Labour Values and the Theory of the Firm: Part I: The Competitive Firm

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Abstract: *This paper is the first part of a Marxian critique of the theory of the firm, focusing on the analysis of labour values. Starting from Adam Smith's example of the deer hunter marginal analysis is introduced, culminating in the derivation of the Labour Value Function as the supply curve of the competitive firm in terms of labour values. The analysis is based on a new definition of labour value, which is Marxian in spirit and respects explicitly production conditions and by this becomes an integral part of modern mathematical optimization methods not found in Marx. The analysis offers a further development and coherent interpretation of Marx's value theory. The analysis is limited to the case of the competitive firm.*

Keywords: Marxian economics; labour theory of value; value theory; marginal analysis; microeconomics; theory of the firm; marginal cost; the labour value function; supply function; Adam Smith;

JEL classification: A13; B14; B24; B51; D21; D41; D46

I. Introduction

Marxists have been particularly week in providing a critique of orthodox microeconomic theory. This is mainly due to the failure of not having developed a consistent approach to the labour theory

of value. This paper is part of an effort to put forward a Marxian theory of production based on a definition of labour value, which is Marxian in spirit and modern in that it takes properly account of production conditions and by this becomes an integral part of modern mathematical optimization methods, which have been absent in Marx's work.

This article begins with examining Adam Smith's example of the deer and beaver hunters, justifying labour as being the natural determinant of price. Surely one would have wished Adam Smith to be more elaborate on his example, but Böhm-Bawerk's claim, Smith had only asserted the labour theory of value without having provided a proof, is simply unjustified, as is the claim that the labour theory of value applies only to the 'early and rude state of society'. This is shown in generalizing Adam Smith's example by introducing diminishing marginal productivity of labour. We show that this is sufficient to explain surplus labour. In a further step, we introduce capital and provide an analysis of labour values within the theory of the firm for the case of perfect competition. The core of the paper is the derivation of the *Labour Value Function* as the minimum labour value, required to produce a commodity, depending on the quantity of output and the factor price ratio. We show that marginal cost is labour value expressed in monetary terms. Finally, implications of the analysis for the theory of capital are briefly addressed.

II. The Morishima – Pasinetti Definition of Labour Value is False!

The elimination of Marxism and Marxists from economic theory proper must end. Western Cold War Marxism, including Sraffians and neo-Ricardians certainly had the effect of creating great confusion in the labour movement and within the ever growing strata of the population acquiring higher education - and this not only in the West - but it did not have any serious impact on orthodox

economics, neither did it contribute in any way to practical ameliorations of the conditions of the labouring classes.

On the other hand, bourgeois economics was challenged by scientific socialist developments, but here again 'Cambridge Marxists'[1] were very effective in denouncing Kantorovich and other progressists, as 'anti-Marxist'. At the core of this 'Cambridge Marxism' is a definition of labour value which has a strong appeal to Marx' original concept, but which is simply false in that it whips out the absolutely important distinction between *labour value* and *labour power*, the difference of which is *surplus labour*.

What the Cambridge Marxists declare as labour value - $\boldsymbol{\lambda} = \mathbf{a_n[I - A]^{-1}}$; $\boldsymbol{\lambda}$ is the vector of labour values (*sic*), $\mathbf{a_n}$ is the vector of labour coefficients, $\mathbf{A}$ is the matrix of technical coefficients (Pasinetti, 1977) - is nothing else, but the proper definition of *labour power*, which shall be designated in this paper as $\boldsymbol{v}$. This is not identical to the Marxian notion of variable capital, but with the notion of vertically integrated labour coefficients in Pasinetti. The difference between *labour value,* $\boldsymbol{\lambda}$, and *labour power,* $\boldsymbol{v}$, is *surplus labour,* $\boldsymbol{s}$. When the Cambridge Marxists claim that their definition is the authentically Marxian definition in mathematical terms one has to admit that this appears to be so, but one should understand that one certainly does not honour Descartes, for example, by insisting that the moon is surrounded by a milky substance! If there is one undeniable shortcoming of Classical and Marx' Political Economy, it is their neglect of mathematical optimization methods in the development of their theories. But the application of these methods is an absolute imperative to economic analysis. When labour is the only *source* of value, its use has to be optimal. The definition of *socially necessary labour* as put forward by Marx (1867, p. 54 ff.) is a

1 We use the term 'Cambridge Marxists' as a substitute for 'Western Cold War Marxists', like Dobb, Meek, Steedman, Okishio, Morishima, including Sraffians and neo-Ricardians, like Pasinetti, Heinz Kurz etc. to Foley and Duménil & Lévy, to name only the most eminent. Whereas the Cold War is over 'Cambridge Marxism' will continue as long as there is an emancipatory labour movement. This follows from a theorem on ideology not stated here.

concept stemming from feudal times. The modern Morishima-Pasinetti definition (the vertical integrated labour coefficients) is simply a summing-up of labour time used directly or indirectly to produce a commodity. Its major shortcoming is the neglect of the production conditions. A certain progress has been made by Sraffa with his concept of *quantities of dated labour* which is based on Sraffa's *cost of production* or in conventional terms on *average cost*. More generally production conditions are properly taken account of only, if one uses marginal cost. We propose the following definition of *labour value:*

The labour value of a commodity is the increment of labour, necessary to increase output by one more unit, leaving all other factors of production constant. The minimum labour value for a given socially determined quantity of a commodity is its socially necessary labour.

