+y-\varepsilon)+\beta\log(1-y+\varepsilon). \tag{22}$$

At even numbered period, the international bond price q equals to the highest MRS across the world,

$$q=\beta\frac{1+x+\varepsilon^p}{1-x-\varepsilon^p}, \tag{23}$$

the domestic bond price in country 1 is q,[6] and the domestic bond price in country 2 equals to the lowest MRS within 2,

$$p=\beta\frac{1-x-\varepsilon^p}{1+x+\varepsilon^p}.$$

As a result, the price sequences for domestically traded bonds are the same as (18) in section 4.1. There is a unique solution to (21) and (22). The optimal solution requires $q=1$, which further implies $x+\varepsilon^p=\frac{1-\beta}{1+\beta}$ using (23) and ε^p is determined implicitly by equation (22). Given that y is large relative to ε, as well as $\varepsilon^p>0$, I know $x^c<x-\varepsilon^p<\frac{1-\beta}{1+\beta}$ from Figure 6.

On the other hand $\varepsilon^p=0$ indicates international financial markets help different types within the same country to match consumption every period. Therefore, no one is constrained domestically and the MRS is equalized in that country. x is then determined by (21). Use (23) again and get the optimal solution $x=\frac{1-\beta}{1+\beta}$, $\varepsilon^p=0$.

4.4 A Comparison of Risk Sharing

Now I can compare the extent of self-enforcing deviation from consumption smoothing in the above three scenarios. In section 4.1 (scenario 1), both types have the same deviation $x^J=\frac{1-\beta}{1+\beta}$ or y. In section 4.2

[6] In contrast to the formal model, the MRS is not equalized within country 1 at period 0 in this simple example.

$$q=\beta\frac{1+x+\varepsilon^p}{1-x-\varepsilon^p}>\beta\frac{1+x-\varepsilon^p}{1-x+\varepsilon^p}, \text{ if } \varepsilon^p>0.$$

The reason is that residents will renege on domestic risk sharing contract as soon as they know their types.

(scenario 2), both types have the same deviation x^c. In section 4.3 (scenario 3), either both types have the same deviation $x = \frac{1-\beta}{1+\beta}$ or type A has deviation $x + \varepsilon^p = \frac{1-\beta}{1+\beta}$ while type B has deviation $x - \varepsilon^p < \frac{1-\beta}{1+\beta}$. To summarize,

$$
\begin{aligned}
0 &< x^c < x + \varepsilon^p \leqslant x^J \leqslant y, \text{ for type A;} \\
0 &< x^c < x - \varepsilon^p \leqslant x^J \leqslant y, \text{ for type B.}
\end{aligned}
$$

Adding domestic enforcement problem rules out the autarky solution, $x^J = y$, in scenario 1. For both types, risk sharing levels are weakly improved. If type A and B cannot match consumption, then B's international risk sharing level is strictly improves by scenario 3, though the increment is smaller than the one by centralization in scenario 2.

In conclusion, given the endowment structure with y relatively larger than ε, I find the domain $\left(\frac{1-y}{1+y}, -\frac{\log(1+y)}{\log(1-y)}\right)$ for discount factor β where scenario 2 is always welfare superior than 1 or 3 and 3 is strictly better than 1. This is the case I studied in section 3. What happens if the discount factor is high enough, $\beta \in \left[-\frac{\log(1+y)}{\log(1-y)}, 1\right]$, then one will observe complete international risk sharing in scenario 2 which can never be an equilibrium allocation in scenario 1 and 3. If agents are extremely impatient, $\beta \in \left[0, \frac{1-y}{1+y}\right]$, then both scenario 1 and 2 lead to the same equilibrium of autarky with complete domestic risk sharing $x^c = x^J = y$. Scenario 3 would result in autarky with no domestic risk sharing, $x = y$ and $\varepsilon^p = \varepsilon$, which is worse than scenario 1 and 2.

5 Conclusion

I have developed an open economy model with heterogeneous residents in each country sharing risk across and within border. Risk of repudiation is pervasive in all debt contracts including both international and domestic ones. The model and analysis is built on Jeske's private international borrowing model except relaxing his assumption that domestic debt contracts are perfectly enforceable. In this paper, besides the difference in price, international debt contracts also differ from domestic ones in punishing strategy defaulters. International debt defaulters are excluded only from international financial markets while domestic debt defaulters are denied from all financial markets.

The main contribution of this paper is to show that an economy with pervasive enforcement problem does better in international capital markets than an economy with enforcement problem on foreign debts alone. The reason is that penalty on international debt repudiation in this model is at least as harsh as in Jeske's model and strictly harsher for some types with smoother endowment overtime. Thus, more international borrowing and higher welfare can be supported for those lucky types of residents. Intuitively, capital control internalizes the externality of individual's default decisions while pervasive commitment problem mitigates the negative externality. The domestic bond pricing rules change in respond to the domestic credit crisis. In my setup, domestic interest rate equals to the reciprocal of the lowest MRS in countries that are participation constrained internationally. This overthrows the well established argument that interest rate should be the lowest to induce repayment in an environment without legal enforcement on financial contracts. This result is due to the crucial ingredient of my model: in equilibrium domestic debt default can never happen without international debt default. Although there is commitment problem within border, domestic debt repayment is secured by preventing international debt default. For countries whose international participation constraint is superfluous, their domestic interest rate equals to prevailing international interest rate as in previous literature.

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A Appendix

A.1 Proof of Proposition 1

Imagine an Arrow-Debreu setup in which there exists a domestic financial market at period 0 for all kinds of bonds that mature at any future period. Denote $P^m(\theta^r) = P^m(\theta^{r-1})p^m(\theta^{r-1}, \theta_r) = \prod_{s=0}^{r} p^m(\theta^s)$ the forward price for a r-period matured domestic contingent bond at period 0, where $r \in (0, \infty)$ and $P^m(\theta^0) = p^m(\theta^0) = 1$. The proof proceeds in three steps:

Step 1: Redefine the resident's international autarky problem (RIA) started at θ^t as follow

$$V_n^m(\theta^t, b_n^m(\theta^t)) \equiv \max_{\{c_n^m(\theta^r)\}_{r\in[t,\infty)}} \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t)U(c_n^m(\theta^r)), \tag{RIA$^{\mathrm{F}}$}$$

subject to the summation of all future budget constraints after history θ^t discounted to period 0

$$\sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r)c_n^m(\theta^r) = \sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r)e_n^m(\theta^r) + P^m(\theta^t)b_n^m(\theta^t),$$

and the participation constraint in domestic financial markets

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r)U(c_n^m(\theta^r)) \geqslant A_n^m(\theta^r),$$

with

$$b_n^m(\theta^t) \text{ and } P^m(\theta^r) \text{ given,}$$

for all histories θ^t with $r \in [t, \infty)$.

