Consumer theory with bounded rational preferences

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Abstract

The neoclassical consumer maximizes utility and makes choices by completely preordering the feasible alternatives and weighing when indifferent. The consumer studied in this paper chooses by weighing when indifferent and also when indecisive, without necessarily preordering the alternatives or exhausting her budget. Preferences therefore need not be complete, transitive or non-satiated but are assumed strictly convex and *adaptive*. The latter axiom is new and parallels that of ambiguity aversion in choice under uncertainty.

Keywords: preferences: incomplete, intransitive, convex, adaptive; representation; demand.

1. Introduction

This paper is concerned with the consumer's problem of choosing the best among the feasible alternatives when her preferences over the latter are not assumed complete, transitive or non-satiated. Completeness and transitivity are the fundamental rationality axioms in decision theory and economics in general. Yet, rationality and reality do not go hand-in-hand. Completeness requires universal decisiveness and transitivity demands perfect consistency from the consumer's preferences. Whether or not, however, preferences should be regarded as of this kind for theoretical model-building purposes has been the subject of a long-lasting debate.

A simple reason why a consumer may be unable to compare two alternatives and thereby be indecisive is ignorance about these alternatives. Another explanation that was put forward by Luce and Raiffa [27] is that if alternatives are multi-attribute and in pairwise comparisons each one dominates the other in some attributes, then the consumer may be indecisive because of the conflicting signals she receives. These authors further argued that if an individual is forced to choose when she is in fact indecisive, this may lead her to inconsistent choices. In other words, violations of completeness and transitivity may co-exist and the former may be causing the latter. To the extent that this is so, attempting to model such behavior is presumably well-motivated.

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Building on the literature generated by the seminal work of Sonnenschein [43], and particularly on the innovative paper of Shafer [39], we outline here a consumer theory for a decision maker whose preferences are not complete, transitive or non-satiated. First, we derive a representation theorem that offers a complete characterization of the strict preference relation by means of a bivariate preference function, which stands in lieu of the utility function that is infeasible in this setting. The representation theorem generalizes that of [39] to the direction of incomplete preferences and substantially improves the partial extensions in relation to incompleteness that were offered by Shafer and Sonnenschein [41] and Bergstrom et. al. [4].

Using this representation theorem and assuming that preferences are convex, we maximize them over competitive budget sets to obtain a demand correspondence for which the optimal feasible alternatives are the maximal (undominated) ones. In light of the remarks in Mas-Colell [30], this result is implicitly present in [43]. However, the method of proof here is very different, and although it relies on (and extends) two arguments of Shafer [39], it also makes use of the celebrated Maximum Theorem and allows for some notion of indirect utility to arise naturally. Not surprisingly, this demand correspondence does not generally satisfy the Weak Axiom of Revealed Preference (WARP). However, an identification of the pattern of WARP-"irrationality" is offered as a simple corollary with the intuitive statement that if two alternatives are violating WARP over two different budget sets, then it must follow that the consumer has no strict preference between them.

The final result shows how the demand correspondence can be reduced to a demand function that satisfies WARP. This is done by assuming strictly convex preferences and by introducing the new axiom of *adaptiveness*. An individual's preferences are said to be adaptive if whenever she is indecisive between two alternatives there exists some convex combination of them that is weakly preferred to both. Despite its simplicity, to our knowledge this axiom is new. It will be argued that it is intuitive and that conceptually it also parallels the axiom of ambiguity aversion in choice under uncertainty introduced by Gilboa and Schmeidler [16].

As far as the literature is concerned, early studies on the representation of incomplete preferences include Aumann [3], Richter [35, 36] and Peleg [33]. With such preferences, equilibrium existence in a pure exchange economy was proved by Schmeidler [37], while Bewley [5] modeled an individual choosing under uncertainty. More recently, Seidenfeld et. al. [38], Shapley and Baucells [42], Ok [31], Dubra et. al. [8], Kochov [22] and Evren and Ok [11] represented incomplete preferences by means of an infinite set of real-valued functions. The problem of distinguishing indifference and incomparability that comes about when preferences are incomplete was studied by Eliaz and Ok [9] and Mandler [28]. With regard to transitivity, apart from the two papers mentioned above, other notable studies that relax this assumption include Kihlstrom et. al. [19], Fishburn [12], Loomes and Sugden [26], as well as Fountain [14], Epstein [10], Al-Najjar [1], Quah [34] and Masatlioglu and Ok [32]. Preferences that are simultaneously incomplete and intransitive have been studied in a general equilibrium, non-cooperative and cooperative game theory context by Mas-Colell [30], Gale and Mas-Colell [15], Shafer and Sonnenschein [41], Fon and Otani [13], Yannelis and Prabhakar [46], Border [6], Kim and Richter [20], Kajii [18] and Martins-da-Rocha and Topuzu [29]. Shafer [40] characterized such preferences that lead to demand functions consistent with WARP and SARP, importantly by weakening completeness with the technical condition of *local* completeness and by retaining non-satiation.