In mathematical terms, the increment of labour is **ΔL** and the increment of output is **ΔQ**. The labour value of that increment of output is $\lambda = \frac{\Delta L}{\Delta Q}$ or for infinitesimal changes $\lambda = \frac{\partial L}{\partial Q}$. We use partial derivatives to indicate that other factors of production remain constant. Notice that *labour value* is just the inverse of the *marginal productivity of labour* $\lambda = \frac{1}{\partial Q / \partial L}$. The usefulness of the definition will become apparent in the following, establishing the validity of the labour theory of value for perfectly competitive markets by applying this definition of labour values to the analysis of the theory of the firm and the determination of equilibrium prices.

13 years before the publication of *Capital* (1867), Hermann Heinrich Gossen (1854) appears to have put forward a marginal analysis, not only of utilities, but also of labour values. It is this type of analysis which offers a coherent treatment of values and prices as it is the proper application of the

economic principle to the use of labour. As labour - apart from Nature[2] - is the only *source of value,* its use in any working process has to be optimized and marginal analysis as one of the most important methods of mathematical optimization is the method naturally to be employed.

The problem is that Political Economy is a social as well as a political science; it is the science which is at the bottom of the analysis of the class struggle and therefore bourgeois economists have banned labour values in their efficient marginal definition, systematically from economics. Unfortunately there are no socialist or Marxist economists, who had understood the concept properly. There is no account in the economic literature on Jevons' remark that commodities exchange according to labour values! (Jevons, 1871, p. 187), a remark which is based on the marginal concept of labour value.

Now we shall develop some of the straight forward properties of the marginal concept of labour values, using Adam Smith's example of the hunters chasing deer and beaver.

III. The Classical Vision of Production[3]

Adam Smith considered:

> "In that early and rude state of society which precedes both the accumulation of stock and the appropriation of land, the proportion between the quantities of labour necessary for acquiring different objects seems to be the only circumstance which can afford any rule for exchanging them for one another. If among a nation of hunters, for example, it usually costs twice the labour to kill a beaver which it does to kill a deer, one beaver should naturally exchange for or be worth two deer. It is natural that what is usually the produce of two days or two hours labour, should be worth double of what is usually the produce of one day's or one hour's labour."
>
> (Smith,1776, Book. I, Chapter VI)

2 In this discussion we ignore Natures contribution to the creation of value, assuming its services to be costless. This is of course absolutely inadmissible for our times and the reader is invited to advance the discussion on this point in particular.

3 Of course this section is a caricature of the Classics to unveil the anti-thesis Marxism – Marginalism as bourgeois ideology. One only has to think of Turgot's S-shaped production function in agriculture.

Applying to this example the modern tool of the production function, we recognize at once that Adam Smith makes a very special assumption of fixed coefficients of production. This appeared to him to be sufficient as he considered a primitive form of production under usual circumstances. This view has been taken over also by Marx. But even under primitive circumstances, there are no usual production conditions which is why the hunter-gatherers were nomads. However, first, we shall elaborate Adam Smith's example using modern techniques of analysis.

Figure 1 demonstrates the fixed coefficient production functions for deer and beaver. The mathematical formula for the production of deer is:

$$Q_D = A_D L_D;$$

$$\begin{gathered} Q_D - \text{amount of deer}, \\ A_D - \text{average productivity of labour}, \\ L_D - \text{amount of labour power used to hunt deer} \end{gathered} \qquad \textbf{(1)}$$

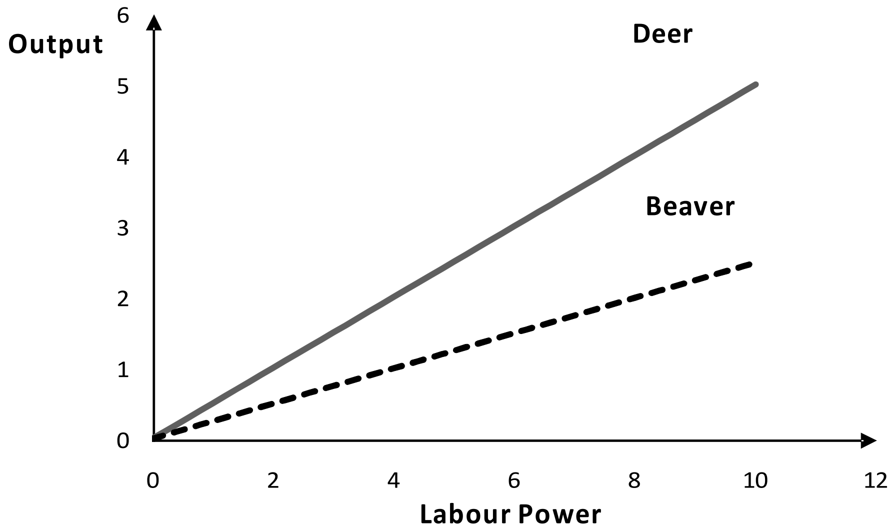

Figure 1: Production Functions for Deer and Beaver Hunting

and for beaver hunting:

$$Q_B = A_B L_B;$$

Q_B − amount of beaver,
A_B − average productivity of labour,
L_B − amount of labour power used to hunt beaver **(1a)**

