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Mynbaev, Kairat

Kazakhstan Institute of Management, Economics, and Strategic Research (KIMEP)

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L_p-Approximable Sequences of Vectors and Limit Distribution of Quadratic Forms of Random Variables*

Kairat T. Mynbaev

Kazakhstan Institute of Management, Economics, and Strategic Research
4, Abai Avenue, Room 207, 480100 Almaty, Kazakhstan
E-mail: kairat@kimep.kz

The properties of L_2-approximable sequences established here form a complete toolkit for statistical results concerning weighted sums of random variables, where the weights are nonstochastic sequences approximated in some sense by square-integrable functions and the random variables are "two-wing" averages of martingale differences. The results constitute the first significant advancement in the theory of L_2-approximable sequences since 1976 when Moussatat introduced a narrower notion of L_2-generated sequences. The method relies on a study of certain linear operators in the spaces L_p and l_p. A criterion of L_p-approximability is given. The results are new even when the weights generating function is identically 1. A central limit theorem for quadratic forms of random variables illustrates the method.

Key Words: linear operators in L_p spaces; central limit theorem; quadratic forms of random variables

1. INTRODUCTION

Among various probabilistic and statistical tools used in econometrics the place of central limit theorems (CLT's) and invariance principles is special. We briefly describe some applications and then discuss the format of CLT's and invariance principles most suited for econometric needs.

Most CLT's treat convergence in distribution of sums $\sum_{t=1}^{n} X_{nt}$ of random variables $X_{n1}, \ldots, X_{nn}$. In linear regression analysis such CLT's are used to study the asymptotics of the ordinary least squares (OLS) estimator and justify various test procedures. Anderson and Kunitomo [3] is both a milestone of the theory and an instructive illustration of application of

*Research was implemented while the author was a visiting professor at the Economics Department of the Federal University of Ceará, Fortaleza, Brazil.

Dvoretzky's CLT to a regression with autoregressive terms in the stable case.

In the unstable case, when the disturbances in the OLS estimator formula are multiplied by some growing factor, a different technique is required. One of the modern approaches, pioneered by Phillips [20], employs the invariance principle for $\sum_{t=1}^{[nx]} X_{nt}$ where $[nx]$ denotes the integer part of nx, $0 \leq x \leq 1$. Hamilton's book [8] is an excellent introduction to the area and contains the principal references, including, but not limited to, the papers by Dickey and Fuller, Chan and Wei, Park and Phillips, Sims, Stock, and Watson, and Phillips and Solo. Subsequent developments have considerably widened the range of applicability of the method. By now the models considered may include polynomial trends with a finite number of structural changes, as in Vogelsang [24]. Nabeya and Tanaka [19] have suggested a different, and very attractive, way to handle the unstable case (based on a CLT for a quadratic form of random variables).

The diversity of the models considered and complexity of the tools used in the above references call for an analysis and generalization of the mathematical basis. To maintain the scope of the ensuing discussion manageable, we leave mixing processes popularized by Phillips [21] alone and concentrate on the methods that take advantage of the martingale theory. One general remark resulting from comparison of results by Anderson and Kunitomo [3] and Vogelsang [24] is that it is better to distinguish from the very beginning exogenous regressors (defined outside the system, such as polynomial trends) from endogenous regressors (such as lagged dependent variables) rather than include them into one stochastic matrix of regressors. Polynomial trends are nonstochastic, and the strength of the theory discussed below is better seen in the case of nonstochastic exogenous regressors, therefore we restrict our attention to this case.

For many econometric problems, in order to separate the influence of heterogeneity and dependence of X_{nt} on convergence in CLT's and invariance principles, it is advantageous to specify X_{nt} as $w_{nt}v_{nt}$ where the nonstochastic weights w_{nt} account for heterogeneity and the basic random variables v_{nt} model dependence over time. As far as we know, there are very few papers devoted specifically to CLT's for

$$\sum_{t=1}^{n} w_{nt}v_{nt} \tag{1.1}$$

Anderson's OLS estimator asymptotics [1, Theorem 2.6.1] implicitly contains one of early examples of a CLT for (1.1). Srinivasan and Zhou [22] consider processes that look like (1.1) but their martingales are continuous and the conditions are too restrictive for econometric applications. Yoshihara [25, 26] develops their method.

In Anderson's result mentioned above the limit distribution involves some limit characteristics of the sequence $\{w_n\}$ where $w_n = (w_{n1}, \dots, w_{nn})$. Since it is hard to grasp the behavior and manage such sequences, it is a good idea to represent them as images of some functions of a continuous argument. This idea has been pursued by Moussatat [13] and Millar [12] (see also Milbrodt [11] for applications) and is realized as follows. For a square-integrable function F and any natural n the vector w_n is defined by

$$w_{nt} = \sqrt{n} \int_{(t-1)/n}^{t/n} F(x)dx, \quad t = 1, \dots, n. \tag{1.2}$$

When $F \equiv 1$, this gives a familiar factor $1/\sqrt{n}$. The sequence $\{w_n\}$ is called L_2-*generated by* F. With volatility of economic data, it is difficult to accept such sequences as weights in econometrics. Therefore Mynbaev [14] has suggested to work with sequences $\{w_n\}$ satisfying

$$\sum_{t=1}^{n} \left(w_{nt} - \sqrt{n} \int_{(t-1)/n}^{t/n} F(x)dx \right)^{1/2} \to 0. \tag{1.3}$$

We call such a sequence L_2-*approximated by* F, and if for a given $\{w_n\}$ there exists an $F \in L_2$ satisfying (1.3), it is said to be L_2-*approximable.*

The conventional approach consists in deriving limit results for (1.1) from those for

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} e_{nt} \tag{1.4}$$

where $e_{n1}, \dots, e_{nn}$ are (say) independent and identically distributed random variables, for each n. The theory has been evolved by relaxing conditions on the weights and widening the class of v_{nt} which can be constructed from e_{nt}. The specification of v_{nt} as moving averages

$$v_{nt} = \sum_{j=0}^{\infty} e_{n,t-j} \psi_j, \tag{1.5}$$

where $\{\psi_j : \ j \geq 0\}$ is a given sequence of real numbers, has become standard. The same approach is applied to obtain convergence of the process

$$\sum_{t=1}^{[nx]} w_{nt} v_{nt}, \quad 0 \leq x \leq 1, \tag{1.6}$$

to a transformed Brownian motion ($[nx]$ denotes the integer part of nx). Vogelsang [24] is a good example of how far the theory has gone along this line.

Despite the many efforts, the existing results have two major drawbacks. The properties of L_2-generated weights established in Moussatat [13] and Millar [12] are not sufficient to include v_{nt} of type (1.5). On the other hand, the authors who work with (1.5) impose such regularity conditions on the weights which exclude nonsmooth elements of L_2. Since the expressions of limit distributions derived in Vogelsang [24] involve integrals of squares of F, the class L_2 is the appropriate one for the problems under consideration. However, Vogelsang's method requires piece-wise continuous weight generating functions. Under L_2-approximability unbounded and discontinuous functions are allowed.

In this paper we show that the solution to these issues lies in the theory of approximation of functions in L_2. The theoretical-functional part of the job is actually done in $L_p,\ 1 \le p < \infty$. Convergence of linear operators involved depends on such characteristics of the weight generating function as the continuity modulus and is therefore a strong convergence (it is not uniform on the unit ball of L_p). Instead of (1.5) we are able to consider "two-wing" averages

$$v_{nt} = \sum_{j=-\infty}^{\infty} e_{n,t-j}\psi_j. \tag{1.7}$$

As to the conditions on $\{\psi_j\}$, the common requirement in the econometrics literature is $\sum_{j\ge 0} j|\psi_j| < \infty$; the existing methods use this requirement in conjunction with the Beveridge and Nelson decomposition [4]. It is less known that Tanaka [23] has succeeded in replacing it by summability $\sum_{j\ge 0} |\psi_j| < \infty$ (the statement can also be found in Nabeya and Tanaka [19]). Our goal is to keep the summability assumption in the more general case (1.7) and justify convergence in distribution in the entire class L_2. Thus, our results are new even when $F \equiv 1$. Due to lack of space we do not consider (1.6) and various extensions such as the error structure used to obtain local-to-unity asymptotics.

Section 2 is devoted to L_p-generated sequences. Theorem 2.1 is a sophisticated version of the property $(P_n x, P_n y) \to (x, y)$ in a Hilbert space H where $\{P_n\}$ is a sequence of projectors strongly convergent to the identity operator. It holds for nonuniform partitions of the segment $[0, 1]$ and can be applied to establish stochastic limit results when the e_{nt}'s are martingale differences (m.d.'s). The use of nonuniform partitions allows for nonequidistant sampling in statistics, as is stressed by Bischoff [5]. Theorem 2.2 treats some linear operators associated with the sequence $\{\psi_j\}$ the main of which is $\Psi_n :\ R^n \to R^n$ defined by

$$\Psi_n z = \left(\sum_{j=1}^{n} z_j \psi_{j-t}\right)_{t=1}^{n}. \tag{1.8}$$

It is a necessary tool to pass from m.d.'s to (1.7). The method of its proof requires uniform partitions.