Empirical evidence against the assumption that preferences are transitive has been provided by, among others, Tversky [44], Grether and Plott [17], Loomes et. al. [25], Kivetz and Simonson

[21] and Lee et. al. [23]. Interestingly, the experimental design in [21] was based on choice alternatives that were both multi-attribute and such that in all pairwise comparisons information was missing for some attributes, making the comparison challenging in the first place. Thus, a possible alternative explanation of the systematic intransitivities detected (30% on average) might be the subjects' inability to compare with confidence the two options each time presented, in accordance with what Luce and Raiffa [27] predicted. Regarding violations of the decisiveness hypothesis associated with preference completeness per se, such patterns are manifested in Danan and Ziegelmeyer [7] in an environment of choice under risk.

2. Preference Representation

2.1. Definitions and Result

Throughout this section the consumer is assumed to have preferences over alternatives belonging to a metric space X. Her strict preferences, indifference and weak preferences are captured by the binary relations $\succ$, $\sim$ and $\succsim := \succ \cup \sim$ on X respectively, while the inverses of $\succ$ and $\succsim$ are denoted $\prec$ and $\precsim$. For notational convenience, $(x, y) \in \succsim$ is usually written $x \succsim y$ and this applies to every relation considered here. It is assumed that $\succ$ is asymmetric and $\sim$ is reflexive and symmetric, so that $\succsim$ is merely reflexive. In particular, the two axioms that are typically imposed on $\succsim$ and which are *not* employed here are *completeness* (CM) and *transitivity* (TR), which require that for all $x, y \in X$ and all $x, y, z \in X$ respectively

CM: $x \succsim y$ or $y \succsim x$

TR: $(x \succsim y, y \succsim z) \Rightarrow x \succsim z.$

In the absence of CM, the relation $\curlywedge := \{(x,y) : x \not\succsim y \text{ and } y \not\succsim x\}$ which captures the consumer's indecisiveness is nonempty and by construction also irreflexive and symmetric. Let us now introduce the relation $\widehat{\sim} := \sim \cup \curlywedge$, which is reflexive and symmetric by definition (Mandler [28] uses this relation independently for distinct purposes). We will refer to $\widehat{\sim}$ as the consumer's *ambivalence* relation because it consists of pairs over which she is either indifferent or indecisive. Ambivalence is identically equal to $(X \times X) \setminus (\succ \cup \prec)$ and is the maximal (with respect to set inclusion) reflexive and symmetric relation on X given the asymmetric relation $\succ$. It is emphasized that ambivalence is not a relabeling of indifference. Indifference is reflexive and symmetric by assumption and in view of the above remark it is contained in the ambivalence relation. The latter is reflexive and symmetric by definition, and this is true even when $\sim$ is an equivalence relation, which happens when $\succsim$ satisfies TR and therefore becomes a preorder. The two notions coincide if and only if $\succsim$ is complete, in which case $\curlywedge \equiv \emptyset$.

The strict preference relation $\succ$ is said to have an open graph if $\succ$ is an open subset of $X \times X$ with the product topology. Importantly, it was shown first by Shafer [39] and then by Bergstrom et. al. [4] that CM and TR are not needed for $\succ$ to have an open graph.

To study the uniqueness properties of the representation provided below we introduce the following transformation concept.

Definition 1
Let $f : X \times X \to \mathbb{R}$ be a function satisfying $f(x,y) = -f(y,x)$ for all $x, y \in X$. A function $g : \mathbb{R} \to \mathbb{R}$ is an odd-monotonic transformation of f if g is odd and strictly increasing.

For example, $g : \mathbb{R} \to \mathbb{R}$ defined by $g(z) = \alpha z^3$ with $\alpha > 0$ is an odd-monotonic transformation of $f : \mathbb{R}^2_+ \to \mathbb{R}$ defined by $f(x,y) = x - y$, because $g(-z) = -g(z)$ and $z > z'$ implies $g(z) > g(z')$ for all $z, z' \in \mathbb{R}$. However, the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(z) = \alpha z + b$ with $\alpha > 0$ and $b \neq 0$ is not an odd-monotonic transformation of f because, clearly, $h(-z) \neq -h(z)$ for some $z \in \mathbb{R}$.