It is very important to realize that the input of the production process is *labour power*, or the amount of *labour force*, which has to be distinguished carefully from *labour value* (Marx, 1867, Chapter 8, Der Arbeitstag, p. 248; Fisher, 1906, p. 175, footnote).

Figure 2 shows the average labour productivities of producing deer and beaver. For the case of fixed coefficients these are equal to the marginal labour productivities. They are constant and independent of the quantity of output and of labour power employed.

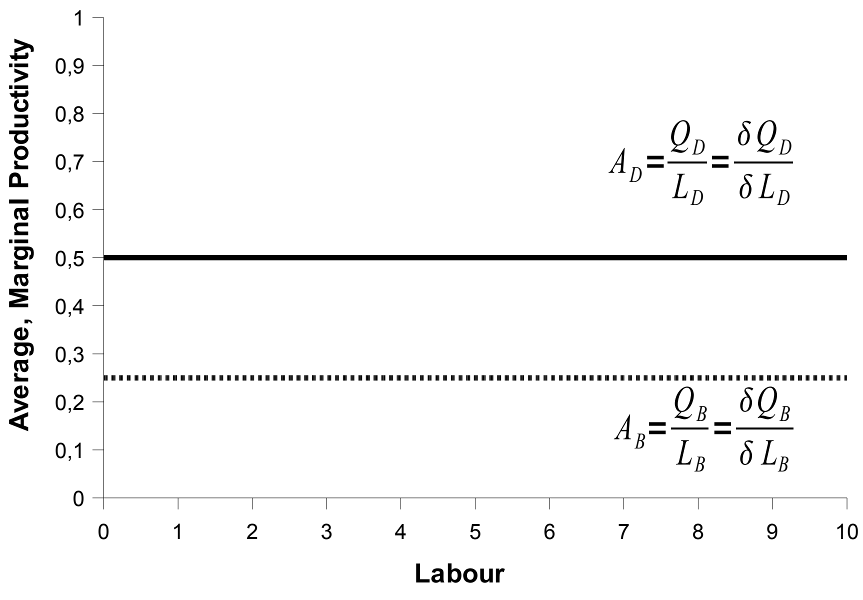

Figure 2: Average and Marginal Productivities of Labour for Deer and Beaver Hunting

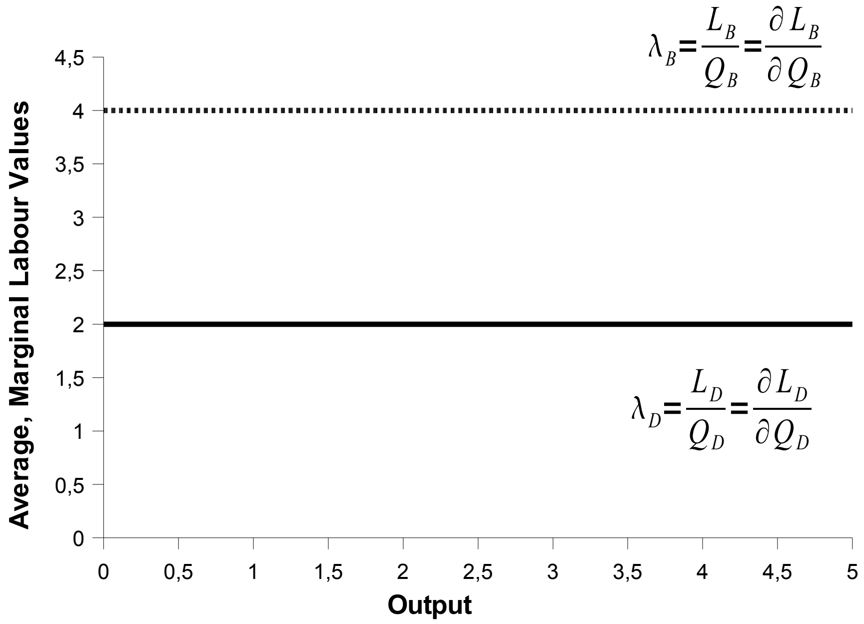

Figure 3: Average and Marginal Labour Values for Deer and Beaver Hunting

To obtain expressions for labour values, we have to take the reciprocals of the average and marginal productivities of labour. These are presented in **Figure 3,** showing labour values as understood by Classical economists and Marx. The labour values are independent of output; but this is a very special case indeed.[4]

Figure 4 shows the production possibility function for deer and beaver for this special case of a constant average productivity of labour. The function shows the feasible combinations of amounts of deer and beaver which can be obtained using a given amount of labour power efficiently. Therefore the negative of the slope of this function,**-α**, shows the price which has to be paid in terms of one good if the other good is increased by one unit. The price is to be understood as the opportunity cost of producing a particular good. The price ratio is equal to the ratio of average or marginal labour values and equal to the inverse of the ratio of average productivities of labour.

