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ABSTRACT

In this note we challenge the non-cooperative foundations of cooperative bargaining solutions on the grounds that the limit operation for approaching a frictionless world is not robust. We show that when discounting almost ceases to play a role, any individually rational payoff can be supported by some subgame perfect equilibrium. To select the "correct" point imposes excessive informational requirements on the analyst.

1. INTRODUCTION

The Strategic Approach to bargaining analyzes such models by means of non-cooperative game theory. The seminal contribution by Rubinstein (1982) considered a two players infinite horizon model with alternating offers which possesses a unique subgame perfect equilibrium as long as players are not infinitely patient (variants of this model are considered in Ståhl (1972)) and Binmore & Dasgupta (1987)). A celebrated result (see Shaked & Sutton (1984), Binmore, Rubinstein & Wolinsky (1986)), establishes that, if the common discount factor tends to one, equilibrium payoffs tend to an equal split of the pie. This has been viewed as providing a non-cooperative foundation of cooperative bargaining solutions such as the Nash Bargaining solution. Analogous results can also be proved if the proposer is chosen by nature in each bargaining round (see Rubinstein & Wolinsky (1985), Binmore (1987)). The latter is what we refer to as the random proposer model[1].

In this paper we show that the limit result is an artifact of the particular discount parameter sequence that has been used. If, instead of supposing identical discount parameters for the two players, a slightly different sequence, converging to the same limit, is considered, any split of the pie can be generated as a subgame perfect equilibrium outcome in both the alternating offer and the random proposer model. We emphasize that the sequence used to prove our result can be chosen *arbitrarily close* to the one considered by Shaked & Sutton (1984) and Binmore, Rubinstein & Wolinsky (1986)

One motivation for our note is the fact that the equal split solution selects only one point among a continuum of limit points, since if the common discount factor is 1 any strictly positive payoff can be supported as a

(1) The determinacy of the solution in all these models contrasts with Folk Theorems for infinitely repeated games (see Friedman (1971), Fudenberg & Maskin (1986)) in which any individually rational payoff is supportable by a subgame perfect equilibrium, if the discount factor is sufficiently close to one. This discrepancy of results is due to the fact that bilateral bargaining models do not have a supergame structure.

subgame perfect equilibrium (see Binmore, Rubinstein & Wolinsky (1986), Proposition 1). Thus, the only argument in favor of the equal split is the observation that it can be approximated by the outcome under sufficiently large interior uniform discount factors. In other words, the correspondence which maps the common discount factor into equilibrium payoffs is not lower hemi-continuous at 1. However, this discontinuity is only an artifact of the particular sequence which is typically chosen. If discount factors of players are *just slightly different*, it is possible to approach *any* strictly positive payoff. In other words, the correspondence mapping *pairs* of discount factors to equilibrium payoffs *is* continuous. Therefore, the asymptotic non-cooperative foundation of the cooperative Nash bargaining solution cannot be regarded as robust, since in a neighbourhood of the point of no discounting the non-cooperative bargaining model with an infinite time horizon loses all its predictive power. In this sense, we are back to the old presumption, challenged by the original limit results, that the outcome of bilateral monopoly is undetermined in a frictionless world.

For the case of three bargaining partners it has already been shown that with sufficient patience any payoff vector can be supported by some subgame perfect equilibrium (Osborne & Rubinstein (1990) credit this result to A. Shaked, see also Herrero (1985)). The nature of the argument is, however, different from the one employed here. The three players example uses non-stationary strategies to establish multiplicity. In this note we will (have to) use the unique (stationary) strategies from Rubinstein's original analysis. Thus our argument is not based on adding strategic options by introducing a third player, but on a careful way of approaching limits.

The phenomenon encountered here may in fact be generated by the infinite time horizon rather than by the special structure of bargaining models. An analogous result was first discovered by Güth & Ritzberger (Jan. 1992) within the context of durable goods monopolies with an infinite time horizon. There, as well as here, it turns out that with an infinite time horizon the slightest difference between the players' time preferences has an enormous impact on the equilibrium, such that when discounting is small anything may happen.

The rest of the paper is as follows. In Section 2 we sketch both the alternating offer and the random proposer model and we prove our main result for these two models. Section 3 is a (somewhat technical) generalization of the previous result. Finally, Section 4 gathers our final comments.