We can now state and prove the representation theorem, which generalizes that of Shafer [39] to the direction of incomplete preferences and substantially improves the partial extensions of that result in relation to incompleteness that were provided by Shafer and Sonnenschein [41] and Bergstrom et. al. [4]. The argument is a variation of Shafer's [39] constructive idea and crucially depends on the observation that $\widehat{\sim}$ is reflexive and symmetric.

Theorem 1
The asymmetric relation $\succ$ has an open graph if and only if there exists a continuous function $\mathcal{P} : X \times X \to \mathbb{R}$, unique up to odd-monotonic transformations, such that for all $x, y \in X$
$\mathcal{P}(x,y) > 0 \Leftrightarrow x \succ y$
$\mathcal{P}(x,y) = 0 \Leftrightarrow x \mathrel{\widehat{\sim}} y$
$\mathcal{P}(x,y) = -\mathcal{P}(y,x)$.

Proof.
Suppose $\succ$ has an open graph and let $\mathcal{P}$ be defined by

$$\mathcal{P}(x,y) := \begin{cases} d\big((x,y), \widehat{\sim}\big) & \text{if } x \succ y \text{ or } x \mathrel{\widehat{\sim}} y \\ -d\big((y,x), \widehat{\sim}\big) & \text{if } y \succ x \text{ or } y \mathrel{\widehat{\sim}} x \end{cases} \tag{1}$$

where $d(\cdot)$ is the product metric on $X \times X$ and $d\big((x,y), \widehat{\sim}\big) = \inf_{(w,z) \in \widehat{\sim}} d\big((x,y),(w,z)\big)$. The open-graph property of $\succ$ and the fact that $\widehat{\sim} \equiv (X \times X) \setminus (\succ \cup \prec)$ establish that $\widehat{\sim}$ is closed in $X \times X$ and therefore that $\mathcal{P}$ is continuous. The three properties of $\mathcal{P}$ follow directly from (1) and the reflexivity and symmetry of $\widehat{\sim}$, while its uniqueness is straightforward. For the converse implication, see the proof of Theorem 4 in [4]. ∎

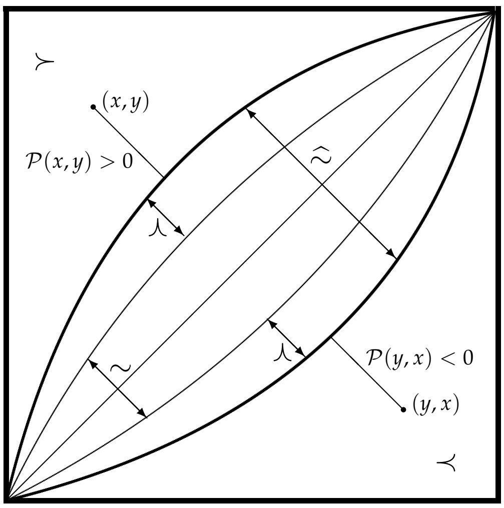

FIGURE 1
An example of a preference function $\mathcal{P}$

The function $\mathcal{P}$ of Theorem 1 generalizes the utility function concept for preferences that are incomplete or intransitive. Given its binary structure, it would be natural for $\mathcal{P}$ to be called a *preference function*, because with each ordered pair $(x,y) \in X \times X$ it associates a positive, negative or zero value whenever $(x,y) \in \succ$, $(x,y) \in \prec$ or $(x,y) \notin \succ$ and $(x,y) \notin \prec$ respectively. This motivates the following definition, which supersedes the one proposed by Vind [45].

Definition 2
If $\succ$ is an asymmetric relation on X, then $\mathcal{P} : X \times X \to \mathbb{R}$ is a preference function if for all $x, y \in X$, $\mathcal{P}(x,y) > 0 \Leftrightarrow x \succ y$, $\mathcal{P}(x,y) = 0 \Leftrightarrow (x \not\succ y, y \not\succ x)$ and $\mathcal{P}(x,y) = -\mathcal{P}(y,x)$.

In light of this definition, $\mathcal{P}$ of Theorem 1 is a preference function satisfying $\mathcal{P}(x,y) = 0 \Leftrightarrow x \approx y$. In Shafer's [39] representation theorem, $\succsim$ satisfies CM but not TR and the preference function $k : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ introduced there is such that $k(x,y) \geq 0 \Leftrightarrow x \succsim y$ and $k(x,y) = -k(y,x)$ so that $k(x,y) = 0 \Leftrightarrow x \sim y$. If $\succsim$ satisfies CM and TR (and X is separable and connected), then the open graph property of $\succ$ implies the existence of a continuous utility function $u : X \to \mathbb{R}$ with $x \succsim y \Leftrightarrow u(x) \geq u(y)$ for all $x, y \in X$, so that one can define $\mathcal{P}^*$ by $\mathcal{P}^*(x,y) := u(x) - u(y)$. It is easily verified that $\mathcal{P}^*$ satisfies the properties of Definition 1. This fact supports the claim made above that preference functions generalize utility functions when CM and/or TR are not assumed. Of course, the original idea behind this generalization (certainly so for intransitive preferences) is due to Shafer [39].