4 This very special case of production conditions - labour power being the only factor of production and its output elasticity, a = 1, with a horizontal supply curve of constant labour value, which represents therefore the *socially necessary labour* whatever the demand conditions - offers *the standard unit of measurement of labour value*. 1 hour of labour power employed under these production conditions is equal to 1 unit of labour value. This standard unit of value is also independent of distribution, i.e. remains constant whatever the values of the wage rate or the rate of interest.

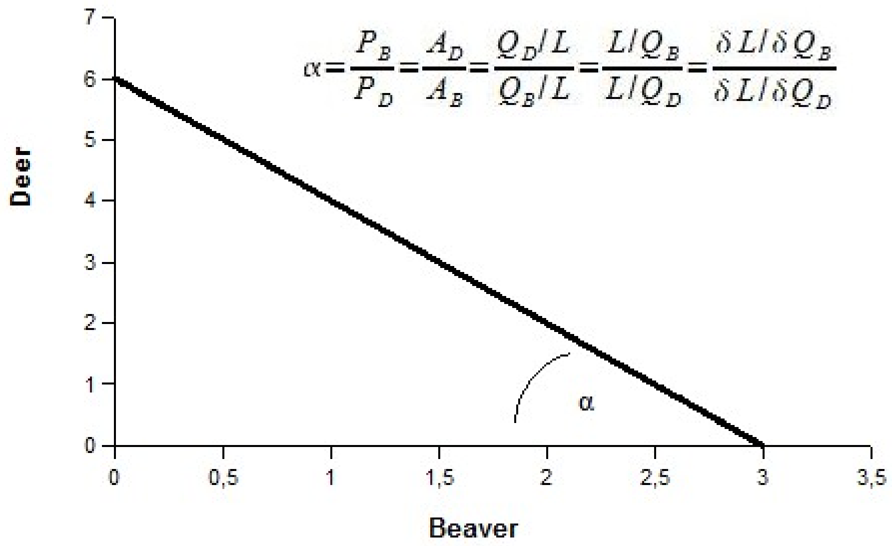

Figure 4: Production Possibility Frontier for Deer and Beaver Hunting

The derivation of the equation for the production possibility frontier is straight forward. From **(1)** and **(1a)** we derive the inverses

$$L_D=\frac{Q_D}{A_D} \qquad \textbf{(2)}$$

and

$$L_B=\frac{Q_B}{A_B} \qquad \textbf{(2a)}$$

A given amount of labour power, **L,** can be employed either in deer or in beaver hunting.

$$L=L_D+L_B \qquad \textbf{(3)}$$

(2) and **(2a)** substituted in **(3)** is:

$$L=\frac{Q_D}{A_D}+\frac{Q_B}{A_B} \qquad \textbf{(4)}$$

Resolved for $\boldsymbol{Q_D}$

$$Q_D = A_D L - \frac{A_D}{A_B} Q_B \qquad \textbf{(5)}$$

The slope of the production possibility frontier **(5)** is the ratio of the average productivities of labour and this is equal to the inverse of the price ratio as shown in equation **(6)**.

$$\frac{P_B}{P_D} = \frac{A_D}{A_B} = \frac{Q_D / L_D}{Q_B / L_B} = \frac{L_B / Q_B}{L_D / Q_D} = \frac{\delta L_B / \delta Q_B}{\delta L_D / \delta Q_D} \qquad \textbf{(6)}$$

Using ***v*** to designate labour power (per unit of output) and **λ** for labour values (per unit of output) respectively and defining ***v*** and ***λ*** as

$$v = L / Q ; \lambda = \partial L / \partial Q \qquad \textbf{(7)}$$

we can express **(6)** as

$$\frac{P_B}{P_D} = \frac{A_D}{A_B} = \frac{Q_D / L_D}{Q_B / L_B} = \frac{v_B}{v_D} = \frac{\lambda_B}{\lambda_D} \qquad \textbf{(6a)}$$

An important consequence of the constancy of average labour productivities and labour values is that the production possibility frontier is a straight line. Whatever efficient combination of output is chosen, the ratio of prices (labour values) remains constant. This is a condition where 'demand' has no impact on labour values and therefore also no impact on prices! One can interpret the horizontal lines of average and marginal labour values in **Figure 3** as supply curves in terms of labour values. The supply curves in terms of money are obtained by multiplying the labour values with the wage rate. Wherever these curves are cut by a demand curve the price remains constant. The commodities exchange according to their labour values. If we multiply the ratio of prices as shown in the equation **(6)** with the ratio of outputs we obtain the ratio of total values, $\mathbf{L_B/L_D}$.

