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多重選擇 Shapley 值的一致性

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本研究計畫推廣 "Shapley值的 Potential" 成為「多重選擇Shapley值的 Potential」，再證明「多重選擇Shapley值」，俱「一致性」。
Abstract

We define the potential of multi-choice cooperative games, find the relationship between the multi-choice Shapley value and the potential, and show that the multi-choice Shapley value is consistent.

Key Words. Potential of Multi-choice Shapley Value, Consistency Property
REFERENCES


Consistency of the Multi-Choice Shapley Value

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Abstract. We define the potential of multi-choice cooperative games, find the relationship between the multi-choice Shapley value and the potential, and show that the multi-choice Shapley value is consistent.

Introduction. In [3], Hsiao and Raghavan started to consider players’ strategies in a cooperative game with side-payments. Henceforth, they extended the traditional cooperative to a multi-choice cooperative game and extended the Shapley value from a vector to a matrix. For brevity, we call the Shapley value for multi-choice cooperative games the multi-choice Shapley value.

In [2] and [3], Hsiao showed that the multi-choice Shapley value is monotone, transferable utility invariant, dummy free of players, dummy free of actions, and independent of non-essential players.

In [1], [2], and [3], Hsiao and Raghavan assume that players have the same number of actions. However, since the multi-choice Shapley value is dummy free of actions, the assumption is inessential. Therefore, by just rewriting the definitions, we may slightly extend the multi-choice Shapley value to a game where players have different numbers of actions.

In this article, we would first rewrite the definition of the multi-choice Shapley value, then we would define the potential of multi-choice cooperative games, show the relationship between the multi-choice Shapley value and the potential, and prove that the multi-choice Shapley value is consistent.

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Definitions and Notations

Let \( N = \{1, 2, \ldots, n\} \) be the set of players. We allow player \( j \) to have \((m_j + 1)\) actions, say \( \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{m_j} \), where \( \sigma_0 \) is the action to do nothing, while \( \sigma_k \) is the action to work at level \( k \), which has higher level than \( \sigma_{k-1} \).

Let \( \beta_j = \{0, 1, \ldots, m_j\} \) and \( \mathbf{m} = (m_1, m_2, \ldots, m_n) \). The action space of \( N \) is defined by \( \Gamma(\mathbf{m}) = \Pi_{j=1}^n \beta_j = \{(x_1, \ldots, x_n) \mid x_i \in \beta_{t_i}, \text{ for all } i \in N\} \). Thus \((x_1, \ldots, x_n)\) is called an action vector of \( N \), and \( x_i = k \) if and only if player \( i \) takes action \( \sigma_k \).

A multi-choice cooperative game in characteristic function form is the pair \((\mathbf{m}, V)\) defined by \( V : \Gamma(\mathbf{m}) \to R \), such that \( V(\mathbf{0}) = 0 \), where \( \mathbf{0} = (0, 0, \ldots, 0) \).

We can identify the set of all multi-choice cooperative games defined on \( \Gamma(\mathbf{m}) \) by, \( G \simeq R^{\Pi_{j=1}^n (m_j + 1) - 1} \).

Let \( m = \max_{j \in N} \{m_j\} \), and let \( w : \{0, 1, \ldots, m\} \to R_+ \) be a non-negative function such that \( w(0) = 0, w(0) < w(1) \leq w(2) \leq \ldots \leq w(m) \), then \( w \) is called a weight function and \( w(i) \) is said to be a weight of \( \sigma_i \).

We may treat the weight of an action as the measure of the "difficulty" of taking the action.

Given a weight function \( w \) for the actions, we define the value of a multi-choice cooperative game \((\mathbf{m}, V)\) by a \( \Pi_{j=1}^n m_j \) dimensional vector \( \phi^w : G \to R^{\Pi_{j=1}^n m_j} \) be such that

\[
\phi^w(V) = \\
(\phi_{11}^w(V), \ldots, \phi_{m_1 1}^w(V), \phi_{12}^w(V), \ldots, \phi_{m_2 2}^w(V), \ldots, \phi_{1n}^w(V), \ldots, \phi_{m_n n}^w(V))
\]

Here \( \phi_{ij}^w(V) \) is the power index or the value of player \( j \) when he takes action \( \sigma_i \) in game \( V \).
In [3] Hsiao and Raghavan showed that when \( w \) is given, there exists a unique \( \phi^w \) satisfying the following four axioms.

**Axiom 1.** Suppose \( w(0), w(1), \ldots, w(m) \) are given. If \( V \) is of the form

\[
V(y) = \begin{cases} 
  c > 0 & \text{if } y \geq x \\
  0 & \text{otherwise},
\end{cases}
\]

then \( \phi^w_{x_i, i}(V) \) is proportional to \( w(x_i) \).