In Section 3 it is shown that L_p-approximable sequences inherit all properties of L_p-generated ones (Theorem 3.1). The criterion of L_p-approximability (Theorem 3.2) gives an idea of how restrictive (or general) the L_p-approximability condition is. Theorems 3.3 and 3.4 contain some easily verifiable sufficient conditions and counter-examples.

In Section 4 we apply the previous results to prove a CLT for (1.1) with L_2-approximable weights, summable $\{\psi_j\}$ and with e_{nt} which have uniformly integrable squares and fixed second conditional moments. Nabeya and Tanaka [18] discovered an interesting link between the limit distribution of a quadratic form of random variables and the theory of integral operators. Theorem 4.2 is an application of our CLT in the spirit of Nabeya and Tanaka. The conditions on the integral operator involved are significantly relaxed; in particular, we require just square-integrability of the kernel instead of their continuity condition. In general, a variety of statistical problems can be considered in a unified manner using the results on L_2-approximable sequences contained here.

Only basic properties of L_p spaces are employed. All probabilistic notions and facts used in the paper can be found in Davidson [6] except where indicated otherwise.

Notation. L_p denotes the space of measurable functions F on $(0,1)$ provided with the norm

$$\|F\|_p = \left(\int_0^1 |F(x)|^p dx\right)^{1/p}, \ 1 \le p < \infty; \ \|F\|_\infty = \operatorname{ess\,sup}_{x\in(0,1)}|F(x)|.$$

Its discrete analogue l_p consists of sequences $\{z_i : \ i \in I\}$ having a finite norm

$$\|z\|_p = \left(\sum_{i\in I} |z_i|^p\right)^{1/p}, \quad 1 \le p < \infty; \quad \|z\|_\infty = \sup_{i\in I} |z_i|.$$

The set of indices I depends on the context. For a subset $A \subset (0,1)$ we denote $\|F\|_{p,A} = (\int_A |F(x)|^p dx)^{1/p}$. R^n provided with the norm $\|\cdot\|_p$ is denoted R^n_p. For $p \in [1,\infty)$ the number or symbol q is defined from $1/p + 1/q = 1$. Z is the set of integers.

We denote $1(A)$ as the indicator of a set A. All random variables in the paper are defined on some probability space (Ω, G, P). For a random variable v, $\|v\|_p$ means $\left(E|v|^p\right)^{1/p}$, $1 \le p < \infty$. $N(0,V)$ stands for the set of normal vectors with mean zero and variance V. Convergence of a sequence $\{X_n\}$ to X in distribution (in probability) is denoted $X_n \xrightarrow{d} X$ ($X_n \xrightarrow{p} X$ or $\operatorname{plim} X_n = X$, respectively).

2. PROPERTIES OF L_P-GENERATED SEQUENCES

Suppose that for each natural n there is a partition $\Pi_n = \{\pi_0, \ldots, \pi_n\}$ which satisfies $0 = \pi_0 < \pi_1 < \ldots < \pi_n = 1$. Its fineness is defined to be $\Lambda_n = \max_t(\pi_t - \pi_{t-1})$. We always assume that $\lim_{n\to\infty} \Lambda_n = 0$. The partition Π_n generates a covering $\{i_t : \ t = 1, \ldots, n\}$ of $[0, 1)$ consisting of intervals $i_t = [\pi_{t-1}, \pi_t), \ t = 1, \ldots, n$, whose lengths are denoted $|i_t| = \pi_t - \pi_{t-1}$. The function

$$\kappa_n(x) = \max\{k : \ k \le n, \ \pi_{k-1} \le x\}, \quad x \in [0, 1)$$

is nondecreasing and possesses the property $x \in i_{\kappa_n(x)}$.

With the partition Π_n one can associate a discretization operator $d_{np} : L_p \to R^n_p$ as

$$(d_{np}F)_t = |i_t|^{-1/q} \int_{i_t} F(x)dx, \quad t = 1, \ldots, n, \quad F \in L_p.$$

For a given $F \in L_p$, the sequence $\{d_{np}F\}$ is called L_p-*generated by* F. Often we denote $f_n = d_{np}F$. The Hölder inequality and absolute continuity of the Lebesgue integral imply

$$|f_{n,t}| \le \|F\|_{p,i_t}, \quad \lim_{n\to\infty} \max_{1\le t\le n} |f_{n,t}| = 0, \tag{2.1}$$

$$\|d_{np}F\|_p \le \|F\|_p \tag{2.2}$$

for all $F \in L_p, \ p < \infty$. The interpolation operator $D_{np} : R^n_p \to L_p$ takes a vector $z \in R^n_p$ to a simple function

$$D_{np}z = \sum_{t=1}^{n} z_t |i_t|^{-1/p} 1(i_t).$$

It is easy to see that

$$\|D_{np}z\|_p = \|z\|_p \tag{2.3}$$

and that the product $D_{np}d_{np}$ coincides with the Haar projector P_n where

$$P_nF = \sum_{t=1}^{n} |i_t|^{-1} \int_{i_t} F(x)dx1(i_t).$$

For $y \in R$, let τ_y be the translation operator, $(\tau_y F)(x) = F(x+y)$, and denote Δ_y as the interval

$$\Delta_y = (\max\{0, -y\},\ \min\{1, 1-y\}).$$

The continuity modulus of $F \in L_p$ is defined by

$$\omega_p(F,\delta) = \sup_{|y|\le\delta} \|F - \tau_y F\|_{p,\Delta_y}, \quad \delta > 0.$$

It is well known that $\omega_p(F,\delta) \to 0$ as $\delta \to 0$ for any $F \in L_p,\ p < \infty$, and that

$$\|P_n F - F\|_p \le 2^{1/p}\omega_p(F,\Lambda_n), \quad \|P_n F\|_p \le \|F\|_p \tag{2.4}$$

(the second of these inequalities is a consequence of (2.2) and (2.3)). The probabilistic methods yield just $\lim_{n\to\infty}\|P_n F - F\|_p = 0$ without an estimate of the rate of convergence (see Millar [12]).

The following theorem has been proved in Mynbaev [15, 16].

Theorem 2.1. *Let $1 < p < \infty$, $X \in L_p$, $Y \in L_q$. For any $(a,b) \subset (0,1)$ denote $\chi = 1\big((a,b)\big)$. Then*

$$\lim_{n\to\infty}\sum_{t=1}^{n}\big(d_{np}(\chi X)\big)_t\big(d_{nq}(\chi Y)\big)_t = \int_a^b X(s)Y(s)ds$$

uniformly with respect to all intervals $(a,b) \subset (0,1)$.

In the rest of this section we consider only uniform partitions in which case $i_t = \big[(t-1)/n, t/n\big),\ t = 1,\dots,n$, and $\kappa_n(x) = [nx]+1$. Let $\{\psi_j : j \in Z\}$ be a given sequence of real numbers such that $\sum_{j\in Z}|\psi_j| < \infty$. In addition to (1.8), define operators $\Phi_n : R_p^n \to l_p$ and $\Upsilon_n : R_p^n \to l_p$ by

$$\Phi_n z = \left(\sum_{j=1}^{n} z_j\psi_{j-t}\right)_{t=-\infty}^{0}, \quad \Upsilon_n z = \left(\sum_{j=1}^{n} z_j\psi_{j-t}\right)_{t=n+1}^{\infty}.$$

In the next theorem we study convergence of these operators. Since convergence is trivial when ψ_k are identically zero, we exclude this case. Besides, it is useful to distinguish (1) zero-tail sequences from (2) non-zero-tail ones defined by

$$(1)\ \exists\, n > 0, \sum_{|k|\ge n}|\psi_k| = 0 \quad \text{and} \quad (2) \sum_{|k|\ge n}|\psi_k| > 0 \quad \forall n > 0,$$

respectively. For a zero-tail nontrivial sequence the number

$$k_\psi = \max\{k \ge 0 : |\psi_k| \ne 0\} + 1 \tag{2.5}$$

is defined.

Let F be a nonzero element of $L_p,\ p < \infty$. Put

$$\mu(\delta) = \mu(F, p, \delta) = \max\{\omega_p(F, \delta),\ \|F\|_{p,(0,\delta)},\ \|F\|_{p,(1-\delta,1)}\}, \quad \delta \in (0, 1],$$

$$\xi(\varepsilon) = \xi(F, p, \varepsilon) = \sup\{\delta \in (0, 1] : \mu(\delta) \le \varepsilon\}, \quad \varepsilon \in (0, 1].$$

Obviously, μ is nonnegative on $(0, 1]$, $\lim_{\delta\to 0} \mu(\delta) = 0$, ξ is positive on $(0, 1]$ and

$$\lim_{\varepsilon\to 0} \xi(\varepsilon) = 0, \quad \mu\big(\xi(\varepsilon)\big) \le \varepsilon. \tag{2.6}$$

Further, denote

$$\alpha_\psi = \sum_{j\in Z} |\psi_j| \ne 0, \quad \beta_\psi = \sum_{j\in Z} \psi_j,$$

$$\begin{aligned} \eta(\varepsilon) &= \eta(\psi, F, p, \varepsilon) \\ &= \begin{cases} \min\{n \ge 1 : \xi(\varepsilon)n \ge k_\psi\}, & \text{if } \{\psi_j\} \text{ is zero} - \text{tail} \\ \min\{n \ge 1 : \sum_{k>\xi(\varepsilon)n} |\psi_k| \le \varepsilon\}, & \text{otherwise} \end{cases} \end{aligned}$$

where $\varepsilon \in (0, 1]$. Evidently, if $\alpha_\psi < \infty$, then η is finite and

$$\lim_{\varepsilon\to 0} \eta(\varepsilon) = \infty, \quad \sum_{k>\xi(\varepsilon)\eta(\varepsilon)} |\psi_k| \le \varepsilon. \tag{2.7}$$

The above definitions serve to prove (2.9) in the case of a nontrivial F. If $F = 0$ a.e., (2.9) becomes obvious if we formally put $\eta(\varepsilon) = 1$ for all $\varepsilon \in (0, 1]$.