$$\frac{P_B Q_B}{P_D Q_D} = \frac{L_B}{L_D} = \frac{\frac{\delta L_B}{\delta Q_B} Q_B}{\frac{\delta L_D}{\delta Q_D} Q_D} \qquad \textbf{(8)}$$

However, we must remember that $\mathbf{L_B}$ and $\mathbf{L_D}$ are designating *labour power*. When the average labour productivity and therefore average labour values are constant, there is no surplus labour, In this particular case put forward by Adam Smith, the *labour values*, $\boldsymbol{\lambda_D}$ and $\boldsymbol{\lambda_B}$, equal the values of *labour power*, $\boldsymbol{v_D}$ and $\boldsymbol{v_B}$.

$$\frac{\lambda_B}{\lambda_D} = \frac{v_B}{v_D} \qquad \textbf{(9)}$$

In the following we show that in the general case with changing average and marginal productivities of labour and consequently changing average and marginal labour values, the ratio of prices equals the ratio of labour values as in the formula:

$$\frac{P_B}{P_D} = \frac{\lambda_B}{\lambda_D} = \frac{\frac{\partial L_B}{\partial Q_B}}{\frac{\partial L_D}{\partial Q_D}} \qquad \textbf{(10)}$$

IV. The Marginal Analysis of Labour Values

Generalizing the model, we introduce production functions for deer and beaver with diminishing marginal productivities of labour. We ignore the case of increasing marginal productivity of labour as this case cannot be regarded as a stable situation, production would be expanded, until decreasing marginal productivities set in. The labour necessary for the "production" of deer or beaver becomes a function of the quantity produced. But in the determination of the cost of output, what matters is not the average amount of labour but the marginal labour necessary to produce a marginal (extra) unit of output. The price (in terms of labour) – the labour per unit of output - becomes a function of the quantity of output. Here we have a supply function in its simplest form based on labour values!

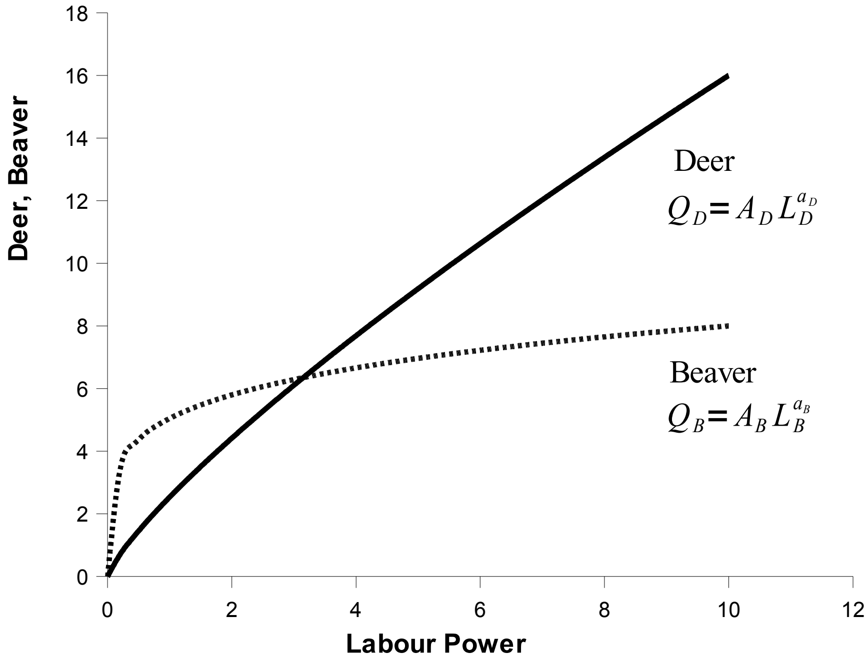

Figure 5: Production Functions for Deer and Beaver Hunting

Figure 5 shows production functions for deer and beaver with diminishing marginal productivities of labour.