2.2. Discussion

We now turn to a comparison between Theorem 1 and other representations of incomplete (and in one case also intransitive) preferences that are already available in the literature. To start with, Peleg (1970), assuming transitivity and imposing further some hemicontinuity and order-separability restrictions on the strict preference relation $\succ$, showed that there exists a continuous "weak" utility function $v : X \to \mathbb{R}$ such that, for any $x, y \in X$, $x \succ y \Rightarrow v(x) > v(y)$. This weak utility representation, however, does not characterize strict preferences, because knowledge of the values of the function does not lead to a reconstruction of the strict preference relation $\succ$. To compare Peleg's weak utility function v with the preference function $\mathcal{P}$ of Theorem 1, consider the function $\phi : X \times X \to \mathbb{R}$ defined by $\phi(x,y) := v(x) - v(y)$. Clearly then,

$$x \succ y \Rightarrow \phi(x,y) > 0 \qquad \text{and} \qquad y \succ x \Rightarrow \phi(x,y) < 0, \tag{2}$$

while the equivalence in the above relations is ruled out by incompleteness. However, it is an implication of the properties of the preference function $\mathcal{P}$ that

$$x \succ y \Leftrightarrow \mathcal{P}(x,y) > 0 \qquad \text{and} \qquad y \succ x \Leftrightarrow \mathcal{P}(x,y) < 0. \tag{3}$$

A comparison of (2) and (3) shows that $\mathcal{P}$ completely characterizes $\succ$ whereas the weak utility function v only partially so.

Applying Shafer's [39] representation theory above to the problem of equilibrium-existence in an abstract economy with incomplete and intransitive preferences, Shafer and Sonnenschein [41] observed that the open-graph property of Theorem 1 implies the existence of a continuous function $\mathcal{V} : X \times X \to \mathbb{R}_+$ defined by $\mathcal{V}(x,y) := d\big((x,y), \succ^c\big)$, where $\succ^c := (X \times X) \backslash \succ$ and $d(\cdot)$ is again the product metric. This function has the properties that, for all $x, y \in X$, $\mathcal{V}(x,y) > 0 \Leftrightarrow x \succ y$ and $\mathcal{V}(x,y) = 0 \Leftrightarrow x \not\succ y$. While these were sufficient for the authors' purposes, one observes that $\mathcal{V}$ takes non-negative values and therefore that it is not a preference function. Thus, $\mathcal{V}$ does not characterize $\succ$ and cannot be used to elicit this relation's maximal elements. Theorem 1 demonstrates that introducing the ambivalence relation $\widehat{\approx}$ allows for this device of Shafer and Sonnenschein [41], which was also later characterized by Bergstrom et. al. [4], to be substantially improved upon at no cost.

Kochov [22] and Evren and Ok [11], in the spirit of multi-function preference representations described in the introduction and building also on the work of Levin [24], showed that $\succsim$ is an incomplete preorder with a closed graph[1] in $X \times X$ if and only if there exists an infinite set $\mathcal{U}$ of continuous real-valued functions u on X such that, for all $x, y \in X$, $x \succsim y \Leftrightarrow u(x) \geq u(y)$ for all $u \in \mathcal{U}$, provided X is compact.[2] This multi-utility result assumes TR, which Theorem 1 does not. Furthermore, the open-graph property of the asymmetric relation $\succ$ and the closed graph property of the preorder $\succsim$ are not related in the present context. Finally, while multi-utility representation allows for a full characterization of the preorder $\succsim$ at the cost of introducing an infinite number of functions, the preference function $\mathcal{P}$ of Theorem 1 fully characterizes the asymmetric relation $\succ$ only, but it does so in a more general (i.e. without assuming transitivity), tractable and perhaps also intuitive way. For demand theory, it is shown next that working with $\mathcal{P}$, or equivalently, focusing on $\succ$ rather than on $\succsim$, is actually sufficient.