A vector \( x^* \in \mathbb{B}^n \) is called a **carrier** of \( V \), if \( V(x^* \land x) = V(x) \) for all \( x \in \mathbb{B}^n \).

**Axiom 2.** If \( x^* \) is a carrier of \( V \) then, for \( m = (m, m, \ldots, m) \) we have

\[
\sum_{x^*_i \neq 0, x^*_i \in x^*} \phi^w_{x_i, i}(V) = V(m).
\]

By \( x^*_i \in x^* \) we mean \( x^*_i \) is the \( i \)-th component of \( x^* \).

**Axiom 3.** \( \phi^w(V^1 + V^2) = \phi^w(V^1) + \phi^w(V^2) \), where \( (V^1 + V^2)(x) = V^1(x) + V^2(x) \).

**Axiom 4.** Given \( x^0 \in \mathbb{B}^n \) if \( V(x) = 0 \), whenever \( x \not\geq x^0 \), then \( \phi^w_{k, i}(V) = 0 \), for all \( k < x_i^0 \) and all \( i \in N \).

**Definition 1.** Given an \( x \in \mathbb{B}^n \), we define \( x^k, i \) as an action vector where player \( i \) takes action \( \sigma_k \) and the other players take exactly the same actions as in \( x \). Sometimes, we would denote \( (x \mid x_i = k) \) as an action vector with \( x_i = k \).

Player \( i \) is said to be a **dummy** player if \( V((x \mid x_i = k)) = V((x \mid x_i = 0)) \) for all \( x \in \mathbb{B}^n \) and for all \( k \in \mathbb{B} \).

An action \( \sigma_k \) is said to be a **dummy** action if \( V((x \mid x_i = k)) = V((x \mid x_i = k - 1)) \) for all \( x \in \mathbb{B}^n \) and for all \( i \in N \).

**Definition 2.** Given \( x \in \mathbb{B}^n \), let \( S(x) = \{ i \mid x_i \neq 0, x_i \text{ is a component of } x \} \). Given \( S \subseteq N \), let \( e(S) \) be the binary vector with components \( e_i(S) \) satisfying

\[
e_i(S) = \begin{cases} 
  1 & \text{if } i \in S \\
  0 & \text{otherwise}.
\end{cases}
\]

For brevity, we let \( e(\{i\}) = e_i \) and let \( |S| \) be the number of elements of \( S \).
Definition 3. Given $\beta^n$ and $w(0) = 0, w(1), \ldots, w(m)$, for any $x \in \beta^n$, we define $\|x\|_w = \sum_{r=1}^n w(x_r)$.

Definition 4. Given $x \in \beta^n$ and $j \in N = \{1, 2, \ldots, n\}$, we define $M_j(x) = \{i \mid x_i \neq m_i, i \neq j\}$.

From Theorem 2 in [3], we have

$$\phi_{ij}^w(V) = \sum_{k=1}^{i} \sum_{j=k}^{T} \sum_{T \subseteq M_j(x)} \frac{(-1)^{|T|} w(x_j)}{\|x\|_w + \sum_{r \in T} [w(x_r + 1) - w(x_r)]} \cdot [V(x) - V(x - e(\{j\}))]. \quad (1.1)$$

The Potential

Let $I_+^n$ denote the set of all non-negative integers. Given $x, y \in I_+^n$ such that $|x|, |y| < \infty$ and $x \leq y$, we call $(x, V)$ a subgame of $(y, V)$, if and only if $(x, V)(z) = (y, V)(z)$ for all $z \in \Gamma(x)$.

Fixed a $m \in I_+^n$ with $|m| < \infty$, we let $G$ denote the set of all $n$ person multi-choice cooperative games defined on a $\Gamma(x)$ with $x \leq m$. Given a weight function $w$ for $\{0, 1, \ldots, m\}$, we define a function $P_w : G \rightarrow R$ which associates a real number $P_w(x, V)$.

Given $i \in \beta_j$, we define $(i_j, x)$ be an action vector such that $(i_j, x) = (x_1, \ldots, x_{j-1}, i_j, x_{j+1}, \ldots, x_n)$. Given $i \in \beta_j$, and $k \in \beta_\ell$, we define $(i_j, k_\ell, x)$ be an action vector whose $j$-th component is $i$ and $\ell$-th component is $k$.