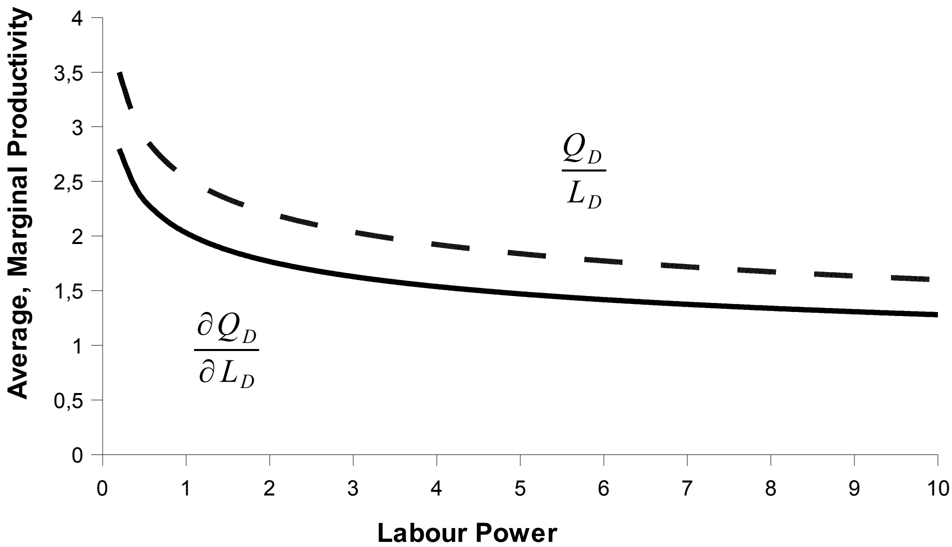

Figure 6a: Average and Marginal Productivities of Labour for Deer Hunting

The average and marginal productivities of labour for deer and beaver hunting, which are now changing with the level of employment, are shown in **Figure 6a** and **Figure 6b** respectively.

To obtain the functions of average and marginal labour values, the production functions have to be inverted and average and marginal labour values have to be constructed on the basis of these

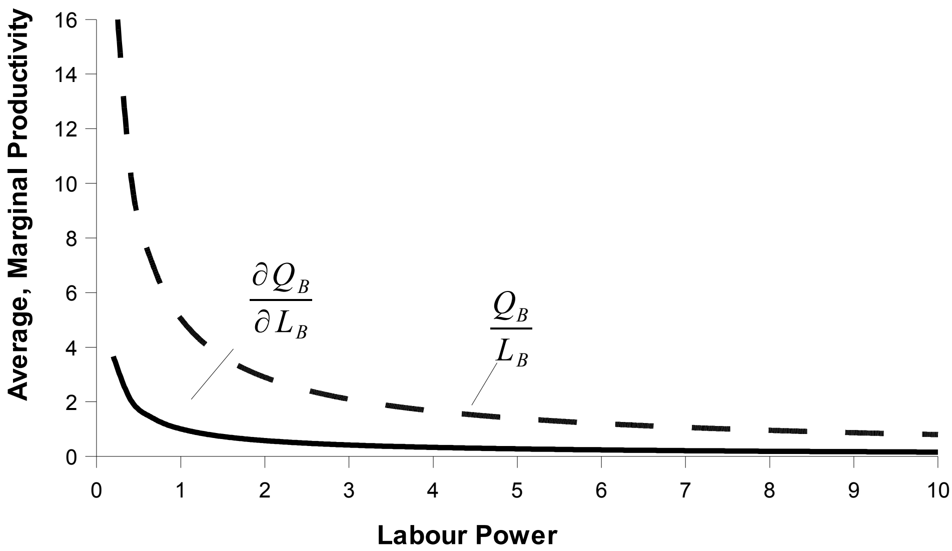

Figure 6b: Average and Marginal Productivities of Labour for Beaver Hunting

functions as demonstrated in **Figure 7a** and **Figure 7b**. One should notice that the marginal labour values are necessarily greater than the average labour values.