3. Consumer Demand

3.1. Demand Correspondence

From now on we assume that the consumption set is $X = \mathbb{R}^n_+$. Furthermore, we let $Y := \mathbb{R}^n_{++}$ denote the set of strictly positive price-income ratios that the consumer faces, with generic element $p \in Y$. We define also the budget correspondence $B : Y \twoheadrightarrow X$ by $B(p) = \{x \in X : px \leq 1\}$, which, as well-known, is continuous and compact-valued. In addition to the *sine qua non* axioms concerning the asymmetry of $\succ$ and the reflexivity and symmetry of $\sim$, the following two axioms will also be imposed on preferences, the restrictions applying to every element $x \in X$.

Axiom 1
$\mathcal{U}_{\succ}(x) := \{y \in X : y \succ x\}$ and $\mathcal{L}_{\succ}(x) := \{y \in X : x \succ y\}$ are open in X.

Axiom 2
$\mathcal{U}_{\succ}(x)$ is convex.

Axioms 1 and 2 are the familiar (upper- and lower-) hemicontinuity and convexity restrictions on the preference relation $\succ$ respectively, although it should be noted that Axiom 2 can be replaced by the weaker requirement that $x \notin \mathrm{co}[\mathcal{U}_{\succ}(x)]$, where $\mathrm{co}(\cdot)$ denotes convex hull. Using Theorem 1 and these two axioms we can prove the existence of an upper-hemicontinuous and compact-valued demand correspondence consisting of the maximal (undominated) elements of the asymmetric relation $\succ$. As noted in the introduction, this result is implicit in Sonnenschein [43]. Instead of using the KKM Theorem and non-empty intersection arguments that were employed in [43], however, preferences here are maximized by means of the preference function $\mathcal{P}$, much in the spirit of neoclassical consumer theory.

[1] We express axiomatically our reservations regarding this assumption elsewhere.

[2] This is Kochov's [22] formulation. Evren and Ok [11] provide a one-way axiomatization assuming that X is locally compact.

Theorem 2
Under Axioms 1 and 2 there exists an upper-hemicontinuous, nonempty- and compact-valued correspondence $x : Y \twoheadrightarrow X$ *such that, for all* $p \in Y$, $x(p) = \{x \in B(p) : \mathcal{P}(x,y) \geq 0 \; \forall \, y \in B(p)\}$.

Proof.
Note first that under Axioms 1 and 2 the relation $\succ$ has an open graph (see Lemma in [39]), so that from Theorem 1 there exists a continuous preference function $\mathcal{P}$ on X. The remainder of the proof is split into three parts, the first two of which rely on (and generalize to the direction of incomplete preferences) Theorems 2 and 3 in [39]. Their proofs appear in the Appendix.

Lemma 1
For $p \in Y$, *there exists* $x \in B(p)$ *such that* $P(x,y) \geq 0$ *for all* $y \in B(p)$.

Lemma 2
For $p \in Y$, *it holds that* $x \in B(p)$ *satisfies* $\mathcal{P}(x,y) \geq 0$ *for all* $y \in B(p)$ *if and only if there exists a continuous function* $h : Y \times X \to \mathbb{R}$ *such that* $h(p,x) = \max\{h(p,z) : z \in B(p)\}$.

Since the function h of Lemma 2 is continuous and so is the budget correspondence B, it follows from Berge's Maximum Theorem that there exists a continuous function $m : Y \to \mathbb{R}$ such that $m(p) = \max\{h(p,z) : z \in B(p)\}$ and an upper-hemicontinuous compact-valued correspondence $x : Y \twoheadrightarrow X$ such that $x(p) = \text{argmax } \{h(p,z) : z \in B(p)\}$. From Lemma 2 we know that $x(p) = \{x \in B(p) : \mathcal{P}(x,y) \geq 0 \; \forall \, y \in B(p)\}$ while Lemma 1 shows that $x(p) \neq \emptyset$. ■

In his framework of intransitive but complete preferences, Shafer [39] interpreted the function h of Lemma 2 as the consumer's "utility" function that is dependent on prices.[3] With this interpretation, and observing that $m : Y \to \mathbb{R}$ defined by $m(p) = \max\{h(p,z) : z \in B(p)\}$ in the proof of Theorem 2 above is an optimal value function, we can clearly interpret it as the consumer's "indirect utility" function in this setting. Therefore, this proof on the existence of a demand correspondence allows for some connection between the consumer's problem when her preferences are bounded-rational and the neoclassical consumer problem when preferences are rational, since in both cases use is made of preference representations by means of numerical functions and of the Maximum Theorem, an approach that in both cases allows for optimal value functions to be derived.

The next observation shows that when the demand correspondence $x(\cdot)$ of Theorem 2 violates WARP, the violation is caused by consumer ambivalence or equivalently, by lack of strict "psychological" preference, which is an intuitive but nevertheless interesting fact.