THEOREM 2.2. *Let the partitions Π_n be uniform and $\alpha_\psi < \infty$.*
(a) *Ψ_n, Φ_n, and Υ_n are uniformly bounded:*

$$\sup_n \max\{\|\Psi_n\|, \|\Phi_n\|, \|\Upsilon_n\|\} \le \alpha_\psi. \tag{2.8}$$

(b) *If $F \in L_p,\ 1 \le p < \infty$, then for all sufficiently small ε one has*

$$\begin{aligned} &\max\{\|(\Psi_n - \beta_\psi)d_{np}F\|_p, \|\Phi_n d_{np}F\|_p, \|\Upsilon_n d_{np}F\|_p\} \\ &\qquad \le 3(\alpha_\psi + \|F\|_p)\varepsilon, \quad \forall n \ge \eta(\varepsilon). \end{aligned} \tag{2.9}$$

Proof. (a) $I_n,\ 0_n$, and J_n will denote the $n\times n$ identity matrix, null matrix, and the matrix whose elements on the secondary diagonal equal one and all other elements are zeros, respectively. Put

$$I_n^+ = \begin{pmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{pmatrix}, \quad I_n^- = (I_n^+)'.$$

Then

$$(I_n^+)^k = \begin{pmatrix} 0_{(n-k)\times k} & I_{n-k} \\ 0_k & 0_{k\times(n-k)} \end{pmatrix}, \quad (I_n^-)^k = \left((I_n^+)^k\right)', \quad k\le n-1;$$
$$(I_n^\pm)^k = 0_n,\ k\ge n; \quad \|(I_n^\pm)^k z\|_p \le \|z\|_p, \quad k\ge 0. \tag{2.10}$$

From the representation

$$\Psi_n = \sum_{k=0}^{n-1}\psi_k (I_n^+)^k + \sum_{k=-n+1}^{-1}\psi_k (I_n^-)^{-k} \tag{2.11}$$

we see that

$$\|\Psi_n z\|_p \le \sum_{|k|\le n-1}|\psi_k|\|z\|_p \le \alpha_\psi\|z\|_p. \tag{2.12}$$

Let

$$A_k = \begin{pmatrix} J_k & 0_{k\times(n-k)} \\ 0 \quad \dots & 0 \\ \dots\dots\dots\dots \end{pmatrix}_{\infty\times n}, \ 1\le k\le n;$$
$$A_k = \begin{pmatrix} 0_{(k-n)\times n} \\ J_n \\ 0 \ \dots \ 0 \\ \dots\dots\dots \end{pmatrix}_{\infty\times n}, \ k>n.$$

Then

$$\Phi_n = \sum_{k\ge 1}\psi_k A_k, \quad \|A_k z\|_p \le \|z\|_p, \quad k\ge 1, \tag{2.13}$$

and

$$\|\Phi_n z\|_p \le \sum_{k\ge 1}|\psi_k|\|z\|_p \le \alpha_\psi\|z\|_p. \tag{2.14}$$

Similarly, with

$$B_k = \begin{pmatrix} \cdots\cdots\cdots\cdots \\ 0 \quad \ldots \quad 0 \\ 0_{k\times(n-k)} \quad J_k \end{pmatrix}_{\infty\times n}, \ 1 \le k \le n;$$

$$B_k = \begin{pmatrix} \cdots\cdots \\ 0 \ \ldots \ 0 \\ J_n \\ 0_{(k-n)\times n} \end{pmatrix}_{\infty\times n}, \ k > n,$$

we have

$$\Upsilon_n = \sum_{k\ge 1} \psi_{-k} B_k, \quad \|B_k z\|_p \le \|z\|_p, \quad k \ge 1, \tag{2.15}$$

so that

$$\|\Upsilon_n z\|_p \le \sum_{k\le -1} |\psi_k| \|z\|_p \le \alpha_\psi \|z\|_p. \tag{2.16}$$

Inequalities (2.12), (2.14), and (2.16) prove part (a).

(b) Denote $f_n = d_{np}F$. Obviously,

$$\|(I_n^-)^k f_n - f_n\|_p = \left(\sum_{t=k+1}^{n} |f_{n,t-k} - f_{nt}|^p + \sum_{t=1}^{k} |f_{nt}|^p \right)^{1/p}.$$

For $t \le k$ we are going to use the inequality in (2.1). For $t > k$

$$|f_{n,t-k} - f_{nt}| = \left| n^{1/q} \int_{i_{t-k}} \big(F(x) - F(x+k/n)\big) dx \right| \le \|F - \tau_{k/n} F\|_{p,i_{t-k}}.$$

Taking into account that

$$\bigcup_{t=k+1}^{n} i_{t-k} = (0, 1-k/n), \quad \bigcup_{t=1}^{k} i_t = (0, k/n),$$

we have

$$\begin{aligned} \|(I_n^-)^k f_n - f_n\|_p &\le \left(\|F - \tau_{k/n}F\|^p_{p,(0,1-k/n)} + \|F\|^p_{p,(0,k/n)}\right)^{1/p} \\ &\le \left(\big(\omega_p(F, k/n)\big)^p + \|F\|^p_{p,(0,k/n)}\right)^{1/p}. \end{aligned} \tag{2.17}$$

Similarly,

$$\begin{aligned}\|(I_n^+)^k f_n - f_n\|_p &= \left(\sum_{t=k+1}^{n} |f_{n,t-k} - f_{nt}|^p + \sum_{t=n-k+1}^{n} |f_{nt}|^p\right)^{1/p} \\ &\le \left(\|F - \tau_{k/n}F\|^p_{p,(0,1-k/n)} + \|F\|^p_{p,(1-k/n,1)}\right)^{1/p} \\ &\le \left(\left(\omega_p(F,k/n)\right)^p + \|F\|^p_{p,(1-k/n,1)}\right)^{1/p}. \qquad (2.18)\end{aligned}$$

Let $n \ge \eta(\varepsilon)$ and denote $n_\varepsilon = [\xi(\varepsilon)n]$. For all sufficiently small ε we have $1/\big(1-\xi(\varepsilon)\big) \le \eta(\varepsilon) \le n$ and

$$n_\varepsilon \le \xi(\varepsilon)n \le n-1, \quad n_\varepsilon + 1 > \xi(\varepsilon)n \ge \xi(\varepsilon)\eta(\varepsilon). \qquad (2.19)$$

For $k \le n_\varepsilon$ (2.6), (2.17), (2.18), and (2.19) imply

$$\begin{aligned}\|(I_n^-)^k f_n - f_n\|_p &\le \left(\left(\omega_p(F,\xi(\varepsilon))\right)^p + \|F\|^p_{p,(0,\xi(\varepsilon))}\right)^{1/p} \le 2^{1/p}\varepsilon, \\ \|(I_n^+)^k f_n - f_n\|_p &\le \left(\left(\omega_p(F,\xi(\varepsilon))\right)^p + \|F\|^p_{p,(1-\xi(\varepsilon),1)}\right)^{1/p} \le 2^{1/p}\varepsilon.\end{aligned}$$

For $k > n_\varepsilon$ by (2.2) and (2.10)

$$\|(I_n^\pm)^k f_n\|_p \le \|f_n\|_p \le \|F\|_p.$$

Therefore, using also (2.11),

$$\begin{aligned}\|(\Psi_n - \beta_\psi)f_n\|_p \le &\sum_{k=0}^{n_\varepsilon} |\psi_k| \|(I_n^+)^k f_n - f_n\|_p \\ &+ \sum_{k=-n_\varepsilon}^{-1} |\psi_k| \|(I_n^-)^{-k} f_n - f_n\|_p + \sum_{n_\varepsilon+1}^{n-1} |\psi_k| \|(I_n^+)^k f_n\|_p \\ &+ \sum_{k=-n+1}^{-n_\varepsilon-1} |\psi_k| \|(I_n^-)^{-k} f_n\|_p + \sum_{|k|>n_\varepsilon} |\psi_k| \|f_n\|_p \\ &\le 2^{1/p}\varepsilon \sum_{|k|\le n_\varepsilon} |\psi_k| + 3 \sum_{|k|>n_\varepsilon} |\psi_k| \|F\|_p.\end{aligned}$$

Now (2.7) and (2.19) yield

$$\|(\Psi_n - \beta_\psi)d_{np}F\|_p \le (2^{1/p}\alpha_\psi + 3\|F\|_p)\varepsilon. \qquad (2.20)$$