2. THE MAIN RESULT IN TWO BILATERAL BARGAINING MODELS

We consider two models in the tradition of Rubinstein (1982). In both models two players, named 1 and 2, negotiate on how to share a pie of unit size. Both players are risk neutral and like pie. Player 1 (resp. 2) discounts future payoffs by a discount factor ρ (resp. δ) $\in [0, 1]$. Time is divided into discrete periods and is assumed to be of infinite length. The last assumption has the advantage that it removes one potential friction, namely a finite horizon, which may otherwise bias the solution via backward induction. Hence the only remaining friction is that players are not infinitely patient. In every period the two players engage in a constituent extensive form game which will be referred to as a bargaining round. The two models differ with respect to how bargaining rounds are organized.

In the version of the model which corresponds to Rubinstein's (1982) original contribution player 1 starts in the first round by making a proposal, denoted by $x \in [0, 1]$, on how to share the pie between him (with share x) and player 2 (with share $1 - x$). Upon hearing player 1's proposal, player 2 then decides whether to accept it, in which case the game ends and the proposal is implemented, or to reject it. In the latter case, a new bargaining round is entered with the roles of players reversed. We will refer to this version as the *alternating offer model*.

In the other version of the model a chance move at the beginning of each round decides whether player 1 (with probability $\alpha \in (0, 1)$) or player 2 (with probability $1 - \alpha$) is the proposer in this round. The rest works as in the previous model except that the roles of the players are never reversed. We will refer to this version as the *random proposer model*.

It is not difficult to show that the equilibrium payoffs from the unique subgame perfect equilibrium for the alternating offer model are

$v_1 = (1 - \delta)/(1 - \delta\rho)$ (for player 1) and $v_2 = \delta(1 - \rho)/(1 - \delta\rho)$ (for player 2)

and for the random proposer model expected equilibrium payoffs are (see Appendix)

$$u_1 = \alpha(1-\delta)/(1-(1-\alpha)\rho - \alpha\delta) \quad \text{and} \quad u_2 = (1 - \alpha)(1 - \rho)/(1-(1-\alpha)\rho - \alpha\delta).$$

Let us introduce some notation. A *smooth path* is a pair of $\mathbb{C}^\infty$ functions $(\delta(t), \rho(t))$ such that $\delta, \rho : \mathbb{R}_{++} \longrightarrow (0, 1)$ and $\lim_{t \to \infty} \delta(t) = \lim_{t \to \infty} \rho(t) = 1$. In the following theorem w will represent the predetermined share of the pie of player 1, ε the maximum distance between $\delta(t)$ and $\rho(t)$, and $u(\delta(t), \rho(t))$ the utility enjoyed by player 1 in the subgame perfect equilibrium when discount factors are $\delta(t)$ and $\rho(t)$.

Theorem 1. *Given $w \in (0, 1)$ and $\varepsilon > 0$, $\exists$ a smooth path $(\rho(t), \delta(t))$ such that*

a) $| \rho(t) - \delta(t) | < \varepsilon \quad \forall t \in \mathbb{R}_{++}$ *and*

b) $u(\delta(t), \rho(t)) = w \quad \forall t \in \mathbb{R}_{++}$.

Proof: *We will start with the proof for the random proposer model. First, let us fix $\varepsilon > 0$ and define the smooth path by*

$$\rho(t) = 1 - \varepsilon\, \alpha\, (1 - w)\, e^{-t} \text{ and } \delta(t) = 1 - \varepsilon\, (1 - \alpha)\, w\, e^{-t}.$$

Clearly $| \rho(t) - \delta(t) | = \varepsilon\, e^{-t} | w - \alpha | < \varepsilon$ *and* $u(\delta(t), \rho(t)) = w$.

For the alternating offer model define the smooth path by

$$\rho(t) = 1 - \varepsilon\, (1 - w)\, e^{-t} \text{ and } \delta(t) = 1 / (1 + \varepsilon\, w\, e^{-t}). \text{ Thus}$$

$$| \rho(t) - \delta(t) | < (1 - \rho(t))/(1 - w\, \rho(t)) \le (1 - \rho(t))/(1 - w) = \varepsilon\, e^{-t} < \varepsilon$$

and again $u(\delta(t), \rho(t)) = w$.■

As the Theorem states, discount factors supporting an arbitrary equilibrium payoff can be chosen arbitrarily close to each other. The asymptotic equal split result quoted above arises only when limits are taken *exactly along the diagonal.* Once slightly different sequences are considered (but still along perfectly smooth paths) the whole interval between zero and one can be traced out as the limit set. Geometrically speaking, equilibrium payoffs as a function of δ and ρ are a continuous correspondence which is set valued only at the point (1, 1), where its value is the whole interval.