Lemma 1 *The redefined resident's international autarky problem (RIA^F) has a unique maximum solution.*

Proof. *Prove by contradiction. Suppose there are two different optimal solutions to problem (RIA^F): $\left\{c_{n,1}^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ and $\left\{c_{n,2}^{m,D}(\theta^r)\right\}_{s\in[t,\infty)}$. Create another consumption allocation $\left\{c_{n,3}^{m,D}(\theta^r)\right\}_{r\in[r,\infty)}$ as a linear combination of the above two, i.e.,*

$$c_{n,3}^{m,D}(\theta^r) = \lambda c_{n,1}^{m,D}(\theta^r) + (1-\lambda)c_{n,2}^{m,D}(\theta^r)$$

for any $\lambda \in (0,1)$ and history θ^r with $r \in [t, \infty)$. Thus, $\left\{c_{n,3}^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ is both affordable and individual rational since strictly concave utility function implies

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r)U(c_{n,3}^{m,D}(\theta^s)) > \sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r)U(c_{n,1(2)}^{m,D}(\theta^s)) \geqslant A_n^m(\theta^r),$$

for all histories θ^r with $r \in [t, \infty)$. But $\left\{c_{n,3}^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ makes one strictly better off at the first place if one sets $r = t$, which contradicts with the assumption that $\left\{c_{n,1}^{m,D}(\theta^r)\right\}_{r\in[t,\infty)}$ and $\left\{c_{n,2}^{m,D}(\theta^r)\right\}_{t\in[t,\infty)}$ are the optimal solutions. ■

STEP 2: For $F_n^m(\theta^t) \in \mathbb{R}$, define another optimization problem (RP^F) as an augmented version of (RIA^F)

$$W_n^{m,F}(\theta^t, b_n^m(\theta^t), F(\theta^t)) \equiv \max_{\{c_n^m(\theta^r)\}_{r\in[t,\infty)}} \sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)), \tag{RP^F}$$

subject to the summation of all future budget constraints after history θ^t discounted to period 0

$$\sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) c_n^m(\theta^r) = \sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) e_n^m(\theta^r) + P^m(\theta^t) b_n^m(\theta^t) + F_n^m(\theta^t),$$

and the participation constraint in domestic financial markets

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^m(\theta^s)) \geqslant A_n^m(\theta^r),$$

with

$$b_n^m(\theta^t), F_n^m(\theta^t) \text{ and } P^m(\theta^r) \text{ given,}$$

for all histories θ^r with $r \in [t, \infty)$.

Consider all histories θ^t and initial bond holdings $b_n^m(\theta^t)$ with $t \in [0, \infty)$. By definition, the value function of problem (RIA^F) equals to the value function of problem (RP^F) given $F_n^m(\theta^t) = 0$.

$$V_n^m(\theta^t, b_n^m(\theta^t)) = W_n^{m,F}(\theta^t, b_n^m(\theta^t), 0),$$

If one defines

$$F_n^m(\theta^t) = \sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right],$$

then the continuation value of the original resident's problem (RP) at θ^t equals to the value function of the newly defined problem (RP^F).

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) = W_n^{m,F}(\theta^t, b_n^m(\theta^t), F_n^m(\theta^t)).$$

But this equality is true only if the international participation constraints (4) in (RP) are also satisfied in (RP^F).

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) \geqslant V_n^m(\theta^t, b_n^m(\theta^t)).$$

Substitute both sides of the constraints for value functions in (RP$^{\mathrm{F}}$) to get

$$W_n^{m,F}(\theta^t, b_n^m(\theta^t), F_n^m(\theta^t)) \geqslant W_n^{m,F}(\theta^t, b_n^m(\theta^t), 0).$$

Since $W_n^{m,F}(\theta^t, b_n^m(\theta^t), \cdot)$ is strictly increasing in $F_n^m(\theta^t)$, the above inequality further implies

$$F_n^m(\theta^t) \geqslant 0.$$

What is more, if (4) holds with equality, then $F_n^m(\theta^r) = 0$. By now, the reasoning suffices to prove the following lemma 2

Lemma 2 *For all m, n and θ^t with $t \in [0, \infty)$, the international participation constraint (4) at θ^t implies*

$$\sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right] \geqslant 0.$$

Moreover, if (4) holds with equality at θ^t, then

$$\sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right] = 0.$$

Step 3: Now I am ready to prove proposition 1. For some n, m and θ^t with $t \in [0, \infty)$, $\mu_n^m(\theta^t) > 0$ implies that the international participation constraint of type n residents in country m holds with equality at θ^t. Lemma 2 concludes $F_n^m(\theta^t) = 0$. By definition, the consumption allocation $\left\{c_n^{m,D}(\theta^r)\right\}_{r \in [t,\infty)}$ solves problem (RIA) started at θ^t and the other one $\{c_n^m(\theta^r)\}_{r \in [0,\infty)}$ solves problem (RP) at period 0. That is to say, $\left\{c_n^{m,D}(\theta^r)\right\}_{r \in [t,\infty)}$ solves $W_n^{m,F}(\theta^t, b_n^m(\theta^t), 0)$ and $\{c_n^m(\theta^r)\}_{r \in [t,\infty)} \subseteq \{c_n^m(\theta^r)\}_{r \in [0,\infty)}$ solves $W_n^{m,F}(\theta^t, b_n^m(\theta^t), F_n^m(\theta^t))$. Since $F_n^m(\theta^t) = 0$, these two optimal allocations both solve $W_n^{m,F}(\theta^t, b_n^m(\theta^t), 0)$, i.e., they both solve the resident's international autarky problem (RIA) and its redefined problem (RIA$^{\mathrm{F}}$) with value function $V_n^m(\theta^t, b_n^m(\theta^t))$. Finally, by Lemma 1, the optimization problem (RIA$^{\mathrm{F}}$) having unique solution proves that $c_n^{m,D}(\theta^r)$ and $c_n^m(\theta^r)$ are identical at all histories θ^r with $r \in [t, \infty)$.

A.2 Proof of Proposition 2

In equilibrium, bond prices are determined by (13) and (14) for all histories (θ^t, θ_{t+1}) with $t \in [0, \infty)$.

$$\begin{cases} p(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + A_2 - \left(1 + \nu(\theta^t, \theta_{t+1})\right) A_1}{1 + A_3}; \\ q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + A_2}{1 + A_3}, \end{cases}$$

where

$$
\begin{aligned}
A_1 &= \mu(\theta^t,\theta_{t+1})\beta^{-t-1}\frac{U'(c^D(\theta^t,\theta_{t+1}))}{U'(c(\theta^t,\theta_{t+1}))}\frac{1}{\pi(\theta^t,\theta_{t+1})};\\
A_2 &= \sum_{s=0}^{t+1}\sum_{\theta^t,\theta_{t+1}|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t,\theta_{t+1}|\theta^s)}{\pi(\theta^t,\theta_{t+1})};\\
A_3 &= \sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}.
\end{aligned}
$$

The proof shows in three steps how the interaction between international and domestic bond prices makes different types reach their upper limits of international borrowing at the same time.