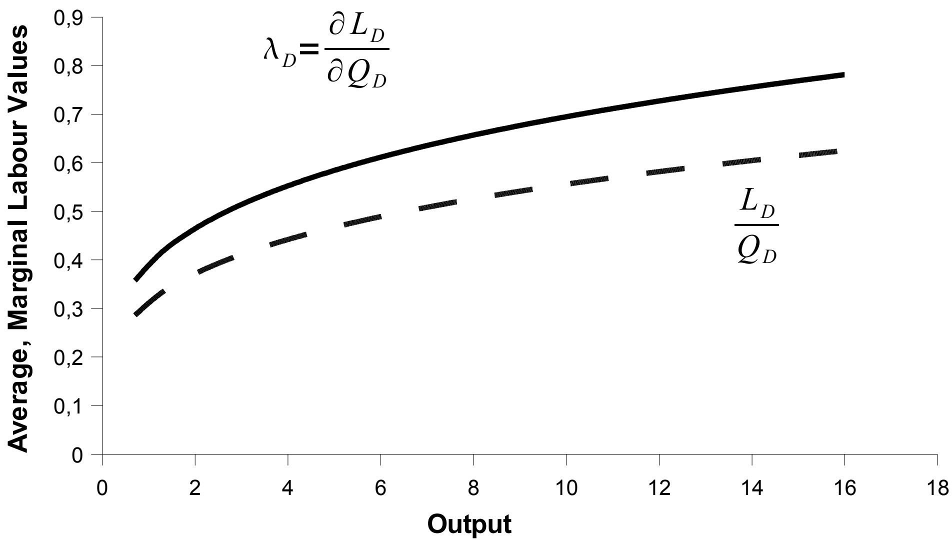

Figure 7a: Average and Marginal Labour Values for Deer Hunting

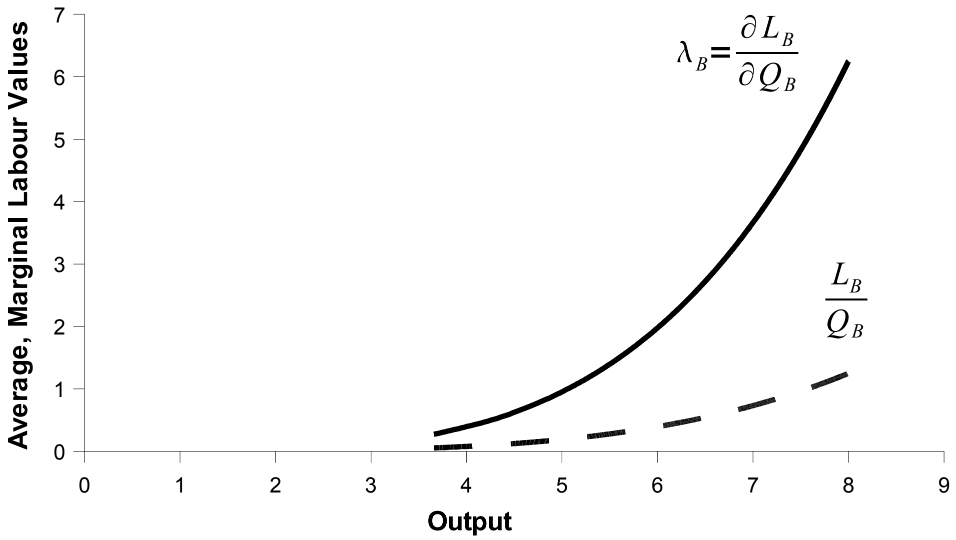

Figure 7b: Average and Marginal Labour Values for Beaver Hunting

In fact, the marginal labour value as a function of output is nothing else but the supply function in terms of labour values.

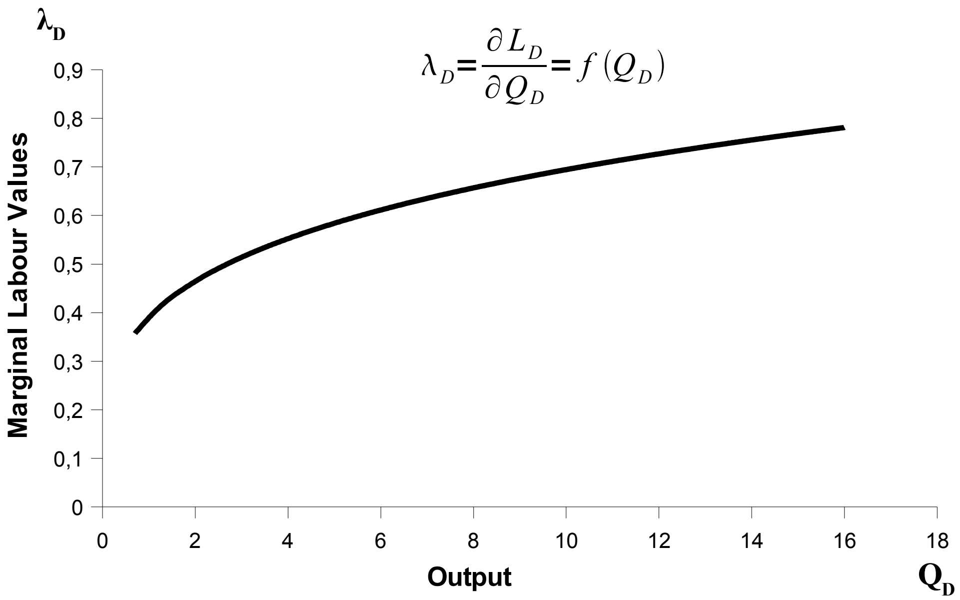

Figure 8a: The Labour Value Function for Deer Hunting

This function shall be named *the Labour Value Function*.