Notice that in Theorem 1, in the case of the random proposer model, the particular sequences chosen to do the job are such that $(1 - \delta)/(1 - \rho)$ is constant along the smooth path. This has the advantage of interpreting the sequence as arising from games in which there is a shorter and shorter delay between offers (i.e. bargaining gets more and more intense) *in a linear way*. In the random proposer model the required sequence could also been chosen such that the previous ratio remains constant along the smooth path.

3. A GENERALIZATION

One may think of the explicit formulae for the players' shares in equilibrium as an equilibrium outcome correspondence mapping the frictions ρ and δ into equilibrium outcomes. The two cornerstones of our result in Section 2 are that

1) the interior of the simplex of the players' shares is contained in the value of the equilibrium outcome correspondence at the point of no frictions, and
2) the equilibrium outcome correspondence is lower hemi-continuous at the point of no frictions.

In this Section we will generalize and sharpen the result from Section 2 for a whole class of games which encompasses many non-cooperative models of bargaining on the division of a unit pie. The emphasis will be on how one obtains predictions on a frictionless world from knowledge on a world with frictions. In our view this is what the "non-cooperative foundations of the cooperative (Nash-bargaining) solution" attempt to do.

For concreteness again consider the alternating offer bargaining model by Rubinstein (1982). But now imagine that the delay between successive bargaining rounds depends on the identity of the responder. Humans tend to have different reaction times under different technologies. Let player *i's, i = 1,2, reaction time* be the time span that it takes player i to respond to an offer with either an acceptance or a counteroffer, and denote this time span by Δ_i. Thus in a subgame that starts with an offer by player i the closest time when a payoff can be had is Δ_j, $j \neq i$, time units in the future. Alternatively one may think of Δ_i, $i = 1, 2$, as the length of time for which a player can commit himself (Sutton (1986) p. 712). The vector $(\Delta_1,\Delta_2) \in \mathcal{R}^2_+$ represents the frictions in this game. But these frictions depend on the technology available to the players, i.e. the players' reaction times will vary with the available technology for computations and communication. Since in the real world only technologies with non-vanishing frictions are feasible, $(\Delta_1, \Delta_2) \in \mathcal{R}^2_{++}$, and an analyst attempting to generate predictions for a frictionless world must be content with extrapolating observations in a world with frictions to the limiting point where frictions vanish.

Still more concretely imagine that experiments are run with the same two players under different technologies that are continually upgraded. First players are only equipped with paper and pencils, to do their calculations, and a messenger service that carries their letter back and forth. Then players are given pocket calculators, but still have to use the mail service. Then the mail is substituted for by fax machines. Eventually players are given PC's instead of pocket calculators, the PC's being equipped with e-mail. Then telephones are introduced, and so on. Given player-specific skills the reaction times of players will vary with the technology. But, if the game has a unique solution for non-vanishing frictions, the analyst can reconstruct *how* the reaction times vary with the technology.

To help the analyst we will allow her to take uncountable infinitely many observations (and technologies) such that she can reconstruct a continuous path on how frictions $\Delta \in \mathcal{R}^2_+$ vanish. The formal reason for this is that the analyst will eventually have to apply l'Hospital's rule which is inapplicable with only countably many observations. However, when in the end the analyst comes up with a single-valued prediction on how the pie will be shared under a perfect computation and communication technology ($\Delta = 0$), we may still not trust her. And we will now explain why.

Consider a class of n-person games $\Gamma(\Delta)$, parameterized by a vector of frictions $\Delta \in \mathcal{R}^n_+$, together with an equilibrium outcome correspondence $F: \mathcal{R}^n_+ \longrightarrow S^{n-1}$, where S^{n-1} is the (n-1)-dimensional simplex. The class of games $\Gamma(\Delta)$ is characterized by the following assumptions:

(Γ.1) $F(0) \supset \text{int } S^{n-1}$;

(Γ.2) $F_i(\Delta) = \dfrac{f_i(\Delta)}{\sum_{j=1}^n f_j(\Delta)}$, $\forall\Delta \in \mathcal{R}_+{}^n \setminus \{0\}$, $\forall i = 1,\ldots,n$;

(Γ.3) $f_i: \mathcal{R}^n_+ \longrightarrow \mathcal{R}_+$ is smooth, i.e. $f_i \in C^\infty(\mathcal{R}^n_{++})$, $f_i(\Delta) > 0$, $\forall\Delta \in \mathcal{R}^n_{++}$, and $f_i(0) = 0$, $\forall i = 1,\ldots,n$.