Step 1: Consider the international bond pricing rule (14), the Lagrange multipliers imposed on the international participation constraints must be non-negative.

$$\mu(\theta^t,\theta_{t+1})\geqslant 0\Rightarrow$$

$$A_2 = A_3 + \mu(\theta^t,\theta_{t+1})\beta^{-t-1}\frac{1}{\pi(\theta^t,\theta_{t+1})}\geqslant A_3,$$

and

$$A_2 = A_3 \text{ if } \mu(\theta^t,\theta_{t+1}) = 0.$$

Therefore, the international bond price represents the highest MRS among all types in all countries.

$$q(\theta^t,\theta_{t+1})\geqslant\beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t),\ \text{with equality iif } \mu_n^m(\theta^t,\theta_{t+1})=0. \tag{A.1}$$

Step 2: Consider the domestic bond pricing rule (13) in any country. Substitute all the A's and rearrange

$$
\begin{aligned}
p(\theta^t,\theta_{t+1}) &= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\times\\
&\left\{1+\frac{(1-\frac{U'(c^D(\theta^t,\theta_{t+1}))}{U'(c(\theta^t,\theta_{t+1}))})\frac{\mu(\theta^t,\theta_{t+1})\beta^{-t-1}}{\pi(\theta^t,\theta_{t+1})}-\frac{U'(c^D(\theta^t,\theta_{t+1}))}{U'(c(\theta^t,\theta_{t+1}))}\frac{\nu(\theta^t,\theta_{t+1})\mu(\theta^t,\theta_{t+1})\beta^{-t-1}}{\pi((\theta^t,\theta_{t+1}))}}{1+\sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}\right\}.
\end{aligned}
$$

For some type n in m, if one observes $\mu_n^m(\theta^t,\theta_{t+1})=0$, then the domestic bond price in m is determined by n's MRS.

$$p^m(\theta^t,\theta_{t+1}) = \beta\frac{U'(c_n^m(\theta^t,\theta_{t+1}))}{U'(c_n^m(\theta^t))}\pi(\theta_{t+1}|\theta^t).$$

If one observes $\mu_n^m(\theta^t, \theta_{t+1}) > 0$ instead, then $c_n^{m,D}(\theta^t, \theta_{t+1}) = c_n^m(\theta^t, \theta_{t+1})$ by proposition 1. The relationship between domestic bond price in m and n's MRS is

$$p^m(\theta^t, \theta_{t+1}) = \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \left[1 - \frac{\frac{\nu_n^m(\theta^t,\theta_{t+1})\mu_n^m(\theta^t,\theta_{t+1})\beta^{-t-1}}{\pi((\theta^t,\theta_{t+1}))}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu_n^m(\theta^s)\beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}} \right]. \tag{A.2}$$

The Lagrange multipliers imposed on domestic participation constraints in problem (RIA) must be non-negative.

$$v_n^m(\theta^t, \theta_{t+1}) \geqslant 0 \Rightarrow$$
$$p^m(\theta^t, \theta_{t+1}) \leqslant \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t), \text{ with equality if } v_n^m(\theta^t, \theta_{t+1}) = 0. \tag{A.3}$$

Step 3: If $\mu_n^m(\theta^t, \theta_{t+1}) > 0$ for some type n residents in country m, then (A.1) and (A.3) together ensure that the international bond price is strictly greater than the domestic one in m.

$$q(\theta^t, \theta_{t+1}) > \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \geqslant p^m(\theta^t, \theta_{t+1}) \Rightarrow$$
$$q(\theta^t, \theta_{t+1}) > p^m(\theta^t, \theta_{t+1}).$$

This strict inequality, the other way around, implies that multipliers $\mu_n^m(\theta^t, \theta_{t+1})$ are positive for all types in m as well.

$$q(\theta^t, \theta_{t+1}) > p^m(\theta^t, \theta_{t+1}) \Rightarrow$$
$$\beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + A_{n,2}^m}{1 + A_{n,3}^m} > \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + A_{n,2}^m - \left(1 + \nu_n^m(\theta^t, \theta_{t+1})\right) A_{n,1}^m}{1 + A_{n,3}^m} \Rightarrow$$
$$\left(1 + \nu_n^m(\theta^t, \theta_{t+1})\right) \mu_n^m(\theta^t, \theta_{t+1}) \beta^{-r-1} \frac{1}{\pi(\theta^t, \theta_{t+1}|\theta^t)} > 0 \Rightarrow$$
$$\left(1 + \nu_n^m(\theta^t, \theta_{t+1})\right) \mu_n^m(\theta^t, \theta_{t+1}) > 0.$$

Since $\nu_n^m(\theta^t, \theta_{t+1}) \geqslant 0$ for all n in m, I have

$$\mu_n^m(\theta^t, \theta_{t+1}) > 0, \text{ for all } n, m.$$

A.3 Proof of Proposition 3

Given $\mu_n^m(\theta^t) > 0$ for some n, m at θ^t with $t \in [0, \infty)$, the international participation constraint in resident's problem is binding at θ^t.

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{\theta^r|\theta^t} \pi(\theta^r|\theta^t) U(c_n^m(\theta^r)) = V_n^m(\theta^t, b_n^m(\theta^t)),$$

where $c_n^m(\theta^r)$ and $b_n^m(\theta^t)$ are the optimal consumption at θ^r and the optimal domestic bond holdings at θ^t in problem (RP). The value function $V_n^m(\theta^t, b_n^m(\theta^t))$ of problem (RIA) started at θ^t affects the optimal consumption allocation after θ^t in problem (RP) through this binding constraint. In addition, $\nu_n^m(\theta^r) > 0$ for the same n and m above with $r \in [t, \infty)$ implies the domestic participation constraint in resident's international autarky problem is bind at θ^r.

$$\sum_{s=r}^{\infty} \beta^{s-r} \sum_{\theta^s|\theta^r} \pi(\theta^s|\theta^r) U(c_n^{m,D}(\theta^s, \theta^t, b_n^m(\theta^t))) = A_n^m(\theta^r),$$

where $c_n^{m,D}(\theta^s, \theta^t, b_n^m(\theta^t))$ denotes the optimal consumption at θ^s in problem (RIA) started at θ^t with initial domestic bond holdings $b_n^m(\theta^t)$. This equation implicitly defines $b_n^m(\theta^t) = \overline{B}_n^m(\theta^t)$.

A.4 Proof of Proposition 4

By lemma 2 and proposition 3, the international participation constraint (4) at θ^t with $t \in [0, \infty)$ implies the following set of inequality and equality constraints

$$\begin{cases} \sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right] \geqslant 0; \\ b_n^m(\theta^t, \theta_{t+1}) = \overline{B}_n^m(\theta^t, \theta_{t+1}) \text{ if } \mu_n^m(\theta^t, \theta_{t+1}) > 0 \text{ and} \\ \nu_n^m(\theta^r, \theta_{r+1}) > 0 \text{ for some } \theta^r \text{ with } r \in [t, \infty). \end{cases} \tag{A.4}$$

The proof proceeds in the following three steps: define, solve and compare.