It is easy to see that

$$\begin{aligned}\|A_k f_n\|_p &= \left(\sum_{t=1}^{k} |f_{nt}|^p\right)^{1/p} \le \|F\|_{p,(0,k/n)}, \quad k \le n; \\ \|A_k f_n\|_p &\le \|f_n\|_p \le \|F\|_p, \quad \forall k.\end{aligned}$$

Hence, by (2.13), (2.19), (2.6), and (2.7)

$$\begin{aligned}\|\Phi_n d_{np} F\|_p &\le \sum_{k=1}^{n_\varepsilon} |\psi_k| \|A_k f_n\|_p + \sum_{k>n_\varepsilon} |\psi_k| \|A_k f_n\|_p \\ &\le \sum_{k=1}^{n_\varepsilon} |\psi_k| \|F\|_{p,(0,k/n)} + \|F\|_p \sum_{k>\xi(\varepsilon)\eta(\varepsilon)} |\psi_k| \\ &\le \alpha_\psi \|F\|_{p,(0,\xi(\varepsilon))} + \|F\|_p \varepsilon \le (\alpha_\psi + \|F\|_p)\varepsilon.\end{aligned} \tag{2.21}$$

Following the same scheme,

$$\begin{aligned}\|B_k f_n\|_p &= \left(\sum_{t=n-k+1}^{n} |f_{nt}|^p \right)^{1/p} \le \|F\|_{p,(1-k/n,1)}, \quad k \le n; \\ \|B_k f_n\|_p &\le \|f_n\|_p \le \|F\|_p, \quad \forall k.\end{aligned}$$

Hence, by (2.15), (2.19), (2.6), and (2.7)

$$\begin{aligned}\|\Upsilon_n d_{np} F\|_p &\le \sum_{k=1}^{n_\varepsilon} |\psi_{-k}| \|B_k f_n\|_p + \sum_{k>n_\varepsilon} |\psi_{-k}| \|B_k f_n\|_p \\ &\le \sum_{k=1}^{n_\varepsilon} |\psi_{-k}| \|F\|_{p,(1-\xi(\varepsilon),1)} + \|F\|_p \sum_{k>\xi(\varepsilon)\eta(\varepsilon)} |\psi_{-k}| \\ &\le (\alpha_\psi + \|F\|_p)\varepsilon.\end{aligned} \tag{2.22}$$

Inequalities (2.20), (2.21), and (2.22) prove (2.9). ∎

Remark 2. 1. Because of (2.1), bound (2.9) does not imply convergence of the terms $\Psi_n d_{np} F$ and $\beta_\psi d_{np} F$. Denote $M_n = D_{np} \Psi_n d_{np}$. Inequality (2.9) implies

$$\|M_n F - \beta_\psi F\|_p \le \|D_{np}(\Psi_n - \beta_\psi) d_{np} F\|_p + |\beta_\psi| \|P_n F - F\|_p \to 0.$$

The $\lim_{n\to\infty} M_n$ is similar to the multiplier operator M in Fourier analysis defined by $(MF)(x) = \sum_{k\in Z} m_k c_k \exp(ikx)$ if a function F on the unit circle is decomposed as $F(x) = \sum_{k\in Z} c_k \exp(ikx)$ and $\{m_k\}$ is a given sequence of numbers.

3. PROPERTIES OF L_p-APPROXIMABLE SEQUENCES

DEFINITION 3.1. A sequence $\{w_n\}$, where $w_n \in R^n$ for all n, is called L_p-*approximable* if there exists a function $F \in L_p$ such that

$$\|w_n - d_{np} F\|_p \to 0, \quad n \to \infty. \tag{3.1}$$

If such is the case, the sequence $\{w_n\}$ is said to be *L_p-approximated by F*.

Note that if $p < \infty$, then (3.1) is equivalent to

$$\|D_{np}w_n - F\|_p \to 0, \quad n \to \infty. \tag{3.2}$$

Indeed, (3.1), (2.3), and (2.4) imply

$$\|D_{np}w_n - F\|_p \le \|D_{np}(w_n - d_{np}F)\|_p + \|P_nF - F\|_p \to 0.$$

Conversely, from (3.2), (2.3), and (2.4) one has

$$\|w_n - d_{np}F\|_p \le \|D_{np}w_n - F\|_p + \|F - P_nF\|_p \to 0.$$

THEOREM 3.1. (a) *If $p < \infty$ and $\{w_n\}$ is L_p-approximable, then*

$$\lim_{n\to\infty} \max_{1\le t\le n} |w_{nt}| = 0. \tag{3.3}$$

(b) *If $1 < p < \infty$, $\{x_n\}$ is L_p-approximated by X, and $\{y_n\}$ is L_p-approximated by Y, then*

$$\lim_{n\to\infty} \sum_{t=\kappa_n(a)}^{t=\kappa_n(b)} x_{nt}y_{nt} = \int_a^b X(s)Y(s)ds \ for\ all\ (a,b) \subset (0,1) \tag{3.4}$$

uniformly with respect to (a,b).

(c) *Let the partitions Π_n be uniform. If $p < \infty$, $\alpha_\psi < \infty$, and $\{w_n\}$ is L_p-approximable, then*

$$\lim_{n\to\infty} \max\left\{\|(\Psi_n - \beta_\psi)w_n\|_p,\ \|\Phi_n w_n\|_p,\ \|\Upsilon w_n\|_p\right\} = 0. \tag{3.5}$$

Proof. Statement (a) is a consequence of (2.1), (3.1), and $|w_{nt}| \le \|w_n - d_{np}F\|_p + |f_{nt}|$.

(b) Denote $\{a,b\} = \{t \in Z : \ \kappa_n(a) \le t \le \kappa_n(b)\}$. We are going to apply Theorem 2.1. Observe that

$$\chi = 1 \text{ on } \bigcup_{t=\kappa_n(a)+1}^{\kappa_n(b)-1} i_t, \quad \chi = 0 \text{ on } \left(\bigcup_{t=1}^{\kappa_n(a)-1} i_t\right) \bigcup \left(\bigcup_{t=\kappa_n(b)+1}^{n} i_t\right).$$

Therefore

$$\sum_{t=1}^{n}\big(d_{np}(\chi X)\big)_t\big(d_{nq}(\chi Y)\big)_t=\sum_{t\in\{a,b\}}(d_{np}X)_t(d_{nq}Y)_t$$
$$+\sum_{t=\kappa_n(a),\kappa_n(b)}\big(d_{np}(\chi X)\big)_t\big(d_{nq}(\chi Y)\big)_t-\sum_{t=\kappa_n(a),\,\kappa_n(b)}(d_{np}X)_t(d_{nq}Y)_t. \tag{3.6}$$

By $L_p(L_q)$-approximability and the Hölder inequality

$$\begin{aligned}&\Big|\sum_{t\in\{a,b\}}(d_{np}X)_t(d_{nq}Y)_t-\sum_{t\in\{a,b\}}x_{nt}y_{nt}\Big|\\ &\le\Big|\sum_{t\in\{a,b\}}\big((d_{np}X)_t-x_{nt}\big)(d_{nq}Y)_t\Big|+\Big|\sum_{t\in\{a,b\}}x_{nt}\big((d_{nq}Y)_t-y_{nt}\big)\Big|\\ &\le\|d_{np}X-x_n\|_p\|d_{nq}Y\|_q+\|x_n\|_p\|d_{nq}Y-y_n\|_q\to 0.\end{aligned}\tag{3.7}$$

Here we have used (2.2) and (3.1). By (2.1)

$$\max\left\{\left|\big(d_{np}(\chi X)\big)_t\right|,\ \left|(d_{np}X)_t\right|\right\}\le\max\left\{\|\chi X\|_{p,i_t},\ \|X\|_{p,i_t}\right\}\le\|X\|_{p,i_t}.$$

A similar bound holds for Y. Therefore of the three sums at the right of (3.6), the last two tend to zero uniformly with respect to a,b. Now (3.6), (3.7), and Theorem 2.1 imply (3.4).

(c) By (2.9), (2.12), and (3.1),

$$\begin{aligned}\|(\Psi_n-\beta_\psi)w_n\|_p&\le\|(\Psi_n-\beta_\psi)(w_n-d_{np}F)\|_p+\|(\Psi_n-\beta_\psi)d_{np}F\|_p\\ &\le 2\alpha_\psi\|w_n-d_{np}F\|_p+\|(\Psi_n-\beta_\psi)d_{np}F\|_p\to 0.\end{aligned}$$

The rest of the proof uses (2.14) and (2.16) and is equally simple. ▮

LEMMA 3.1. *Let the partitions Π_n be uniform and $p<\infty$.*

(a) *If $\{f_n\}$ is generated by $F\in L_p$, then*

$$\lim_{\delta\to 0,\,m\to\infty}\ \sup_{n\ge m,\,0<y\le\delta}\|(I_n^-)^{[yn]}f_n-f_n\|_p=0. \tag{3.8}$$

(b) *Conversely, suppose that a sequence $\{w_n\}$, such that $w_n\in R^n$ $\forall n$, satisfies*

$$\lim_{\delta\to 0,\,m\to\infty}\ \sup_{n\ge m,\,0<y\le\delta}\|(I_n^-)^{[yn]}w_n-w_n\|_p=0. \tag{3.9}$$

Then the functions $W_n=D_{np}w_n$ possess the property

$$\lim_{\delta\to 0}\sup_{n\ge 1}\omega_p(W_n,\delta)=0. \tag{3.10}$$

Proof. Let $0 < y \le \delta < 1$ and denote $k = [yn] \le yn < n$.
(a) By (2.17)

$$\|(I_n^-)^k f_n - f_n\|_p \le \left(\omega_p^p(F,\delta) + \|F\|_{p,(0,\delta)}^p\right)^{1/p}.$$

This implies

$$\lim_{\delta\to 0} \sup_{n\ge 1,\, 0<y\le\delta} \|(I_n^-)^{[yn]} f_n - f_n\|_p = 0$$

which is stronger than (3.8).