All the models mentioned in the previous Section fall into this class. Even the three-players example with subgame perfect *stationary* equilibrium strategies as the solution concept is a member of this class. Also, it can be shown that the model of commitment studied in Muthoo (1992) falls in this class.

Since a world where cooperative solution concepts make sense is presumably one *without* frictions, the non-cooperative foundations of cooperative solutions consist of letting Δ approach zero. The last part of (Γ.3) makes explicit that this is not a trivial operation, because at $\Delta = 0$ each share $F_i(\Delta)$ becomes an indeterminate number. Indeed it has been shown (Binmore, Rubinstein and Wolinsky, 1986, Proposition 1) that *in* the limit, $\Delta = 0$, any distribution $x \in S^{n-1}$ can be supported by a subgame perfect equilibrium: (Γ.1). To obtain a determinate solution in the limit thus requires a special way of approaching the limit $\Delta = 0$. This way is traditionally an application of l'Hospital's rule.

Within the present framework an application of l'Hospital can be characterized as follows: A *path* of frictions, π, is a stratifiable and 1-dimensional subset of $\mathcal{R}^n_+$ with the property that the origin $0 \in \mathcal{R}^n_+$ is an element of π. Let $\prod$ denote the set of all paths π of frictions.

DEFINITION. *An outcome $x \in S^{n-1}$ is attributable to the frictionless world by l'Hospital, if there exists $\pi \in \prod$ such that*

$$lim_{\Delta\downarrow 0,\Delta\in\pi} F(\Delta) = x$$

Some consequences of the definition of paths are worth noticing. First, because π is stratifiable, it is a finite union of smooth manifolds with maximum dimension 1 (because it is a 1-dimensional stratifiable set). Consequently, there exists a neighborhood $\mathcal{O}_\pi$ of the origin $0 \in \mathcal{R}^n_+$ and (n-1) independent and smooth functions $g_1,\ldots,g_{n-1}$, $g = (g_1,\ldots,g_{n-1})$, $g: \mathcal{O}_\pi \cap \mathcal{R}^n_{++} \to \mathcal{R}^{n-1}$ such that $g^{-1}(0) = \pi \cap \mathcal{O}_\pi \cap \mathcal{R}^n_{++}$ (Guillemin and Pollack, 1974, p. 24). We will say that a function $g_\pi: \mathcal{O}_\pi \cap \mathcal{R}^n_{++} \longrightarrow \mathcal{R}^{n-1}$ locally *cuts out* π, if $g_\pi^{-1}(0) = \pi \cap \mathcal{O}_\pi \cap \mathcal{R}^n_{++}$ and the Jacobian $D_\Delta g_\pi(\Delta)$ is of rank n-1 for all $\Delta \in g_\pi^{-1}(0)$. By definition for every $\pi \in \Pi$ there exists a function g_π which locally cuts out π.

Denote by $f(\Delta) = [f_1(\Delta),\ldots, f_n(\Delta)]'$ the (column vector of the f_i's corresponding to some $\Delta \in \mathcal{R}^n_+$ and let $D_\Delta f(0)$ be the Jacobian matrix of f evaluated at $\Delta = 0 \in \mathcal{R}^n_+$. Denote by $e' = (1,\ldots,1)$ the (row) summation vector. It is now easy to see that:

Lemma 1. *An outcome $x \in S^{n-1}$ is attributable to the frictionless world by l'Hospital, if and only if*

$$x \in Image(D_\Delta f(0)) \cap S^{n-1}.$$

Proof: *(i) Suppose*

$$\exists \pi \in \Pi: \lim_{\Delta \downarrow 0, \Delta \in \pi} F(\Delta) = x \in S^{n-1}$$