STEP 1: For all histories θ^t with $t \in [0, \infty)$, replacing (4) in problem (RP) with weaker constraints (A.4) creates an alternative (convex) resident's problem (RP$^{\mathrm{a}}$).

$$\max_{\{c_n^m(\theta^t), b_n^m(\theta^t, \theta_{t+1}), f_n^m(\theta^t, \theta_{t+1})\}_{t\in[0,\infty)}} \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \pi(\theta^t) U(c_n^m(\theta^t)), \tag{RP$^{\mathrm{a}}$}$$

subject to the budget constraint

$$\begin{aligned} & e_n^m(\theta^t) + b_n^m(\theta^t) + f_n^m(\theta^t) \\ \geqslant \;& c_n^m(\theta^t) + \sum_{\theta_{t+1}} p^m(\theta^t, \theta_{t+1}) b_n^m(\theta^t, \theta_{t+1}) + \sum_{\theta_{t+1}} q(\theta^t, \theta_{t+1}) f_n^m(\theta^t, \theta_{t+1}), \end{aligned} \tag{A.5}$$

the weaker version of participation constraint in international financial market

$$\sum_{r=t}^{\infty} \sum_{\theta^r|\theta^t} P^m(\theta^r) \left[f_n^m(\theta^r) - \sum_{\theta_{r+1}} q(\theta^r, \theta_{r+1}) f_n^m(\theta^r, \theta_{r+1}) \right] \geqslant 0 \tag{A.6}$$

and

$$
\begin{aligned}
b_n^m(\theta^t, \theta_{t+1}) &= \overline{B}_n^m(\theta^t, \theta_{t+1}) \text{ if } \mu_n^m(\theta^t, \theta_{t+1}) > 0 \text{ and} \\
\nu_n^m(\theta^r, \theta_{r+1}) &> 0 \text{ for some } \theta^r \text{ with } r \in [t, \infty), \qquad \text{(A.7)}
\end{aligned}
$$

the no-Ponzi conditions

$$
b_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{B}, f_n^m(\theta^t, \theta_{t+1}) \geqslant -\overline{F},
$$

with initial bond holdings

$$
b_n^m(\theta^0), f_n^m(\theta^0) \text{ given}
$$

and bond price sequences

$$
p^m(\theta^t, \theta_{t+1}), q(\theta^t, \theta_{t+1}) \text{ given,}
$$

for all histories θ^t and all states $\left(\theta^t, \theta_{t+1}\right)$ with $t \in [0, \infty)$. Let $\kappa\left(\theta^t\right), \mu_f\left(\theta^t\right)$ and $\mu_b\left(\theta^t\right)$ be the Lagrange multiplier on budget constraint (A.5), non-negative foreign capital inflow condition (A.6), domestic bond holding restriction (A.7), respectively.

First order conditions are: with respect to $c(\theta^t)$,

$$
\kappa(\theta^t) = \beta^t \pi(\theta^t) U'(c(\theta^t));
$$

with respect to $b(\theta^t, \theta_{t+1})$,

$$
p(\theta^t, \theta_{t+1})\kappa(\theta^t) = \begin{cases} \kappa(\theta^t, \theta_{t+1}) - \mu_b(\theta^t, \theta_{t+1}) \text{ if } \mu(\theta^t, \theta_{t+1}) > 0, \nu(\theta^r, \theta_{r+1}) > 0; \\ \kappa(\theta^t, \theta_{t+1}) \text{ otherwise;} \end{cases}
$$

and with respect to $f(\theta^t, \theta_{t+1})$,

$$
q(\theta^t, \theta_{t+1})\kappa(\theta^t) = \kappa(\theta^t, \theta_{t+1}) + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \mu_f(\theta^s) P(\theta^t, \theta_{t+1}) - \sum_{s=0}^{r} \sum_{\theta^t|\theta^s} \mu_f(\theta^s) P(\theta^t) q(\theta^t, \theta_{t+1}).
$$

Use all first order conditions to generate domestic and international bond pricing rules as follow

$$
p(\theta^t, \theta_{t+1}) = \begin{cases} \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t)[1 - \frac{\mu_b(\theta^t, \theta_{t+1})}{\kappa(\theta^t, \theta_{t+1})}] \text{ if } \mu(\theta^t, \theta_{t+1}) > 0, \nu(\theta^r, \theta_{r+1}) > 0; \\ \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ otherwise,} \end{cases} \qquad \text{(A.8)}
$$

and

$$
q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \frac{\mu_f(\theta^s) P(\theta^t, \theta_{t+1})}{\kappa(\theta^t, \theta_{t+1})}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \frac{\mu_f(\theta^s) P(\theta^t)}{\kappa(\theta^t)}}. \qquad \text{(A.9)}
$$

STEP 2: Consider any one country in the world and all histories (θ^t, θ_{t+1}) with $\mu(\theta^t, \theta_{t+1}) > 0$ since non-convexity only becomes problematic when the international participation constraints are binding.

For types of residents with $\mu(\theta^t, \theta_{t+1}) > 0$ and $\nu(\theta^r, \theta_{r+1}) = 0$ at all θ^r with $r \in [t, \infty)$, or n_A types as in the model, the domestic bond pricing rule (A.8) degenerates into

$$p(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) = \frac{\kappa(\theta^t, \theta_{t+1})}{\kappa(\theta^t)}. \tag{A.10}$$

The Arrow-Debreu price of a $(t+1)$-period matured domestic bond at period 0 can be, therefore, written as

$$P(\theta^t, \theta_{r+1}) = P(\theta^t) p(\theta^t, \theta_{t+1}) = P(\theta^t) \frac{\kappa(\theta^t, \theta_{t+1})}{\kappa(\theta^t)}.$$

Plug it into the international bond pricing rule (A.9) to get

$$q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \frac{\mu_f(\theta^s) P(\theta^t)}{\kappa(\theta^t)}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \frac{\mu_f(\theta^s) P(\theta^t)}{\kappa(\theta^t)}}.$$

To rescale the Lagrange multipliers $\mu_f(\theta^s)$, define

$$\mu'_f(\theta^s) = \mu_f(\theta^s) \beta^s \frac{P(\theta^t) \pi(\theta^t, \theta_{t+1})}{\kappa(\theta^t) \pi(\theta^t, \theta_{t+1}|\theta^s)}, \tag{A.11}$$

where $\kappa(\theta^t) > 0$ since the budget constraint always binds in equilibrium and notice

$$\frac{\pi(\theta^t, \theta_{t+1}|\theta^s)}{\pi(\theta^t, \theta_{t+1})} = \frac{\pi(\theta_{t+1}|\theta^t)\pi(\theta^t|\theta^s)}{\pi(\theta_{t+1}|\theta^t)\pi(\theta^t)} = \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}.$$

Using the definition of (A.11) to replace $\mu_f(\theta^s)$ yileds

$$q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) \frac{1 + \sum_{s=0}^{t+1} \sum_{\theta^t, \theta_{t+1}|\theta^s} \mu'_f(\theta^s) \beta^{-s} \frac{\pi(\theta^t, \theta_{t+1}|\theta^s)}{\pi(\theta^t, \theta_{t+1})}}{1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu'_f(\theta^s) \beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}. \tag{A.12}$$

For types of residents in the same country with $\mu(\theta^t, \theta_{t+1}) > 0$ and $\nu(\theta^r, \theta_{r+1}) > 0$ at some θ^r with $r \in [t, \infty)$, or n_B types as in the model, they must confront with the same international and domestic bond prices as determined above. To rescale the Lagrange multipliers $\mu_b(\theta^t, \theta_{t+1})$, define

$$\mu'_b(\theta^t, \theta_{t+1}) \equiv \frac{\mu_b(\theta^t, \theta_{t+1})}{U'(c(\theta^t, \theta_{t+1}))\mu'_f(\theta^t, \theta_{t+1})\beta^{-t}} \left[1 + \sum_{s=0}^{t} \sum_{\theta^t|\theta^s} \mu'_f(\theta^s) \beta^{-s} \frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)} \right], \tag{A.13}$$

where $\mu'_f(\theta^t, \theta_{t+1}) > 0$ since the non-negative foreign capital inflow condition always binds in equilibrium. Then the domestic bond pricing rule (A.8) for types n_B suggests

$$p(\theta^t, \theta_{t+1}) = \beta \frac{U'(c(\theta^t, \theta_{t+1}))}{U'(c(\theta^t))} \pi(\theta_{t+1}|\theta^t) [1 - \frac{\mu_b(\theta^t, \theta_{t+1})}{\kappa(\theta^t, \theta_{t+1})}].$$