(b) The identity $\|F - \tau_y F\|_{p,\Delta_y} = \|F - \tau_{-y} F\|_{p,\Delta_{-y}}$ implies

$$\omega_p(F,\delta) = \sup_{0<y\le\delta} \|F - \tau_y F\|_{p,\Delta_y}, \quad \forall F \in L_p. \tag{3.11}$$

By the definition of k

$$k/n \le y < (k+1)/n. \tag{3.12}$$

Begin with

$$\|W_n - \tau_y W_n\|_{p,\Delta_y}^p = \sum_{t=1}^n \int_{i_t\cap\Delta_y} |W_n(x) - W_n(x+y)|^p dx. \tag{3.13}$$

Let $x \in i_t \cap \Delta_y$, that is

$$(t-1)/n \le x < t/n, \quad 0 < x < 1-y. \tag{3.14}$$

Inequalities (3.12) and (3.14) imply $(t+k-1)/n \le x+y < (t+k+1)/n$, that is, $x+y \in i_{t+k} \cup i_{t+k+1}$. Denote $c_t = n^{1/p} w_{nt}$ the value that W_n assumes on i_t, $t = 1, \dots, n$. Then $W_n(x+y)$ may take only values c_{t+k} and c_{t+k+1}. It follows that

$$\begin{aligned} &\int\limits_{i_t\cap\Delta_y} |W_n(x) - W_n(x+y)|^p dx \\ &= \int\limits_{i_t\cap i_{t+k}\cap\Delta_y} |c_t - c_{t+k}|^p dx + \int\limits_{i_t\cap i_{t+k+1}\cap\Delta_y} |c_t - c_{t+k+1}|^p dx. \end{aligned} \tag{3.15}$$

If $t+k \ge n+1$, then $(t+k-1)/n \ge 1 > 1-y$ and $i_{t+k} \cap \Delta_y = \emptyset$. This means that the first term at the right of (3.15) may be nonzero only if $t+k \le n$. Similarly, the second term may be nonzero only if $t+k+1 \le n$. Consequently, from (3.15)

$$\begin{aligned} &\int\limits_{i_t\cap\Delta_y} |W_n(x) - W_n(x+y)|^p dx \\ &\le \tfrac{1}{n}\{|c_t - c_{t+k}|^p 1(t+k \le n) + |c_t - c_{t+k+1}|^p 1(t+k+1 \le n)\}. \end{aligned}$$

This bound together with (3.13) and the definition of c_t's gives

$$\begin{aligned}\|W_n-\tau_yW_n\|_{p,\Delta_y}^p &\le \sum_{t=1}^{n-k}|w_{nt}-w_{n,t+k}|^p+\sum_{t=1}^{n-k-1}|w_{nt}-w_{n,t+k+1}|^p\\ &\le \|(I_n^-)^kw_n-w_n\|_p^p+\|(I_n^-)^{k+1}w_n-w_n\|_p^p.\end{aligned}$$

Taking into account the inequalities $(a^p+b^p)^{1/p}\le 2^{1/p}(a+b)$ and (see (2.10))

$$\begin{aligned}\|(I_n^-)^{k+1}w_n-w_n\|_p &\le \|(I_n^-)^k(I_n^-w_n-w_n)\|_p+\|(I_n^-)^kw_n-w_n\|_p\\ &\le \|I_n^-w_n-w_n\|_p+\|(I_n^-)^kw_n-w_n\|_p,\end{aligned}$$

we have

$$\|W_n-\tau_yW_n\|_{p,\Delta_y}\le 2^{1/p}\big(2\|(I_n^-)^kw_n-w_n\|_p+\|I_n^-w_n-w_n\|_p\big).$$

Combining this with (3.11) we get

$$\omega_p(W_n,\delta)\le 2^{1/p}\left(2\sup_{0<y\le\delta}\|(I_n^-)^{[yn]}w_n-w_n\|_p+\|I_n^-w_n-w_n\|_p\right). \tag{3.16}$$

Let us prove that (3.9) implies

$$\lim_{n\to\infty}\|(I_n^-)^kw_n-w_n\|_p=0 \tag{3.17}$$

for any natural k. By (3.9) for any $\varepsilon>0$ there exist $\delta>0$ and $m\ge 1$ such that

$$\|(I_n^-)^{[yn]}w_n-w_n\|_p<\varepsilon \quad \forall n\ge m, \quad \forall y\in(0,\delta].$$

For a natural k consider $n\ge m_0\equiv\max\{m,k/\delta\}$ and put $y=k/n\le\delta$. Then $[yn]=k$ and the preceding bound gives (3.17):

$$\|(I_n^-)^kw_n-w_n\|_p<\varepsilon \quad \forall n\ge m_0.$$

Choosing $k=1$ in (3.17), from (3.9) and (3.16) we see that for any $\varepsilon>0$ there exist $\delta>0$ and $m\ge 1$ such that $\omega_p(W_n,\delta)\le\varepsilon$ $\forall n\ge m$. Reducing δ, if necessary, we can satisfy also $\omega_p(W_n,\delta)\le\varepsilon$ for $n<m$. This proves the statement. ▮

Lemma 3.2. *Let $1<p<\infty$ and let $\{w_n\}$ be a sequence such that $w_n\in R^n$ $\forall n$. If some subsequence $\{w_{n_m}\}$ of $\{w_n\}$ is L_p-approximable, that is, $\|w_{n_m}-d_{n_m,p}F\|_p\to 0$ with some $F\in L_p$, then*

$$\sup_m\|w_{n_m}\|_p<\infty \tag{3.18}$$

and

$$\lim_{m\to\infty}\sum_{t=\kappa_{n_m}(a)}^{\kappa_{n_m}(b)} w_{n_m,t}|i_t|^{1/q}=\int_a^b F(s)ds. \tag{3.19}$$

Proof. L_p-approximability of the subsequence implies by (2.2)

$$\|w_{n_m}\|_p \le \|w_{n_m}-d_{n_m,p}F\|_p+\|F\|_p$$

which proves (3.18).

Note that Theorem 3.1(b) can be modified to read: if just subsequences $\{x_{n_m}\}$ and $\{y_{n_m}\}$ are $L_p(L_q)$-approximated by X and Y, respectively, then (3.4) is true with n replaced by n_m. Applying this statement to $x_{n_m}=w_{n_m}$ and $y_{n_m}=d_{n_m,q}1$, we obtain the lemma because $(y_{n_m})_t=|i_t|^{1/q}$ for all t. ▮

THEOREM 3.2. *Suppose all partitions Π_n are uniform. Let $1<p<\infty$ and let $\{w_n\}$ be a sequence of vectors such that $w_n\in R_p^n$ $\forall n$. Then $\{w_n\}$ is L_p-approximable if and only if the conditions*

$$\text{the limit } \lim_{n\to\infty} n^{-1/q}\sum_{t=[na]}^{[nb]} w_{nt} \text{ exists for any } 0<a<b<1, \tag{3.20}$$

$$\sup_n \|w_n\|_p<\infty \tag{3.21}$$

and (3.9) *hold.*

Proof. Necessity. If $\{w_n\}$ is L_p-approximable, then (3.20) and (3.21) follow from Lemma 3.2, the limit in (3.20) being uniform with respect to a and b.

By Lemma 3.1(a) for any $\varepsilon>0$ there exist δ and m such that

$$\sup_{n\ge m,\,0<y\le\delta}\|(I_n^-)^{[yn]}f_n-f_n\|_p<\varepsilon.$$

Due to L_p-approximability, the choice of m can also be subject to

$$\sup_{n\ge m}\|f_n-w_n\|_p<\varepsilon.$$

Therefore by (2.10) for $n\ge m$ and $0<y\le\delta$

$$\begin{aligned}\|(I_n^-)^{[yn]}w_n-w_n\|_p &\le \|(I_n^-)^{[yn]}(w_n-f_n)\|_p+\|(I_n^-)^{[yn]}f_n-f_n\|_p\\ &+\|f_n-w_n\|_p\le 2\|f_n-w_n\|_p+\|(I_n^-)^{[yn]}f_n-f_n\|_p\le 3\varepsilon.\end{aligned}$$

We have proved (3.9).