Corollary 1
Suppose there exist $p, p' \in Y$ *and* $x, y \in X$ *such that (i)* $x \in x(p)$ *and* $y \in B(p) \setminus x(p)$; *(ii)* $y \in x(p')$ *and* $x \in B(p')$. *Then* $x \stackrel{\frown}{\sim} y$.

Proof.
By assumption, $\mathcal{P}(x,y) \geq 0$ and $\mathcal{P}(y,x) \geq 0$. Thus, $\mathcal{P}(x,y) = 0$ so that $x \stackrel{\frown}{\sim} y$. ■

[3]Note, however, that in both this framework as well as Shafer's [39], for $p \in Y$ and $x, y \in B(p)$, $x \succ y$ neither implies nor is implied by $h(p,x) > h(p,y)$. It is the case, however, that in any budget set, the maximizers of h coincide with the maximal elements of $\succ$.

We finally note that it cannot be distinguished whether the consumer is indifferent or indecisive between any two optimal alternatives in relation to the demand correspondence $x(\cdot)$ of Theorem 2 without adding more structure to the analysis, and this is in line with the relevant findings in Mandler [28].

3.2. Demand Function

In this subsection we reduce the demand correspondence of Theorem 2 to a single-valued demand function that satisfies WARP. To do so, the following two axioms are used in addition to Axioms 1 and 2.

Axiom 3
$(x \succsim z, y \succsim z, x \neq y)$ implies that for all $\alpha \in (0,1)$, $\alpha x + (1-\alpha)y \succ z$.

Axiom 4
$x \curlywedge y$ implies there exists $\lambda \in (0,1)$ such that $z := \lambda x + (1-\lambda)y$ satisfies $z \succsim x$ and $z \succsim y$.

Axiom 3 is the standard assumption of strict convexity. Axiom 4 is a new preference axiom, which we will refer to as *adaptiveness*. If a consumer is indecisive between two alternatives x and y and her preferences are adaptive, then some weighted average of x and y (which will generally depend on x and y) makes her at least as well off as x and y. If, for example, an individual is about to buy ice-cream for the first time and is offered the choice between a cup of vanilla, a cup of chocolate and a cup containing both in equal shares, then given that she's never tried vanilla and chocolate before, she may prefer the mixture because it gives her the opportunity to diversify across the two primitive alternatives so that whichever of the two turns out to be undesirable for her, she can consume the part containing the other (presumably more desirable) one. Her preferences therefore adapt to indecisiveness, and they do so in a way that parallels the axiom of ambiguity aversion (Gilboa and Schmeidler [16]) in choice under uncertainty and the principle of hedging against risk.

A testable implication of this new axiom is that when the decision maker chooses between two unfamiliar alternatives at the same time when mixtures of them are also feasible, certain mixtures will be preferable to the original options and the weights that determine these mixtures will generally be biased toward the option that is more familiar to her. In the ice-cream example, if the consumer likes chocolate but hasn't tried vanilla before, she may prefer a cup where the analogy is 3/4 chocolate and 1/4 vanilla. This prediction is also in line with Ellsberg-paradox findings, where preferences have been shown to favor risky alternatives that are better understood compared to uncertain ones. The framework there, of course, is one of genuine probabilistic uncertainty whereas here uncertainty is implicit, but there is an analogy between the two that is perhaps noteworthy.

Theorem 3
Under Axioms 1–4, $p \in Y$ implies there exists $x \in B(p)$ such that $\mathcal{P}(x,y) > 0$ for all $y \in B(p)$ with $y \neq x$.

Proof.

Let $p \in Y$. From Theorem 2 there exists $x \in B(p)$ such that $\mathcal{P}(x,y) \geq 0$ for all $y \in B(p)$. Suppose $w \in B(p)$ also satisfies $\mathcal{P}(w,y) \geq 0$ for all $y \in B(p)$. We will show that $x = w$. Note first that, by assumption, $\mathcal{P}(x,w) \geq 0$ and $\mathcal{P}(w,x) \geq 0$, so that $\mathcal{P}(x,w) = 0$ which in turn is equivalent to $x \mathrel{\widehat{\sim}} w$ and ($x \sim w$ or $x \curlywedge w$). Suppose $x \sim w$. Convexity of the set $B(p)$ and Axiom 3 imply that every convex combination z of x and w satisfies $z \in B(p)$, $z \succ x$ and $z \succ w$, which imply $\mathcal{P}(z,x) > 0$ and $\mathcal{P}(z,w) > 0$, a contradiction. Now suppose $x \curlywedge w$. Convexity of $B(p)$ and Axiom 4 imply the existence of a nontrivial convex combination z of x and w such that $z \in B(p)$, $z \succsim x$ and $z \succsim w$. If $z \sim x$ or $z \sim w$, Axiom 3 leads to the previous contradiction. If $z \succ x$ or $z \succ w$, then the optimality of x and w is directly violated. It follows then that $x = w$. ■

Thus, under the conditions of Theorem 3 the upper-hemicontinuous demand correspondence $x(\cdot)$ of Theorem 2 becomes a continuous demand function. Given prices $p, p' \in Y$, if $x(p) = x$, $x \neq y$ and $y \in B(p)$, then $\mathcal{P}(x,y) > 0$ and hence $x(p') = y$ implies $x \notin B(p')$. Therefore, the demand function associated with Theorem 3 satisfies WARP.