Using the definition of (A.13) to replace $\mu_b(\theta^r, \theta_{r+1})$ generates

$$\begin{aligned} p(\theta^t,\theta_{t+1}) &= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t) \times \\ & \left[1-\frac{\mu_b'(\theta^t,\theta_{t+1})\mu_f'(\theta^t,\theta_{t+1})\beta^{-t-1}\frac{1}{\pi(\theta^t,\theta_{t+1})}}{1+\sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu_f'(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}\right] \\ \text{if } \mu(\theta^t,\theta_{t+1}) &> 0, \nu(\theta^r,\theta_{r+1})>0. \end{aligned} \tag{A.14}$$

Step 3: Now recall bond prices (14) and (13) in trade equilibrium with all the A's substituted away.

$$\begin{aligned} q(\theta^t,\theta_{t+1}) &= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1+A_2}{1+A_3} \\ &= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1+\sum_{s=0}^{t+1}\sum_{\theta^t,\theta_{t+1}|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t,\theta_{t+1}|\theta^s)}{\pi(\theta^t,\theta_{t+1})}}{1+\sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}, \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} p(\theta^t,\theta_{t+1}) &= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\frac{1+A_2-\left(1+\nu(\theta^t,\theta_{t+1})\right)A_1}{1+A_3} \\ &= \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\times \\ & \left[\frac{1+\sum_{s=0}^{t+1}\sum_{\theta^t,\theta_{t+1}|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t,\theta_{t+1}|\theta^s)}{\pi(\theta^t,\theta_{t+1})}-\frac{\left(1+\nu(\theta^t,\theta_{t+1})\right)\mu(\theta^t,\theta_{t+1})}{\pi(\theta^t,\theta_{t+1})}\beta^{-t-1}\frac{U'(c^D(\theta^t,\theta_{t+1}))}{U'(c(\theta^t,\theta_{t+1}))}}{1+\sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}\right] \\ &= \begin{cases} \begin{array}{c}\beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t)\times \\ \left[1-\frac{\nu(\theta^t,\theta_{r+1})\mu(\theta^t,\theta_{t+1})\beta^{-t-1}\frac{1}{\pi(\theta^t,\theta_{t+1})}}{1+\sum_{s=0}^{t}\sum_{\theta^t|\theta^s}\mu(\theta^s)\beta^{-s}\frac{\pi(\theta^t|\theta^s)}{\pi(\theta^t)}}\right]\end{array} & \text{if } \mu(\theta^t,\theta_{t+1})>0, \nu(\theta^r,\theta_{r+1})>0; \\ \beta\frac{U'(c(\theta^t,\theta_{t+1}))}{U'(c(\theta^t))}\pi(\theta_{t+1}|\theta^t) \text{ otherwise.} \end{cases} \end{aligned} \tag{A.16}$$

Notice that the domestic bond pricing rules (A.14) plus (A.10) and international bond pricing rule (A.12) in the alternative (convex) resident's problem (RP$^{\mathrm{a}}$) are, respectively, identical to the correspnding ones (A.16) and (A.15) from the original (non-convex) resident's problem (RP). Both maximization problems have identical first order conditions, hence same optimal solution. I have defined an alternative maximization problem with the same objective function and a convex constraint set that is a super set of the original (non-convex in general) constraint set. The optimal solution is a global maximum for problem (RP$^{\mathrm{a}}$) because of convexity. Therefore, it must be the global maximum for the original problem (RP), which has the same objective function and first order conditions as in problem (RP$^{\mathrm{a}}$) except for a smaller constraint set. This proves the sufficiency of first order conditions for a global max in the original problem (RP).

A.5 Proof of Proposition 5

As the proposition itself, the proof has been divided into three parts:

Part 1: For the analysis below, consider any history (θ^t, θ_{t+1}) with $t \in [0, \infty)$. The first part of proposition 5 can be easily read off from inequality (A.1) in the proof of proposition 2.

$$q(\theta^t, \theta_{t+1}) \geqslant \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t), \text{ for all } n, m. \tag{A.17}$$

Part 2: Without loss of generality, suppose residents with the highest MRS across the world live in some countries m as some types n, where m and n are subsets of $\{1, 2, ..., M\}$ and $\{1, 2, ..., N\}$, respectively. Then their MRS determines the international bond price.

$$q(\theta^t, \theta_{t+1}) = \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ and } \mu_n^m(\theta^t, \theta_{t+1}) = 0.$$

By proposition 2, $\mu_n^m(\theta^t, \theta_{t+1}) = 0$ implies that $\mu_{n^-}^m(\theta^t, \theta_{t+1}) = 0$ for all other types in m, where $n^- = \{1, 2, ..., N\} \setminus n$. Equation (A.2) in the proof of proposition 2 tells us

$$p^m(\theta^t, \theta_{t+1}) = \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ for all } n \text{ in } m.$$

The MRS is equalized within country m. Therefore,

$$p^m(\theta^t, \theta_{t+1}) = q(\theta^t, \theta_{t+1}) \text{ for all } m. \tag{A.18}$$

All the other countries $m^- = \{1, 2, ..., M\} \setminus m$ which are internationally participation constrained as a whole have

$$q(\theta^t, \theta_{t+1}) < \beta \frac{U'(c_n^{m^-}(\theta^t, \theta_{t+1}))}{U'(c_n^{m^-}(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ and } \mu_n^{m^-}(\theta^t, \theta_{t+1}) > 0, \text{ for all } n \text{ in } m^-.$$

Inequality (A.3) in the proof of proposition 2 yields

$$p^{m^-}(\theta^t, \theta_{t+1}) \leqslant \beta \frac{U'(c_n^{m^-}(\theta^t, \theta_{t+1}))}{U'(c_n^{m^-}(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ for all } n \text{ in } m^-. \tag{A.19}$$

For all countries, combining equation (A.18) and inequality (A.19) together to get

$$p^m(\theta^t, \theta_{t+1}) = \begin{cases} \leqslant \beta \frac{U'(c_n^m(\theta^t, \theta_{t+1}))}{U'(c_n^m(\theta^t))} \pi(\theta_{t+1}|\theta^t) \text{ if } \mu_n^m(\theta^t, \theta_{t+1}) > 0; \\ = q(\theta^t, \theta_{t+1}) \text{ if } \mu_n^m(\theta^t, \theta_{t+1}) = 0, \end{cases}$$

or more general like the second part of proposition 5

$$p^m(\theta^t,\theta_{t+1}) \leqslant \beta \frac{U'(c_n^m(\theta^t,\theta_{t+1}))}{U'(c_n^m(\theta^t))}\pi(\theta_{t+1}|\theta^t), \text{ for all } n, m. \tag{A.20}$$

PART 3: Connecting inequality (A.17) and (A.20) in one direction,

$$p^m(\theta^t,\theta_{t+1}) \leqslant \beta \frac{U'(c_n^m(\theta^t,\theta_{t+1}))}{U'(c_n^m(\theta^t))}\pi(\theta_{t+1}|\theta^t) \leqslant q(\theta^t,\theta_{t+1}),$$

or more directly,

$$p^m(\theta^r,\theta_{r+1}) \leqslant q(\theta^r,\theta_{r+1}), \text{ for all } m,$$

which proves the third part of proposition 5.