Sufficiency. Put $W_n = D_{np}w_n$. Conditions (3.9) and (3.21) by Lemma 3.1(b) and (2.3) imply

$$\sup_n \|W_n\|_p < \infty, \quad \lim_{\delta\to 0}\sup_{n\ge 1}\omega_p(W_n,\delta) = 0.$$

According to the Frechet-Kolmogorov theorem (see Iosida [9, Section X.1]) $\{W_n\}$ is precompact and there exist a subsequence $\{W_{n_k}\}$ and a function $F \in L_p$ such that $\|W_{n_k} - F\|_p \to 0$. Then $\{w_{n_k}\}$ is L_p-approximable and (3.19) is true. We need to show that the whole sequence $\{W_n\}$ converges to F. Suppose it does not. Then there exist another subsequence $\{X_{n_m}\}$, a number $\varepsilon > 0$ and a function $G \in L_p$ such that

$$\|X_{n_m} - F\|_p \ge \varepsilon, \quad \|X_{n_m} - G\|_p \to 0, \tag{3.22}$$

$$\lim_{m\to\infty} n_m^{-1/q} \sum_{t=[n_m a]}^{[n_m b]} w_{n_m,t} = \int_a^b G(x)dx \quad \forall (a,b) \subset (0,1). \tag{3.23}$$

By condition (3.20), Eqs. (3.19) and (3.23) entail

$$\int_0^1 (F-G)1\big((a,b)\big)dx = 0 \quad \forall (a,b) \subset (0,1).$$

Since the set of simple functions is dense in L_q, it follows that $F = G$ a.e. which contradicts (3.22). ∎

Remark 3. 1. Simple statistical problems require less properties of L_2-approximable sequences which (properties) one might want to impose directly. For example, the asymptotics of the ordinary least squares (OLS) estimator for the linear regression model $y = X\beta + e$ with a nonstochastic $n \times L$ matrix X is obtained under the conditions: (1) $\max_t |w_{nt}^l| \to 0$, (2) the limits $\lim_{n\to\infty}(w_n^k)'w_n^l$ exists, (3) $w_n^1, \ldots, w_n^L$ are asymptotically linearly independent, where w_n^l are normalized columns of X (see Anderson [1, Theorem 2.6.1]). Condition (1) coincides with (3.3), condition (2) is a very particular case of (3.4) (with $(a,b) = (0,1)$). Condition (3) for L_2-approximable $w_n^1, \ldots, w_n^L$ means that the corresponding generating functions are linearly independent. For more complex problems one would need more of the properties of L_2-approximable sequences. For example, we cannot indicate applications of (3.20) but it is derived from (3.4), and

(3.4) with arbitrary $(a,b) \subset (0,1)$ is used to prove convergence of (1.6). We also do not know applications of (3.9) but its corollary, (3.17), is used to study the asymptotics of the OLS estimator for an autoregressive model with exogenous nonstochastic regressors. The last two applications are from the author's unpublished work.

The next theorem contains easily verifiable conditions of practical interest.

THEOREM 3.3. *Let $p < \infty$.*

(a) *Suppose that for a given $\{w_n\}$ there exists $F \in L_\infty$ such that $\|D_{np}w_n - F\|_\infty \to 0$. Then $\{w_n\}$ is L_p-approximated by F.*

(b) *Let F be continuous on $[0,1]$ and suppose that a sequence $\{z_n\}$ satisfies $\max_{1\le t\le n} |z_{nt} - F(t/n)| \to 0,\ n \to \infty$. Denote $w_n = n^{-1/p}z_n$. Then $\{w_n\}$ is L_p-approximated by F.*

(c) *Let x_n be defined by one of the expressions*

(i) $x_n = (1^{k-1}, 2^{k-1}, \dots, n^{k-1})$, *$k$ is natural (polynomial trend),*

(ii) $x_n = (\ln^k 1, \dots, \ln^k n)$, *$k$ is natural (logarithmic trend),*

(iii) $x_n = (a^0, a^1, \dots, a^{n-1})$, *$a$ is real (geometric progression),*

(vi) $x_n = (e^a, \dots, e^{na})$, *$a$ is real (exponential trend).*

Put $w_n = x_n/\|x_n\|_p$. Then, respectively,

(i′) *$\{w_n\}$ is L_p-approximated by $F(x) = \big((k-1)p+1\big)^{1/p}x^{k-1}$,*

(ii′) *$\{w_n\}$ is L_p-approximated by $F(x) \equiv 1$ (for any natural k and $p < \infty$),*

(iii′) *$\{w_n\}$ is not L_p-approximable, unless $a = 1$,*

(vi′) *$\{w_n\}$ is not L_p-approximable, unless $a = 0$.*

Proof. (a) obviously follows from the equivalence of (3.1) and (3.2) where $\|D_{np}w_n - F\|_p \le \|D_{np}w_n - F\|_\infty$.

(b) By uniform continuity $\max_t \max_{x\in i_t} |F(t/n) - F(x)| \to 0,\ n \to \infty$. Since $D_{np}w_n = \sum_{t=1}^n z_{nt}1(i_t)$, we see that

$$\|D_{np}w_n - F\|_\infty = \max_{1\le t\le n} \max_{x\in i_t} |z_{nt} - F(x)|$$
$$\le \max_t |z_{nt} - F(t/n)| + \max_t \max_{x\in i_t} |F(t/n) - F(x)| \to 0,$$

and it remains to apply part (a).

The proof of part (c) given in the case $p = 2$ by Mynbaev and Castelar [17] easily generalizes for $p < \infty$. ∎

In Section 4 we shall need two-dimensional analogues of some of the facts obtained so far. Considering only uniform partitions of $(0,1)$, put

$$q_{st} = i_s \times i_t, \quad |q_{st}| = 1/n^2.$$

For a given $F \in L_p\big((0,1)^2\big)$, $d_{np}F$ is defined as a matrix with elements

$$(d_{np}F)_{st} = |q_{st}|^{-1/q} \int_{q_{st}} F(x,y)dxdy, \quad 1 \le s,t \le n, \tag{3.24}$$

and for a $n \times n$ matrix z, the function $D_{np}z$ is defined by

$$D_{np}z = \sum_{s,t=1}^{n} z_{st}|q_{st}|^{-1/p}1(q_{st}).$$

It is easy to check that

$$\|d_{np}F\|_p \le \|F\|_p, \quad \|D_{np}z\|_p = \|z\|_p \tag{3.25}$$

and that $D_{np}d_{np} = P_n$ where

$$P_nF = \sum_{s,t=1}^{n} |q_{st}|^{-1} \int_{q_{st}} F(x,y)dxdy1(q_{st}).$$

P_n possesses the property

$$\|P_nF - F\|_p \le 4^{1/p}\omega_p(F, \sqrt{2}/n), \quad \|P_nF\|_p \le \|F\|_p$$

(with the definition of the continuity modulus properly modified). A sequence of matrices $\{w_n\}$, where w_n is $n \times n$ for all n, is L_p-approximable if there exists a function $F \in L_p\big((0,1)^2\big)$ such that

$$\|w_n - d_{np}F\|_p \to 0. \tag{3.26}$$

THEOREM 3.4. *Let all partitions be uniform.*
(a) *If* $p < \infty$, *then* (3.26) *is equivalent to*

$$\|D_{np}w_n - F\|_p \to 0. \tag{3.27}$$

(b) *If* F *is symmetric,* $F(x,y) = F(y,x)$ *for all* $(x,y) \in (0,1)^2$, *then* $d_{np}F$ *is symmetric.*

(c) *In order to distinguish the two- and one-dimensional cases, denote* (3.24) *by* d^2_{np} *and its one-dimensional cousin from Section 2 by* d^1_{np}. *If* $F(x,y) = G(x)H(y)$, *then* $(d^2_{np}F)_{st} = (d^1_{np}G)_s(d^1_{np}H)_t$ *for all* s,t.

(d) *Let* $\{B_n\}$ *be a sequence of matrices such that* B_n *is of dimension* $n \times n$ *for all* n *and there exists a continuous function* $K \in C([0,1]^2)$ *such that*

$$\max_{1 \le s,t \le n} |B_{nst} - K(s/n, t/n)| \to 0. \tag{3.28}$$

Put $w_n = n^{-2/p} B_n$. *If* $p < \infty$, *then* w_n *is* L_p*-approximated by* K.

Proof. (a) Equivalence of (3.26) and (3.27) is proved as in the one-dimensional case.

(b) Observe that $(x, y) \in q_{st}$ if and only if $(y, x) \in q_{ts}$ and therefore $(d_{np}F)_{st} = (d_{np}F)_{ts}$ for all s, t.

(c) This is straightforward.