Let g_π be a function that locally cuts out π and denote by $D_\Delta g_\pi(0)$ its Jacobian matrix evaluated at $\Delta = 0 \in \pi$, viz. the limit of $D_\Delta g_\pi(\Delta)$ along $g_\pi^{-1}(0)$ as $\Delta \longrightarrow 0$ (this exists by virtue of the existence of a continuous extension of the Jacobian to the boundary of $\mathcal{O}_\pi \cap \mathcal{R}^n_+$). Let $ker(D_\Delta g_\pi(0))$ denote the kernel of the Jacobian evaluated at $\Delta = 0$. Then by l'Hospital

$$x = [e' \; D_\Delta f(0)y]^{-1} \; D_\Delta f(0)y,$$

for some $y \in ker(D_\Delta g_\pi(0))$. This implies $x \in Image(D_\Delta f(0)) \cap S^{n-1}$.

(ii) Suppose $x \in Image(D_\Delta f(0)) \cap S^{n-1}$. Choose an $(n{\times}n - 1)$ matrix A with rank $n-1$ such that $ker(A) \subset \mathcal{R}^n_+ \cup \mathcal{R}^n_-$ and

$$D_\Delta f(0)ker(A) = \lambda x,$$

for any $\lambda \in \mathcal{R}$. Define the function g: $\mathcal{R}^n_+ \longrightarrow \mathcal{R}^{n-1}$ by $g(\Delta) = A\Delta$. Then g cuts out the path $\pi = ker(A) \in \Pi$ and

$$lim_{\Delta \downarrow 0, \Delta\in\pi} F(0) = [e' D_\Delta f(0)ker(A)]^{-1} D_\Delta f(0)ker(A) = x. \blacksquare$$

All the models that were mentioned in Section 2 do in fact satisfy $\det(D_\Delta f(0)) \neq 0$ such that $\text{Image}(D_\Delta f(0)) \cap S^{n-1} = S^{n-1}$. In other words: for all these models the set of outcomes attributable to the frictionless world by l'Hospital is the full simplex S^{n-1}. This does *not* mean that anything can be attributed to the frictionless world. If our analyst has sufficient knowledge on how the frictions become smaller as the technology improves, she may still be able to select a single point from S^{n-1}.

Still, at least in our view, the operation of approaching the limit, i.e. the operation of attributing a solution to the frictionless world, should satisfy some basic robustness property. The most basic robustness property that comes to mind in this context is, of course, *continuity*. However, this is something that unfortunately fails whenever continuity is required on the mapping from Π to S^{n-1} defined by $\lim_{\Delta\downarrow 0, \Delta\in\pi} F(\Delta)$. To show this we proceed in two steps.

Lemma 2. *If $\det(D_\Delta f(0)) \neq 0$, then for any function g_π that locally cuts out $\pi \in \Pi$*

$$dim\left(lim_{\Delta\downarrow 0, \Delta\in\pi} F(\Delta)\right) = dim\left(ker(\Delta_\Delta g_\pi(0))\right) - 1.$$

Proof: *Define the set $H_f = \{\Delta \in \mathcal{R}^n \mid e' D_\Delta f(0)\Delta = 1\}$. If $D_\Delta f(0)$ is non-singular, then $e' D_\Delta f(0) \neq 0$ such that H_f is a $(n-1)$-dimensional hyperplane in $\mathcal{R}^n$ which does not contain the origin. Let g be a function that locally cuts out π and note that by definition $dim\left(ker(D_\Delta g(0))\right) \geq 1$. Since H_f is a $(n-1)$-dimensional hyperplane which does not contain the origin and $ker(D_\Delta g(0))$*

is a linear subspace of $\mathcal{R}^n$ *of minimum dimension 1 which does contain the origin, one has* $H_f + ker(D_\Delta g(0)) = \mathcal{R}^n$ *whenever* $H_f \cap ker(D_\Delta g(0)) \neq 0$*, such that the intersection* $H_f \cap ker(D_\Delta g(0))$ *is transversal. We show that this intersection is non-empty: If it would be empty, then* $\exists y \in \mathcal{R}^n \setminus \{0\}$ *such that* $ker(D_\Delta g(0)) + y \in H_f$*, because* $ker(D_\Delta g(0))$ *must then be parallel to the hyperplane* H_f *such that an affine translation will make them coincide. Since* $0 \in ker(D_\Delta g(0))$ *this implies* $y \in H_f$ *with the consequence that*

$$e'D_\Delta f(0)[ker(D_\Delta g(0)) + y] = 1 \Rightarrow e'D_\Delta f(0)ker(D_\Delta g(0)) = \{0\}.$$