A.6 Proof of Proposition 6

STEP 1: First of all, assume that bond price sequence $\left\{p^m(\theta^t,\theta_{t+1}), q(\theta^t,\theta_{t+1})\right\}_{t\in[0,\infty)}$ in problem (RP) and (RP^{J}) are identical. I will show later that they are indeed the same in the two maximization problem's corresponding equilibria. Consider any type n residents in country m at θ^t with initial domestic bond holdings $b_n^m\left(\theta^t\right)$ and bond prices $\{p^m(\theta^r,\theta_{r+1}), q(\theta^r,\theta_{r+1})\}_{r\in[t,\infty)}$ given. The constraint set of the post-default maximization problem (RIA) is a subset of problem ($\mathrm{RIA}^{\mathrm{J}}$)'s constraint set since the former set contains constraint (2) to prevent domestic debt default. Therefore, the optimal solution to the (RIA) problem is always feasible in the ($\mathrm{RIA}^{\mathrm{J}}$) problem, which proves the following

$$V_n^{m,J}(\theta^t, b_n^m(\theta^t)) \geqslant V_n^m(\theta^t, b_n^m(\theta^t)), \text{ for all } \theta^t, b_n^m(\theta^t).$$

Using the same argument in the other direction, that is, noting that the constraint set of (RP^{J}) is a subset of (RP)'s constraint set proves the result that adding domestic enforcement problem makes every type in all countries weakly better.

Maximization problem (RP) and (RP^{J}) will have the same consumption allocation if

$$V_n^{m,J}(\theta^t, b_n^m(\theta^t)) = V_n^m(\theta^t, b_n^m(\theta^t)), \text{ for all } \theta^t, b_n^m(\theta^t).$$

For them to be equal in all histories, the domestic participation constraint (2) must be slack at all future histories θ^r with $r\in[t,\infty)$ in problem (RIA) started at any present hisotry θ^t with $t\in[0,\infty)$. In other words, the optimal consumption path for n_A types residents with $\nu_n^m(\theta^r)=0$ for all θ^r stays unchanged in the above two resident's problems. And I know from the domestic bond pricing rule (13) that $p^m(\theta^t,\theta_{t+1})$ is determined by the MRS of type n_A residents. As a result, the domestic price sequences in the two problems are identical and so are the international bond price sequences since international price is the maximum among all domestic prices.

Step 2: I can now prove the main result that adding domestic enforcement problem strictly improve type n_B's utility. Remember type n_B residents are defined as those who are indifferent between repaying and reneging domestic debt in resident's international autarky problem. By assumption, we have

$$\frac{U'\left(c_n^m(\theta^t,\theta_{t+1})\right)}{U'\left(c_n^m\left(\theta^t\right)\right)} \neq \frac{U'\left(c_1^m(\theta^t,\theta_{t+1})\right)}{U'\left(c_1^m\left(\theta^t\right)\right)}. \tag{A.21}$$

Without loss of generality, I assume type 1 residents belong to types n_A, then the domestic bond price in m is determined by type 1's MRS

$$p^m(\theta^t,\theta_{t+1}) = \beta\frac{U'(c_1^m(\theta^t,\theta_{t+1}))}{U'(c_1^m(\theta^t))}\pi(\theta_{t+1}|\theta^t).$$

Compare it with equation (A.2) and use (A.21), I can conclude

$$\nu_n^m(\theta^t,\theta_{t+1}), \mu_n^m(\theta^t,\theta_{t+1}) > 0.$$

As a result, type n residents are really type n_B residents whose continuation value after (θ^t,θ_{t+1}) in resident's problem (RP) equals to the one in resident's international autarky problem (RIA) since the international participation constraint (4) binds, and again equals to the one in resident's autarky problem (RA) started at (θ^t,θ_{t+1}) since the domestic participation constraint (2) binds.

Also by assumption $f_n^{m,J}(\theta^r,\theta_{r+1}) < 0$, which implies two things for the same type n residents above but in problem (RP$^{\text{J}}$). One thing is that type n's consumption allocation is neither resident's international autarky nor resident's autarky after (θ^r,θ_{r+1}), and another is that the international participation constraint (15) binds because otherwise one can always attain higher utility by borrowing more from foreigners as long as (15) is slack.

Next, recall proposition 1, the result extends to (RP$^{\text{J}}$). That is, with a binding international participation constraint, the equilibrium consumption path in problem (RP$^{\text{J}}$) from (θ^r,θ_{r+1}) onward is identical to the path after international debt default in the (RIA$^{\text{J}}$) problem. Since the consumption path after (θ^r,θ_{r+1}) in (RIA$^{\text{J}}$) is not resident autarkic while it sure is in (RIA), the continuation values are such that $V_n^m(\theta^t,b_n^m(\theta^t)) < V_n^{m,J}(\theta^t,b_n^{m,J}(\theta^t))$ because of strictly concave objective function.

Thus, type n in problem (RP) without domestic commitment can relax the international participation constraint (4) in history that has a stictly positive Lagrange multiplier, thereby borrow more from outside the country and increase utility to a new level which is strictly greater than the one in problem ($\mathrm{RP^J}$) with full domestic commitment.

A.7 Derivation of the Zero Net Payment condition (16)

As in the proof of proposition 1. Denote $P_t = \prod_{r=0}^{t} p_r$ the Arrow-Debreu price at date 0 for a domestic contingent bond matured after t periods in country 1. The one-period domestic bond price sequence in country 1 is known as

$$p_r = \begin{cases} p, & r = 0, 2, 4, ...; \\ q, & r = 1, 3, 5, ..., \end{cases}$$

then use it to get

$$\{P_t\}_{t\in[0,\infty)} = \left\{p, pq, p^2q, p^2q^2, p^3q^2, p^3q^3, ...\right\}.$$

By symmetry, the net payment to foreign country 2 at period t is denoted by N_t with the following pattern

$$N_t = \begin{cases} x^J - y, & t = 0, 2, 4, ... \\ y - x^J, & t = 1, 3, 5, ... \end{cases}.$$

According to lemma 2, when I discount all future net payments to period 0 using $\{P_t\}_{t\in[0,\infty)}$ and add them up, I will get zero if country 1 is internationally participation constrained at present date 0.

$$\sum_{t=0}^{\infty} N_t P_t = 0.$$

Substitute the expressions for N_t and P_t into the above equation to get the zero captial inflow condition (16) in section 4.1.

$$\begin{aligned}
0 &= (x^J - y)(P_0 + P_2 + P_4 + ...) + (y - x^J)(P_1 + P_3 + P_5 + ...) \\
&= (x^J - y)(p + p^2q + p^3q^2 + ...) + (y - x^J)(pq + p^2q^2 + p^3q^3 + ...) \\
&= (x^J - y)p\sum_{s=1}^{\infty} p^{s-1}q^{s-1} + (y - x^J)\sum_{s=1}^{\infty} p^s q^s \\
&= \frac{(x^J - y)p}{1 - pq} + \frac{(y - x^J)pq}{1 - pq} \\
&= \frac{p\left[(x^J - y) + q(y - x^J)\right]}{1 - pq} \\
&= \frac{(x^J - y) + q(y - x^J)}{1 - pq}.
\end{aligned}$$

The same method is used to derive equation (21) in section 4.3.