(d) This is proved as Theorem 3.3(b). ∎

4. APPLICATIONS

Suppose that the e_{nt} in (1.7) are martingale differences with respect to σ-fields G_{nt}, the v_{nt} are defined by (1.7) with a summable sequence $\{\psi_j\}$, W_n is a $n \times L$ matrix whose columns are denoted w_n^l, and the lth column w_n^l is L_2-approximated by a function $F_l \in L_2(0,1)$, $l = 1, \dots, L$. Denote

$$e_n = (e_{n1}, \dots, e_{nn})', \; v_n = (v_{n1}, \dots, v_{nn})', \; V = \left(\int_0^1 F_k(x) F_l(x) dx \right)_{k,l=1}^L .$$

THEOREM 4.1. *Suppose that*

(A) $\{w_n^l\}$ *is* L_2*-approximated by* F_l, $l = 1, \dots, L$, *where the functions* $F_1, \dots, F_L$ *are linearly independent,*

(B) $E(e_{nt}^2 | G_{n,t-1}) = \sigma^2$ *for all* t, n *and the* e_{nt}^2 *are uniformly integrable. Then*

$$W_n' e_n \xrightarrow{d} N(0, \sigma^2 V), \quad \lim_{n\to\infty} \mathrm{Var}(W_n' e_n) = \sigma^2 V. \tag{4.1}$$

If, additionally,

(C) $\alpha_\psi < \infty$ *and* $\beta_\psi \neq 0$,

(D) *all partitions are uniform,*

then

$$W_n' v_n \xrightarrow{d} N(0, (\sigma\beta_\psi)^2 V), \quad \lim_{n\to\infty} \mathrm{Var}(W_n' v_n) = (\sigma\beta_\psi)^2 V. \tag{4.2}$$

Proof of (4.1). By Theorem 3.1(b)

$$\lim_{n\to\infty} W_n' W_n = \lim_{n\to\infty} \left((w_n^k)' w_n^l\right)_{k,l=1}^L = V. \tag{4.3}$$

The martingale difference definition, law of iterated expectations, and condition (B) imply

$$Ee_n = 0, \quad Ee_n e_n' = \sigma^2 I_n. \tag{4.4}$$

Since $EW_n' e_n = 0$, $EW_n' e_n e_n' W_n = \sigma^2 W_n' W_n$, (4.3) proves the second equation in (4.1).

By the Cramér-Wold theorem the first relation in (4.1) will follow if we establish

$$a' W_n' e_n \xrightarrow{d} N(0, \sigma^2 a' V a) \tag{4.5}$$

for any $a \in R^L$, $a \neq 0$ (all vectors are written as columns). By (4.4)

$$Ea' W_n' e_n = 0, \quad E(a' W_n' e_n)^2 = \sigma^2 a' W_n' W_n a.$$

Equation (4.3) and linear independence of $F_1, \dots, F_L$ imply

$$\lim_{n\to\infty} a' W_n' W_n a = a' V a \neq 0. \tag{4.6}$$

Hence, for all sufficiently large n we may define

$$c_{nt} = \sum_{l=1}^L a_l w_{nt}^l (\sigma^2 a' W_n' W_n a)^{-1/2}, \quad S_n = \sum_{t=1}^n c_{nt} e_{nt}.$$

With these definitions

$$ES_n = 0, \quad ES_n^2 = 1, \quad S_n = a' W_n' e_n (\sigma^2 a' W_n' W_n a)^{-1/2}. \tag{4.7}$$

By (3.3) with some $c_1 > 0$

$$\max_t |c_{nt}| \leq c_1 \max_{t,l} |w_{nt}^l| \to 0. \tag{4.8}$$

Besides,

$$\sum_{t=1}^n c_{nt}^2 = \sum_{k,l=1}^L a_k a_l (w_n^k)' w_n^l (\sigma^2 a' W_n' W_n a)^{-1} = \sigma^{-2}. \tag{4.9}$$

Let us show that

$$\operatorname{plim} \sum_{t=1}^{n} c_{nt}^2 e_{nt}^2 = 1. \tag{4.10}$$

By condition (B), $\{(e_{nt}^2 - \sigma^2)c_{nt}^2, G_{nt}\}$ is an m.d. array and the functions $e_{nt}^2 - \sigma^2$ are uniformly integrable. This fact, (4.9), and

$$\sum_{t=1}^{n} c_{nt}^4 \le \max_t c_{nt}^2 \sum_{t=1}^{n} c_{nt}^2 \to 0$$

allow us to apply the Chow-Davidson theorem (see Davidson [6, Theorem 19.7]) to yield $\| \sum_t (e_{nt}^2 - \sigma^2)c_{nt}^2 \|_1 \to 0$. By the Chebyshev inequality and (4.9) this implies (4.10).

Now we verify that

$$\operatorname{plim} \max_{1 \le t \le n} |c_{nt} e_{nt}| = 0. \tag{4.11}$$

By uniform integrability, for any $\varepsilon > 0$ one can choose $M > 0$ such that

$$\sup_{t,n} E e_{nt}^2 1(|e_{nt}| > M) \le \varepsilon \sigma^2.$$

From (4.8) one can see that there exists n_0 such that

$$M^2 \max_t c_{nt}^2 \le \varepsilon, \quad n \ge n_0.$$

Denote

$$A_0 = \emptyset,\ A_t = \left\{ |c_{nt} e_{nt}| = \max_t |c_{nt} e_{nt}| \right\} \setminus \bigcup_{j=0}^{t-1} A_j,\ B_t = \{|e_{nt}| > M\}, \quad t = 1, \dots, n.$$

Then $A_1, \dots, A_n$ form a disjoint covering of Ω and

$$\begin{aligned} E \max\nolimits_t |c_{nt} e_{nt}|^2 &= \sum_{t=1}^{n} E c_{nt}^2 e_{nt}^2 1(A_t) \\ &= \sum_{t=1}^{n} c_{nt}^2 E e_{nt}^2 1(A_t \cap B_t) + \sum_{t=1}^{n} c_{nt}^2 E e_{nt}^2 1(A_t \setminus B_t) \\ &\le \sum_{t=1}^{n} c_{nt}^2 E e_{nt}^2 1(|e_{nt}| > M) + \max_t c_{nt}^2 M^2 \sum_{t=1}^{n} E 1(A_t) \le 2\varepsilon,\ n \ge n_0. \end{aligned}$$

This bound proves (4.11).

Equations (4.7), (4.10), and (4.11) allow us to apply the McLeish theorem (see McLeish [10] or Davidson [6, Theorem 24.3]). Hence, $S_n \xrightarrow{d} N(0,1)$. By the Cramér theorem, this convergence in combination with (4.6) and (4.7) leads to (4.5).

Proof of (4.2). By (4.1), $(\beta_\psi W_n)' e_n \xrightarrow{d} N(0, (\sigma\beta_\psi)^2 V)$. Hence, to prove the first relation in (4.2) it suffices to show that

$$\operatorname{plim}\big(W_n' v_n - (\beta_\psi W_n)' e_n\big) = 0. \tag{4.12}$$

Using (1.7) and the definitions of $\Psi_n,\ \Phi_n, \Upsilon_n$, it is easy to derive the identity

$$\sum_{t=1}^n z_t v_{nt} = \sum_{t=1}^n e_{nt}(\Psi_n z)_t + \sum_{t<1} e_{nt}(\Phi_n z)_t + \sum_{t>n} e_{nt}(\Upsilon_n z)_t, \tag{4.13}$$

where $z \in R^n$. Therefore the lth component of the vector $\rho \equiv W_n' v_n - (\beta_\psi W_n)' e_n$ equals

$$\begin{aligned} \rho_l &= \sum_{t=1}^n w_{nt}^l v_{nt} - \beta_\psi \sum_{t=1}^n w_{nt}^l e_{nt} \\ &= \sum_{t=1}^n e_{nt}\big((\Psi_n - \beta_\psi) w_n^l\big)_t + \sum_{t<1} e_{nt}(\Phi_n w_n^l)_t + \sum_{t>n} e_{nt}(\Upsilon_n w_n^l)_t. \end{aligned}$$

Hence, by orthogonality of m.d.'s and Theorem 3.1(c)

$$E\rho_l^2 = \sigma^2\big(\|(\Psi_n - \beta_\psi) w_n^l\|_2^2 + \|\Phi_n w_n^l\|_2^2 + \|\Upsilon_n w_n^l\|_2^2\big) \to 0$$

which proves (4.12).

Now we turn to the second relation in (4.2). The matrix $Ev_n v_n'$ has as its elements

$$Ev_{ns}v_{nt} = \sum_{i,j\in Z} \psi_i \psi_j E e_{n,s-i} e_{n,t-j} = \sigma^2 \sum_{j \in Z} \psi_{s-j}\psi_{t-j}.$$

Therefore the elements of $\operatorname{Var}(W_n' v_n) = W_n' E v_n v_n' W_n$ are

$$\begin{aligned} \sum_{s,t=1}^n w_{ns}^k w_{nt}^l E v_{ns} v_{nt} &= \sigma^2 \sum_{j\in Z} \sum_{s,t=1}^n w_{ns}^k w_{nt}^l \psi_{s-j}\psi_{t-j} \\ &= \sigma^2 \sum_{j \in Z} \left(\sum_{s=1}^n w_{ns}^k \psi_{s-j}\right)\left(\sum_{t=1}^n w_{nt}^l \psi_{t-j}\right) \\ &= \sigma^2 \big\{(\Psi_n w_n^k, \Psi_n w_n^l) + (\Phi_n w_n^k, \Phi_n w_n^l) + (\Upsilon_n w_n^k, \Upsilon_n w_n^l)\big\}. \end{aligned} \tag{4.14}$$