Since this would imply that $ker(D_\Delta g(0)) = \{0\}$ *in contradiction to* $dim\left(ker(D_\Delta g(0))\right) \geq 1$*, the conclusion is that the intersection* $H_f \cap ker(D_\Delta g(0))$ *is non-empty and transversal.*

For a non-empty and transversal intersection we have

$$codim\left(H_f \cap ker(D_\Delta g(0))\right) = codim(H_f) + codim\left(ker(D_\Delta g(0))\right)$$

(Guillemin and Pollack, 1974, p. 30) such that we have to conclude $dim\left(H_f \cap ker(D_\Delta g(0))\right) = dim\left(ker(D_\Delta g(0))\right) - 1$*. Together with the hypothesis* $rank(D_\Delta f(0)) = n$ *this yields the statement of the Lemma, because*

$$lim_{\Delta \downarrow 0, \Delta \in \pi} F(\Delta) = D_\Delta f(0)[H_f \cap ker(D_\Delta g(0))].\blacksquare$$

Lemma 2 says that our analyst, in order to extrapolate the solutions $F(\Delta)$ to $\Delta = 0$, needs to extend a function g_π which locally cuts out the path π of her observations to $g_\pi(0)$ -or at least she needs to extend the derivatives of this function g_π which cuts out π. Between two different functions g_π and $\hat{g}_\pi$ which both cut out π our analyst cannot distinguish. And this is the reason, why we would not trust her when she comes up with a point prediction for the frictionless world.

Theorem 2. *If the limit operation, defined by*

$$\pi \longmapsto lim_{\Delta \downarrow 0, \Delta \in \pi} F(\Delta),$$

is continuous, then its value is S^{n-1} *for all* $\pi \in \Pi$.

Proof: *Assuming that the limit operation is continuous we first show that for any $\pi \in \Pi$ its value must be (n-1)-dimensional. Fix some $\pi \in \Pi$ and let g be any function which locally cuts out π. Define a new function* $\hat{g}$ by

$$\hat{g}(\Delta) = \left(\sum_{i=1}^{n} \Delta_i \right) g(\Delta)$$

Then, $\hat{g}^{-1}(0) = g^{-1}(0)$ and

$$D_\Delta \hat{g}(\Delta) = \left(\sum_{i=1}^{n} \Delta_i \right) D_\Delta g(\Delta), \quad \forall \Delta \in g^{-1}(0),$$

implies from $\sum_{i=1}^{n} \Delta_i > 0$, $\forall \Delta \in g^{-1}(0)$, *that* $\hat{g}$ *also cuts out* π. *However,* $D_\Delta \hat{g}(0) = 0$ *implies* $\ker(D_\Delta \hat{g}(0)) = \mathcal{R}^n$, *such that from Lemma 2 the dimension of the image set (of the limit operation) is* $n - 1$.

By the explicit representation of the limit operation at the end of the proof of Lemma 2 we obtain from $H_f \cap \mathcal{R}^n = H_f$ *that*

$$\lim_{\Delta \downarrow 0, \Delta \in \pi} F(\Delta) = D_\Delta f(0) H_f = S^{n-1}$$

Since, therefore, all paths $\pi \in \Pi$ *map into* S^{n-1} *this is the only value which can render the limit operation continuous.*■

Thus no topology on Π can make the limit operation continuous whenever it attempts to generate singleton predictions. Such single-valued predictions for the frictionless world are only feasible in a continuous fashion, if the knowledge on π extends smoothly beyond what can be observed: If π would be a smooth manifold (of dimension 1) that passes through the origin from $\mathcal{R}^n_{++}$ into $\mathcal{R}^n_{--}$, then a topology on the set of paths can be constructed (by measuring angles or distances between the kernels of the Jacobians of functions that locally cut out paths) with respect to which the limit operation can be continuous even if its predictions are singletons. But how is the knowledge on such paths generated?.