To interpret Theorem 3 consider the following example, where although alternatives are indivisible and therefore, strictly speaking, no convex combinations exist, the central idea is preserved. Suppose a consumer is about to buy a computer as a gift for her child. She goes to a computer store and tells the salesman about her plan. He informs her that there are three types of computers, "desktops" (d), "laptops" (l) and "palmtops" (p). She understands that there are two criteria on which to base her choice, computing power and portability. It is clear to her that desktops dominate laptops and laptops dominate palmtops in computing power, while palmtops dominate laptops and laptops dominate desktops in portability.

Keeping this in mind, to make a choice she first assigns a numerical value from 0 (lowest) to 2 on each of the two attributes of the three available alternatives as follows (the first entry in brackets corresponds to the assigned value for computing power and the second to that for portability): $d = (2,0)$, $l = (1,1)$, $p = (0,2)$. With perfect information about her child's attitude toward computing power and portability a transitive ordering of the above triple would occur, either such that

$$d \succ l, \;\; l \succ p, \;\; d \succ p \tag{4}$$

or

$$p \succ l, \;\; l \succ d, \;\; p \succ d. \tag{5}$$

Thus, she would buy the desktop in the first case and the palmtop in the latter case. However, she is unaware of the criterion that matters most for her child and is therefore reluctant to make a choice according to either (4) or (5). In particular, due to the conflicting values over the two attributes, in all three pairs she is indecisive à la Luce and Raiffa [27], so that $d \curlywedge l$, $l \curlywedge p$ and $d \curlywedge p$. Since her preferences are adaptive, she looks for weighted averages in every pair. There is none for (d,l) and (l,p) but there is one for (d,p), namely the laptop, since $l = \frac{1}{2}d + \frac{1}{2}p$. This option, in particular, offers a balanced choice in terms of both computing power and portability. Reasoning in this way, she leaves the store with the laptop.

The only paper that we are aware of which has provided demand-existence results similar to Theorem 3 is Shafer [40], where the preference relations that generate WARP- and SARP-rational demand functions that exhaust the budget were characterized. However, completeness in Shafer's case was replaced by *local* completeness, which requires that if $x \notin \mathrm{cl}[\mathcal{U}_{\succ}(y)]$, where $\mathrm{cl}(\cdot)$ denotes closure, then there exists $\beta \in [0,1)$ such that $\beta x + (1-\beta)y \succ x$. Yet, it generally holds that $\mathrm{cl}[\mathcal{U}_{\succ}(y)] \neq \mathcal{U}_{\succsim}(y)$ and therefore the behavioral restrictions that this axiom places are not clear. In contrast, the adaptiveness axiom that is used in Theorem 3 is arguably behavioral in nature and although similar to local completeness in its convexity-like structure, the two are independent. Furthermore, Shafer's [40] results rely on the budget-exhaustive property of the demand functions. By contrast, in Theorem 3 existence is proved without this assumption and therefore local non-satiation restrictions on preferences are also not imposed. Finally, unlike [40], the demand function here is ultimately established through the use of a preference function.

4. Concluding Remarks

Motivated by the scepticism on intuitive and empirical grounds that has been surrounding the principal rationality axioms of complete and transitive preferences for long, we studied here consumer theory when both these axioms were removed. It was shown that the two basic pillars of this theory, preference representation and demand, can be appropriately modified to accommodate this weaker axiomatic framework. In particular, the existence of a single-valued and well-behaved demand function was established under conditions that allow for a new interpretation of the consumer's problem and its solution. In sharp contrast to the neoclassical consumer who maximizes utility and makes choices by completely preordering the alternatives and weighing when indifferent, the proposed consumer makes choices by weighing both when indecisive and when indifferent. She is therefore "rational" in that she maximizes her preferences to single out the best alternative, but also "bounded rational" because her preferences incorporate the imperfections of indecisiveness and inconsistency, both of which are inherent properties of human decision making. In future work we plan to extend the representation theorem presented here to choice environments of risk and uncertainty. It may also be of interest to study the possible effects of this demand theory on the existence and optimality of general equilibrium.