($(\cdot,\cdot)$ denotes the scalar product in l_2). By Theorem 3.1(c)

$$|(\Phi_n w_n^k, \Phi_n w_n^l)| \le \|\Phi_n w_n^k\|_2 \|\Phi_n w_n^l\|_2 \to 0,$$

$$|(\Upsilon_n w_n^k, \Upsilon_n w_n^l)| \le \|\Upsilon_n w_n^k\|_2 \|\Upsilon_n w_n^l\|_2 \to 0,$$

$$|(\Psi_n w_n^k, \Psi_n w_n^l) - \beta_\psi^2 (w_n^k, w_n^l)|$$

$$\le \left|\left((\Psi_n - \beta_\psi) w_n^k, \Psi_n w_n^l\right)\right| + \left|\left(\beta_\psi w_n^k, (\Psi_n - \beta_\psi) w_n^l\right)\right|$$

$$\le \|(\Psi_n - \beta_\psi) w_n^k\|_2 \|\Psi_n w_n^l\|_2 + |\beta_\psi| \|w_n^k\|_2 \|(\Psi_n - \beta_\psi) w_n^l\|_2 \to 0.$$

In the last line we have also used (2.12) and (3.21). Taking into account (4.3), we see that the limit of (4.14) equals $(\sigma\beta_\psi)^2 \int_0^1 F_k F_l dx$. ▮

Remark 4. 1. Some econometrics papers contain CLT's as intermediate steps. All of them impose regularity conditions on the weights that are stronger than L_2-approximability, as it was mentioned in Introduction. The first statement of the theorem, (4.1), holds true if (A) is replaced by Anderson's conditions (1)-(3) cited in Remark 3.1. The novelty of (4.2) is that it has been justified in the entire class L_2 with the basic variables of form (1.7). The Beveridge and Nelson [4] decomposition (see also Hamilton [8, Chap. 17]) is obtained from (4.13) by choosing $z_1 = \ldots = z_n = 1$, $\psi_j = 0$ for $j < 0$.

In Theorem 4.2 below we shall need the following assumption:

(E) $\{w_n\}$ *is a sequence of matrices such that* w_n *is* $n \times n$ *for all* n *and there exists a symmetric function* $K \in L_2\big((0,1)^2\big)$ *such that*

$$\|w_n - d_{n2} K\|_2 = o(1/n) \tag{4.15}$$

and the integral operator

$$(\mathcal{K} f)(x) = \int_0^1 K(x,y) f(y) dy$$

is nuclear.

Under this condition $\mathcal{K}$ is self-adjoint and compact. Let $\{\lambda_i : \ i \ge 1\}$ and $\{f_i : \ i \ge 1\}$ be its systems of eigenvalues and eigenfunctions, respectively, such that $\mathcal{K} f_i = \lambda_i f_i$. The eigenvalues are real and listed according to their multiplicity; the system of eigenfunctions is complete and orthonormal in $L_2(0,1)$. The square-integrability of K implies that $\mathcal{K}$ belongs to the Hilbert-Schmidt class, that is

$$\sum_{i\geq 1}\lambda_i^2<\infty \tag{4.16}$$

and that the kernel can be decomposed as

$$K(x,y)=\sum_{i\geq 1}\lambda_i f_i(x)f_i(y)$$

with the series converging in $L_2\big((0,1)^2\big)$. The nuclearity assumption means that

$$\sum_{i\geq 1}|\lambda_i|<\infty \tag{4.17}$$

which is stronger than (4.16). The last condition ensures convergence of the infinite product

$$D(z)=\prod_{i=1}^{\infty}(1-z\lambda_i),$$

called a Fredholm determinant. $D(z)$ is an entire function with zeros at $1/\lambda_i$. See Gohberg and Kreĭn [7] for more information.

Theorem 4.2. *Suppose that conditions* (B), (C), (D) *of Theorem* 4.1 *and condition* (E) *above hold. Then the quadratic form* $Q_n(w_n)\equiv v_n' w_n v_n$ *converges in distribution to* $(\sigma\beta_\psi)^2\sum_{i\geq 1}\lambda_i u_i^2$ *where* $u_i\in N(0,1)$ *are independent.*

Proof. First consider convergence of $Q_n(d_{n2}K)$. We approximate it by $Q_n(d_{n2}K_L)$ where $K_L(x,y)=\sum_{i=1}^L\lambda_i f_i(x)f_i(y)$. By Theorem 3.4(c)

$$(d_{n2}^2K-d_{n2}^2K_L)_{st}=\sum_{i>L}\lambda_i(d_{n2}^1f_i)_s(d_{n2}^1f_i)_t.$$

Therefore

$$\begin{aligned}Q_n(d_{n2}^2K)-Q_n(d_{n2}^2K_L)&=\sum_{i>L}\lambda_i\sum_{s,t=1}^n(d_{n2}^1f_i)_s v_{ns}(d_{n2}^1f_i)_t v_{nt}\\&=\sum_{i>L}\lambda_i\left(\sum_{t=1}^n(d_{n2}^1f_i)_t v_{nt}\right)^2.\end{aligned}$$

From (4.13), (2.8), and (2.2) by orthogonality of m.d.'s

$$E\left(\sum_{t=1}^{n}(d_{n2}f_i)_t v_{nt}\right)^2 = \sigma^2\{\|\Psi_n d_{n2}f_i\|_2^2 + \|\Phi_n d_{n2}f_i\|_2^2 + \|\Upsilon_n d_{n2}f_i\|_2^2\}$$
$$\le 3(\sigma\alpha_\psi)^2\|d_{n2}f_i\|_2^2 \le 3(\sigma\alpha_\psi)^2\|f_i\|_2^2 = 3(\sigma\alpha_\psi)^2.$$

Hence,

$$E|Q_n(d_{n2}K) - Q_n(d_{n2}K_L)| \le 3(\sigma\alpha_\psi)^2\sum_{i>L}|\lambda_i| \to 0, \quad L\to\infty,$$

and $\text{plim}E|Q_n(d_{n2}K) - Q_n(d_{n2}K_L)| = 0$ uniformly with respect to n.

By Theorem 4.1 and orthonormality of $\{f_i\}$ for any L

$$\begin{pmatrix} \sum_{t=1}^n (d_{n2}f_1)_t v_{nt} \\ \dots \\ \sum_{t=1}^n (d_{n2}f_L)_t v_{nt} \end{pmatrix} \xrightarrow{d} N\big(0, (\sigma\beta_\psi)^2 I_L\big).$$

By the continuous mapping theorem then

$$\begin{aligned} Q_n(d_{n2}^2K_L) &= \sum_{i=1}^{L}\lambda_i\sum_{s,t=1}^{n}(d_{n2}^1 f_i)_s v_{ns}(d_{n2}^1 f_i)_t v_{nt} \\ &= \sum_{i=1}^{L}\lambda_i\left(\sum_{t=1}^{n}(d_{n2}^1 f_i)_t v_{nt}\right)^2 \xrightarrow{d} (\sigma\beta_\psi)^2\sum_{i=1}^{L}\lambda_i u_i^2. \end{aligned}$$

where $u_i \in N(0,1)$ are independent. Because of condition (4.17) the variables $\sum_{i=1}^L \lambda_i u_i^2$ converge to $\sum_{i=1}^\infty \lambda_i u_i^2$ in L_1 and in distribution. We have verified all the conditions of Anderson [1, Theorem 7.7.1] wherefrom

$$Q_n(d_{n2}K) \xrightarrow{d} (\sigma\beta_\psi)^2\sum_{i=1}^{\infty}\lambda_i u_i^2.$$

By the Cauchy-Schwarz inequality

$$|Q_n(w_n) - Q_n(d_{n2}K)| = |v_n'(w_n - d_{n2}K)v_n|$$

$$\le \|w_n - d_{n2}K\|_2\left(\sum_{s,t=1}^{n} v_{ns}^2 v_{nt}^2\right)^{1/2} = \|w_n - d_{n2}K\|_2\sum_{t=1}^{n} v_{nt}^2.$$

Here

$$E\sum_{t=1}^{n} v_{nt}^2 = \sum_{t=1}^{n}\sum_{i,j\in Z}E(e_{n,t-i}\psi_i e_{n,t-j}\psi_j) = \sigma^2\sum_{t=1}^{n}\sum_{j\in Z}\psi_j^2 = \sigma^2 n\sum_{j\in Z}\psi_j^2.$$

Therefore (4.15) implies

$$E|Q_n(w_n) - Q_n(d_{n2}K)| \le cn\|w_n - d_{n2}K\|_2 \to 0.$$

This proves that $Q_n(w_n)$ has the same limit in distribution as $Q_n(d_{n2}K)$. ▮

Remark 4. 2. Our conditions on the integral operator are much weaker and the error structure is more general than in Nabeya and Tanaka [19]. They impose the following condition on the matrices. Assuming continuity of the kernel, instead of $Q_n(w_n)$ they consider $\bar{Q}_n(B_n) = (1/n)v_n'B_nv_n$ where the matrix B_n satisfies (3.28). This condition in general is not comparable to ours: (3.28) is not applicable to nonsmooth kernels, and for continuous ones, by Theorem 3.4(d), (3.28) implies just (3.26) with $p = 2$ and $w_n = n^{-1}B_n$ instead of (4.15). The result itself can be expressed in other useful forms, see Nabeya and Tanaka [19]. In particular, the characteristic function of $(\sigma\beta_\psi)^2\sum_{i=1}^\infty \lambda_i u_i^2$ equals $\left(D(2it(\sigma\beta_\psi)^2)\right)^{-1/2}$ for $t \in R$ (see Anderson and Darling [2]).

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