Given $P_w(x, V)$, we define the following operators.

$$D_{i,j}P_w(x, V) = w(i) \cdot \left[ P_w((i_j, x), V) - P_w((i - 1)_j, x), V) \right],$$

and

$$H_{x_j} = \sum_{\ell = 1}^{t = x_j} D_{\ell,j}.$$
Definition 2.1. A function $P_w : G \rightarrow R$ with $P_w(0, V) = 0$ is called a w-potential function if it satisfies the following condition: for each fixed $x \in \Gamma(m)$

$$
\sum_{j \in S(x)} H_{x, j} P_w(x, V) = (x, V)(x) \tag{2.1}
$$

Given $j \in N$ and $V(x)$, we define

$$
d_j V(x) = V(x) - V(x - e_j),
$$

then $d_j$ is associative, i.e. $d_k(d_j V(x)) = d_j(d_k V(x))$. For convenience, we denote $d_id_j = d_{ij}$, $d_{ijk} = d_idjd_k$, ..., etc. We also denote $d_{i_1, i_2, \ldots, i_t} = d_S$ whenever \{i_1, i_2, \ldots, i_t\} = S. Furthermore, we denote $d_S(x)$ by $d_x$.

Theorem 2.1. The Potential of multi-choice cooperative games is unique, and

$$
P_w(x, V) = \sum_{\substack{y \leq x \\ y \neq 0}} \frac{1}{||y||_w} d_y(x, V)(y).
$$

Proof. It is easy to see that $P_w(0, V) = 0$. Let $|x| = \sum_{i \in N} x_i$, by mathematical induction the proof is completed.

Theorem 2.2. Given a multi-choice cooperative game $(m, V)$ then the Shapley value and the Potential of $(m, V)$ have the following relationship.

$$
\phi^w_{ij}((m, v)) = H_{ij} P_w((m, V)).
$$

Proof. From formulas (1.1) and (2.1), we can easily see the result.

Let $x, y \in \Gamma(m)$, we say that $x$ is adjacent to $y$ if and only if $y - x = e_j$ for some $j \in N$. Given an action vector $z \in \Gamma(m)$, we can always find a finite sequence of $p$ action vectors $0 = x_0, x_1, \ldots, x_p = z$ such that $x_j$ is adjacent to $x_{j+1}$ for $j = 0, 1, 2, \ldots, p$. We call $x_0, \ldots, x_p$ an adjacent sequence of $z$. 5
**Theorem 2.3.** Given a multi-choice cooperative game \((z, V)\), let \(x_1, \ldots, x_p\) be an adjacent sequence of \(z\) such that \(x_1 - x_0 = e_{j_1}, x_2 - x_1 = e_{j_2}, \ldots, x_p - x_{p-1} = e_{j_p}\). Then

\[
P_w((z, V)) = \sum_{\ell=1}^{\ell=p} \phi_{x_{j_{\ell}}, j_{\ell}}((x_{\ell}, V)).
\]

**Proof.** By mathematical induction on \(|x|\), we can easily completed the proof.

**Consistency Property of the Multi-choice Shapley Value**

Given a multi-choice cooperative game \((m, V)\) and its solution,

\[
(\psi_{11}^w(V), \ldots, \psi_{m_1, 1}^w(V), \psi_{12}^w(V), \ldots, \psi_{m_2, 1}^w(V), \ldots, \psi_{1n}^w(V), \ldots, \psi_{m_n, n}^w(V)),
\]

for each \(z \in \Gamma(m)\), we define an action vector \(z^* = (z_1^*, z_2^*, \ldots, z_n^*)\) where

\[
\begin{cases}
z_j^* = m_j & \text{if } z_j < m_j \\
z_j^* = 0 & \text{if } z_j = m_j.
\end{cases}
\]

Furthermore, we define a new game \(V_{z}^\phi : \Gamma(m) \to R\) such that

\[
V_{z}^\psi(y) = V(y \lor z^*) - \sum_{j \in S(z^*)} \psi_{m_j, j}((y \lor z^*), V).
\]

We call \(V_{z}^\psi\) a reduced game of \(V\) with respect to \(z\) and the solution \(\psi\). Furthermore, we say that the solution \(\psi\) is **consistent** if \(\psi_{i, j}(V) = \psi_{i, j}(V_{z}^\phi)\) for all \(i \leq z_j\) and all \(j \in N - S(z^*)\).

**Theorem 3.1.** The multi-choice Shapley value is consistent.

**Proof.** By formulas (1.1), (2.1) and (3.1), we can easily see \(\phi_{i, j}(V) = \phi_{i, j}(V_{z}^\phi)\) for all \(i \leq z_j\) and all \(j \in N - S(z^*)\).

Hence, the Shapley value is consistent.
REFERENCES