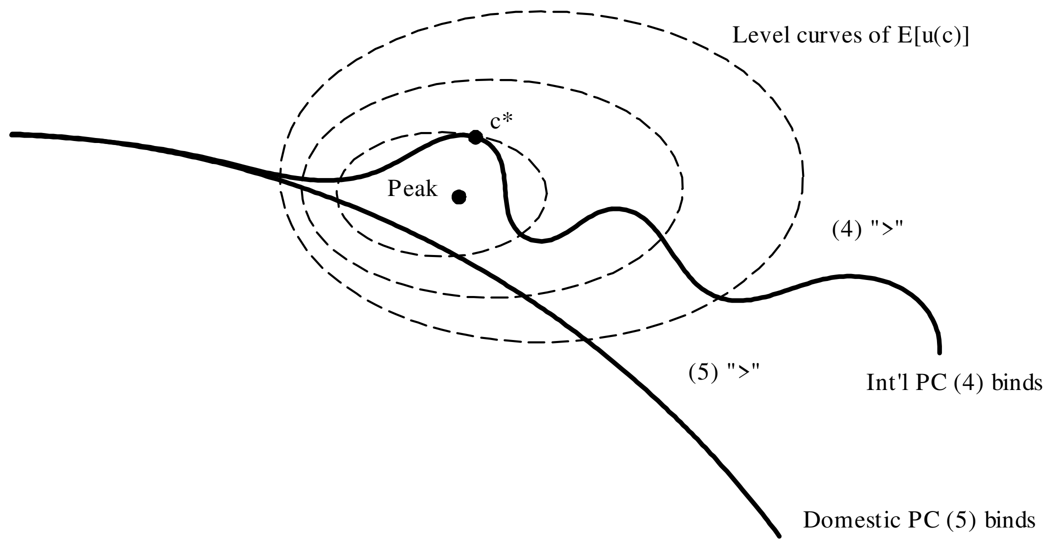

Figure 1: Redundant Domestic Constraint

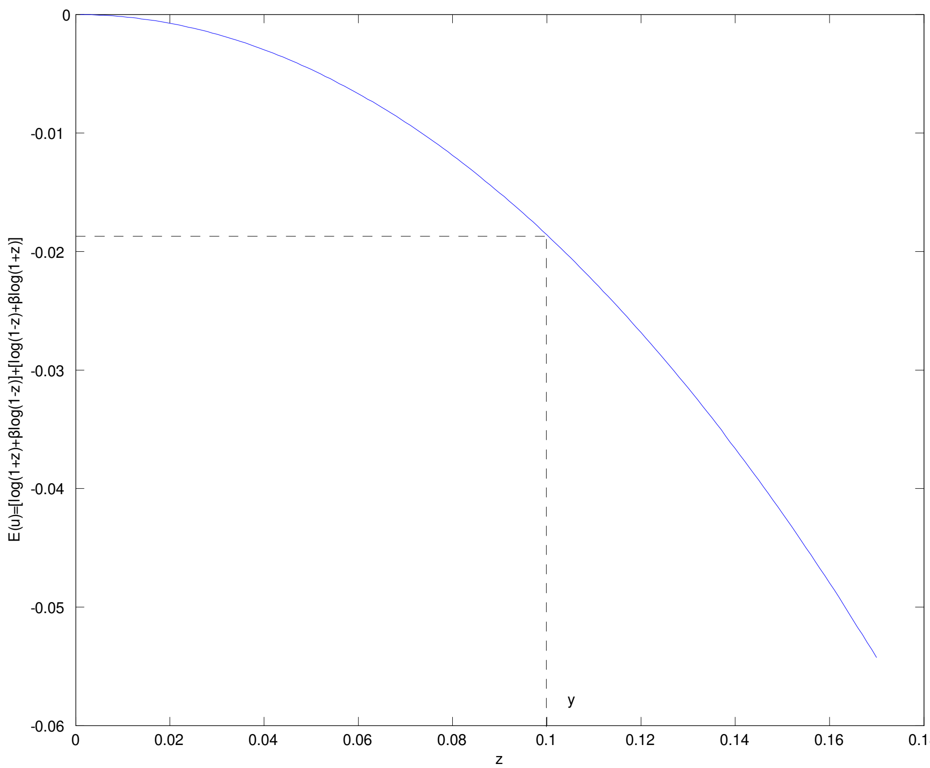

Figure 2: Ex-Ante Expected Utility

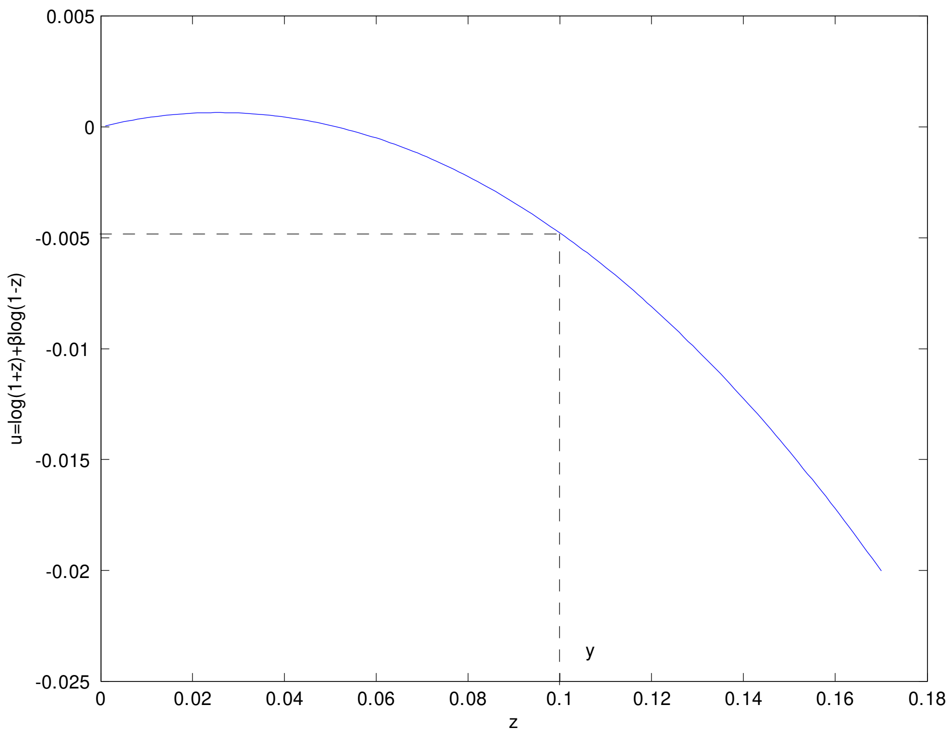

Figure 3: Full Consumption Smoothing

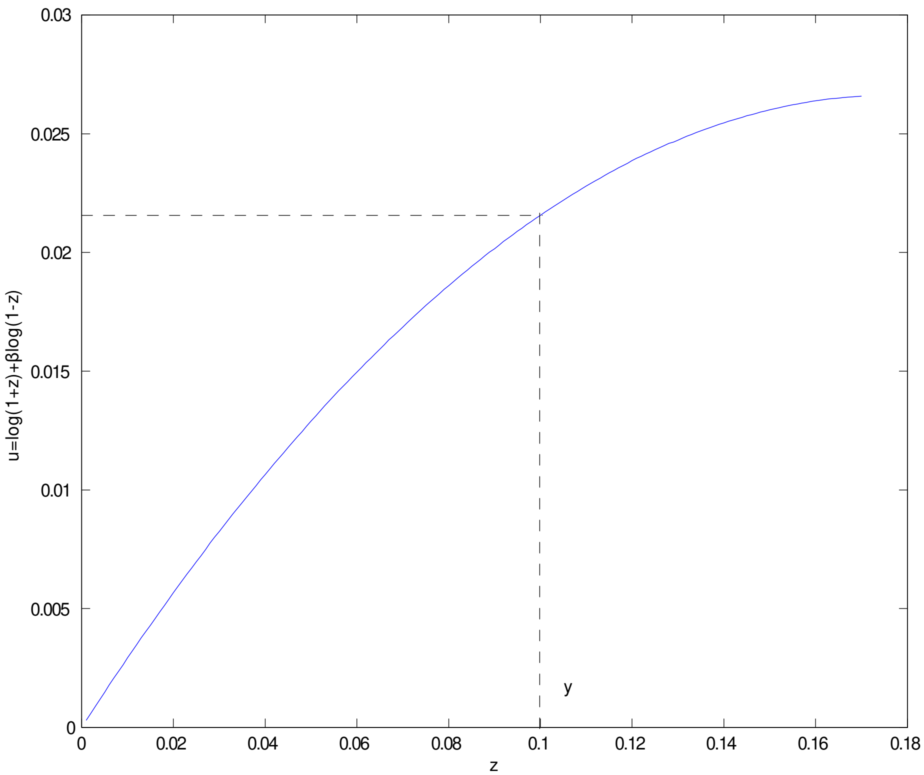

Figure 4: Autarky

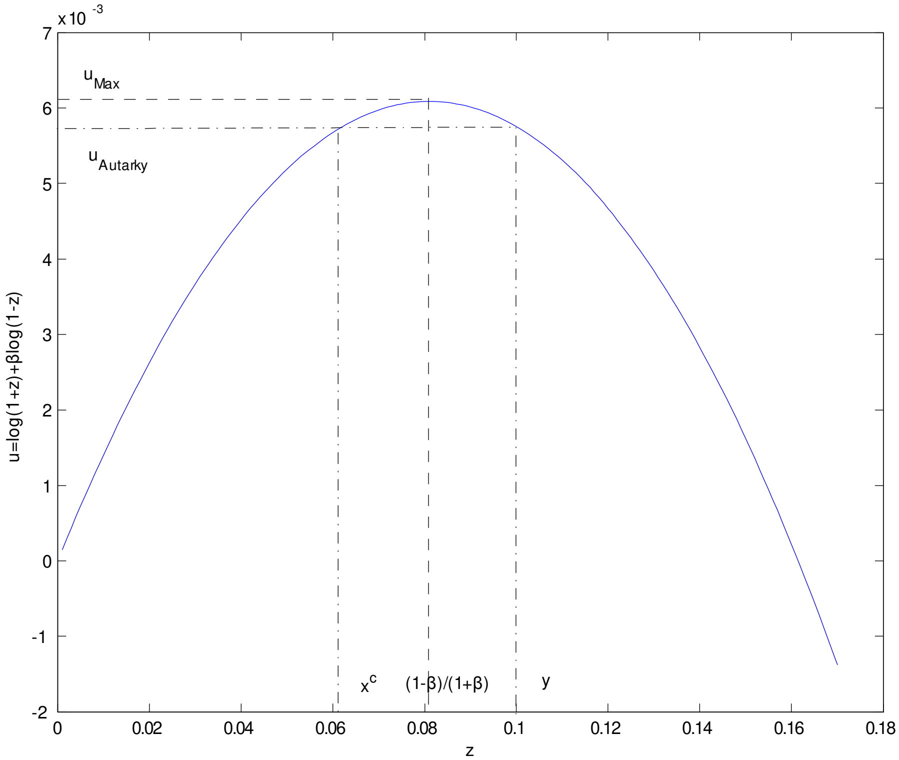

Figure 5: Some International Risk Sharing

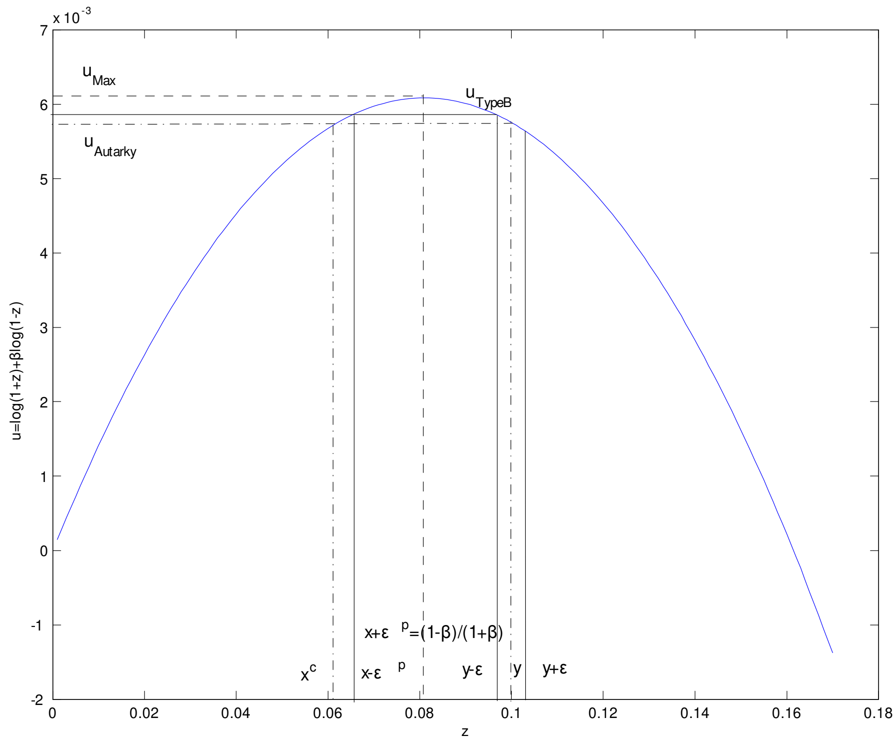

Figure 6: Trade Equilibrium Allocation

Measure	Type n	Country m	
		$m=1$	$m=2$
$\frac{1}{2}$	$n=A$	$1+y+\varepsilon$	$1-y-\varepsilon$
$\frac{1}{2}$	$n=B$	$1+y-\varepsilon$	$1-y+\varepsilon$
$\frac{1}{2}(A+B)$		$1+y$	$1-y$

Table 1: Endowment Structure at Even Numbered Period

Measure	Type n	Country m	
		$m=1$	$m=2$
$\frac{1}{2}$	$n=A$	$1-y-\varepsilon$	$1+y+\varepsilon$
$\frac{1}{2}$	$n=B$	$1-y+\varepsilon$	$1+y-\varepsilon$
$\frac{1}{2}(A+B)$		$1+y$	$1-y$

Table 2: Endowment Structure at Odd Numbered Period