4. CONCLUSIONS

In this note we have shown that, if we allow for the discount factors of players to be just slightly different, any partition of the pie can be supported as a subgame perfect equilibrium when discount factors are close to one. From an intuitive point of view the result is driven by the fact that when players are very patient, the smallest discrepancy in time preferences causes enormous differences in equilibrium payoffs. This intuition carries over to other models (e.g. the durable monopoly model considered by Güth & Ritzberger (Jan. 1992)) and the authors have the strong suspicion that it might carry over to more infinite horizon models. In particular, in bargaining models with more than two players, even if they have a unique equilibrium (see Chae & Yang (1988), Yang (1992) and Asheim (1992)), the problem revealed in this note is likely to reappear.

The conclusion from the above is that, if the cooperative bargaining solution is to have any non-cooperative foundations, then it takes a world with non-vanishing frictions. However, our result points out that to choose the "right" frictions in infinite horizon models is a delicate task, because what is obtained in the limit depends very much on the sequence under consideration (for an alternative way to remove frictions see Sjöstrom (1991)). That such delicacy with respect to limit operations is not shared by other parts of economic theory is exemplified by core convergence theorems which demonstrate that the core shrinks to the competitive equilibrium under fairly general circumstances. In Hildenbrand's words "The conclusion *(that the difference between the core and the competitive equilibria tends to zero when the economy is large enough)*, to be of general relevance, should be robust to small deviations from the strict replication procedure" (italics added) (Hildenbrand (1987) p. 116). The non-cooperative models of bargaining considered in this note do not exhibit the analogous robustness property with respect to discounting.

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APPENDIX

In this Appendix we derive the corresponding formulae for the random proposer model.

Let $\bar{x}$ resp. $\underline{x}$ ($\bar{y}$, resp. $\underline{y}$) be the supremum (resp. infimum) of accepted equilibrium offers made by player 1 (2). Let x (y) denote an accepted equilibrium offer by player 1 (2). Denote by u_i $i = 1, 2$, player i expected equilibrium payoff. An offer which satisfies

$$1 - x > \delta[\alpha(1 - \underline{x}) + (1 - \alpha)\bar{y}]$$

will certainly be accepted by player 1, because in none of the (identical) subgames of the future she can get more than $\alpha(1 - \underline{x}) + (1 - \alpha)\bar{y}$ by the definition of $\underline{x}$ and $\bar{y}$. But then $1 - \delta[\alpha(1 - \underline{x}) + (1 - \alpha)\bar{y}] > x$ implies that there exists some $\varepsilon > 0$ such that the offer $x + \varepsilon$ is strictly preferable for player 1 and $x + \varepsilon$ will still be accepted by player 2. Consequently

$$\underline{x} \geq 1 - \delta[\alpha(1 - \underline{x}) + (1 - \alpha)\bar{y}] \tag{1.1}$$

By an analogous argument with the roles of players reversed

$$\underline{y} \geq 1 - \rho\,[\alpha\,\bar{x} + (1 - \alpha)(1 - \underline{y})] \tag{1.2}$$

On the other hand, if the offer x by player 1 is to be accepted by player 2, then it must satisfy $1 - x \geq \delta[\alpha(1 - \bar{x}) + (1 - \alpha)\underline{y}]$, because otherwise player would be better off with waiting for the next period. Consequently, the largest accepted equilibrium offer which player 1 can make in any equilibrium must satisfy

$$1 - \delta[\alpha(1 - \bar{x}) + (1 - \alpha)\underline{y}] \geq \bar{x} \tag{2.1}$$

And analogously for player 2

$$1 - \rho\,[\alpha\,\underline{x} + (1 - \alpha)(1 - \bar{y})] \geq \bar{y} \tag{2.2}$$

Substituting (1.2) into (2.1) and lengthy calculations yield

$$(1 - \delta)\;[1 - (1 - \alpha)\rho]/[1 - (1 - \alpha)\rho - \alpha\delta] \geq \bar{x} \tag{3.1}$$

Also, substituting (2.2) into (1.1) we get

$$(1 - \delta)\;[1 - (1 - \alpha)\rho]/[1 - (1 - \alpha)\rho - \alpha\delta] \leq \bar{x} \tag{3.2}$$

which shows that $x = \underline{x} = \bar{x}$.

An analogous reasoning for player 2 shows that

$$y = \underline{y} = \bar{y} = (1 - \rho)\,(1 - \alpha\delta)/[1 - (1 - \alpha)\rho - \alpha\delta]$$

Finally since $u_1 = \alpha x + (1 - \alpha)(1 - y)$ and $u_2 = \alpha(1 - x) + (1 - \alpha)y$ we obtain the desired result.■