Appendix

Proof of Lemma 1.

$B : Y \twoheadrightarrow X$ is a continuous compact-valued correspondence and $\mathcal{P} : X \times X \to \mathbb{R}$ is a continuous function. Fix $p \in Y$ and let $S : B(p) \to \mathbb{R}$ be defined by $S(y) = \max\{\mathcal{P}(w, y) : w \in B(p)\}$. Since $\mathcal{P}$ is continuous and $B(p)$ is compact, S is continuous. In view of these facts, from Berge's Theorem follows that $E : B(p) \twoheadrightarrow B(p)$ defined by $E(y) = \text{argmax}\{S(y) : y \in B(p)\}$ is an upper-hemicontinuous and compact-valued correspondence, and from Weierstrass' Theorem it is also nonempty-valued. Let $G : B(p) \twoheadrightarrow B(p)$ be defined by $G(y) = \text{co}[E(y)]$, where $\text{co}(\cdot)$ denotes convex hull. G is convex-valued and from the above it is also nonempty-valued, upper-hemicontinuous and compact-valued (see Corollary 5.33 and Theorem 17.35 in [2]). From Kakutani's Theorem there exists $x \in B(p)$ such that $x \in G(x)$.

Since $G(x) = \text{co}[E(x)]$ is convex and $X = \mathbb{R}^n_+$, from Carathéodory's Theorem there exist $m \leq n + 1$ points $x_1, \ldots, x_m$ in $B(p)$ and real numbers $\alpha_1, \ldots, \alpha_m$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{m} = 1$ such that $x_i \in E(x)$ for all $i \leq m$ and $x = \sum_{i=1}^{m} \alpha_i x_i$. Suppose $\mathcal{P}(x_i, x) > 0$ for all $i \leq m$. From Theorem 1 this is equivalent to $x_i \in \mathcal{U}_{\succ}(x)$ and since $\mathcal{U}_{\succ}(x)$ is convex from Axiom 2 and $x = \sum_{i=1}^{m} \alpha_i x_i$, it also implies $x \in \mathcal{U}_{\succ}(x)$, a contradiction. There exists then some x_j, $j \in \{1, \ldots, m\}$, such that $\mathcal{P}(x_j, x) \leq 0$. But since $x_i \in E(x)$ for all $i \leq m$, it is true that $\mathcal{P}(x_j, x) = \mathcal{P}(x_i, x)$ for all $i \neq j$. Thus, $\mathcal{P}(x_i, x) \leq 0$, or equivalently, $\mathcal{P}(x, x_i) \geq 0$ for all $i \leq m$. From the definition of x_i then follows that $\mathcal{P}(x, y) \geq 0$ for all $y \in B(p)$. ■

Proof of Lemma 2.

Let the function $h : Y \times X \to \mathbb{R}$ be defined by $h(p, x) = \min\{\mathcal{P}(x, y) : y \in B(p)\}$, where $x \in B(p)$. Since $B(p)$ is compact, $h(p, x)$ is well-defined. Since $\mathcal{P}$ and B are continuous, h is continuous. Suppose $x \in B(p)$ is such that $\mathcal{P}(x,y) \geq 0$ for all $y \in B(p)$. Since $\mathcal{P}(x,x) = 0$, it follows that $h(p,x) = \min\{\mathcal{P}(x,y) : y \in B(p)\} = 0$. Furthermore, for $z \in B(p)$, $z \neq x$, it holds that $h(p,z) = \min\{\mathcal{P}(z,y) : y \in B(p)\} \leq \min\{\mathcal{P}(x,y) : y \in B(p)\} = h(p,x) = 0$. Thus, if $x \in B(p)$ is such that $\mathcal{P}(x,y) \geq 0$ for all $y \in B(p)$, then $h(p,x) \geq h(p,z)$ for all $z \in B(p)$.

Now suppose $x \in B(p)$ and $h(p,x) \geq h(p,z)$ for all $z \in B(p)$. Let $w \in B(p)$ be such that $\mathcal{P}(w,y) \geq 0$ for all $y \in B(p)$. It follows from above that $h(p,w) = 0$, while $h(p,x) \geq h(p,w)$ is also true by assumption. But since it is necessarily true that $h(p,x) \leq 0$ because $x \in B(p)$, this implies $h(p,x) = h(p,w) = 0$, so that $\mathcal{P}(x,y) \geq 0$ for all $y \in B(p)$ too. From this also follows that $\mathcal{P}(x,w) = 0$ and therefore $x \mathrel{\widehat{\sim}} w$. ■

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