$$\boldsymbol{\lambda = f(Q)} \qquad \textbf{(11)}$$

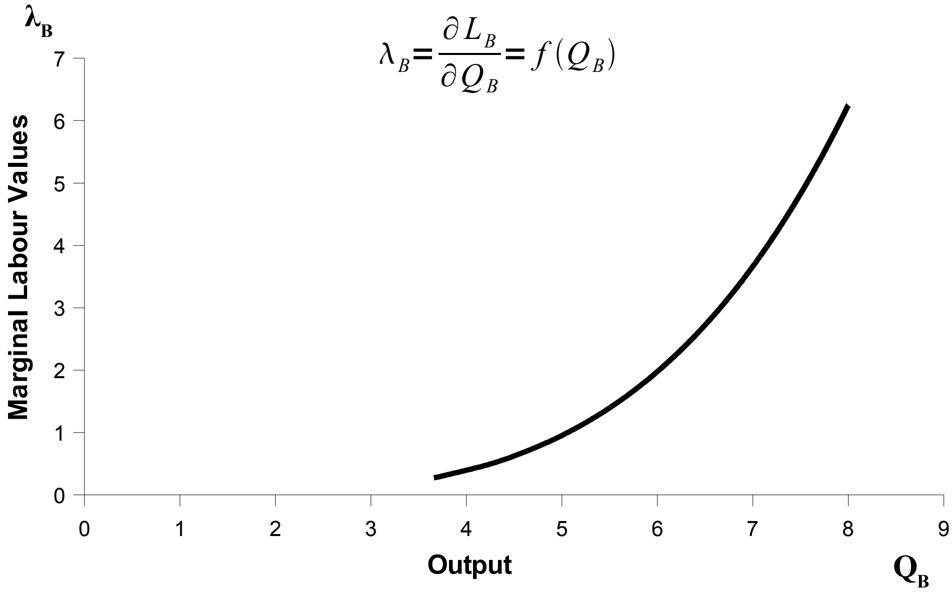

Figure 8b: The Labour Value Function for Beaver Hunting

All points on the *Labour Value Function* represent the minimum labour value required for the production of the corresponding level of output. The point where the *demand function*[5] intersects this function defines the *socially necessary labour* for the production of the commodity.

Now we leave Adam Smith's example for a moment and consider a market situation for which we have derived the Labour Value Function. The multiplication of the labour values of the Labour Value Function with the wage rate, **w**, gives the marginal cost curve, showing marginal cost as a function of output.

$$\frac{dC}{dQ} = w\lambda = w f(Q) \qquad \textbf{(12)}$$

where $\boldsymbol{\frac{dC}{dQ}}$ – marginal cost, **w** – wage rate,

5 A demand function in terms of labour values attributes to each quantity of a commodity an amount of labour the consumer is willing to sacrifice in order to obtain an additional unit of the commodity $\Lambda = \frac{\partial L}{\partial Q} = f(Q)$. These labour values are 'labour commanded', (**Λ = p/w**) in the sense of the Classics.

Figure 9 shows an ordinary diagram with the *supply* and *demand functions* of an industry for the case of perfect competition. The equilibrium price is well determined by the labour values. In

equilibrium, labour commanded, $\boldsymbol{\Lambda} = \boldsymbol{p_e/w}$, is equal to labour embodied, λ_e. The equilibrium price, $\mathbf{p_e}$, is *the socially necessary labour,* $\boldsymbol{\lambda_e}$, multiplied by the wage rate, $\boldsymbol{w}$.

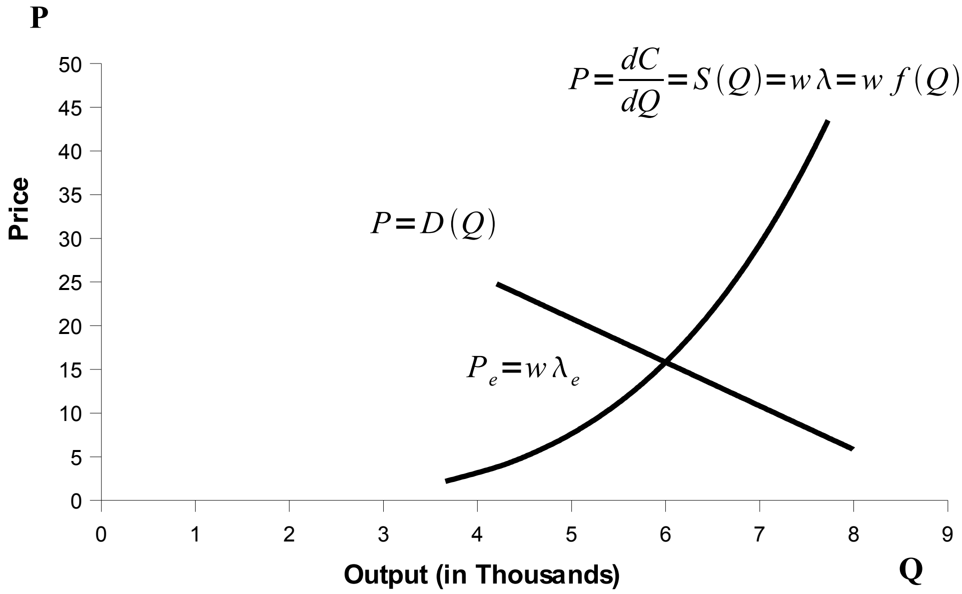

Figure 9: Supply and Demand Functions of an Industry

Under perfect competition the price is proportional to the labour value. The supply function of the industry is the sum of the quantities supplied by the firms of the industry.

Now we derive the *Production Possibility Frontier* for the general case of *diminishing marginal productivities of labour*. The procedure is the same as before:

From the production functions

$$Q_D = A_D\, L_D^{a_D} \qquad \textbf{(13a)}$$

and $$Q_B = A_B L_B^{a_B} \quad \textbf{(13b)}$$

we derive the demand functions for *labour power*.

$$L_D = \left(\frac{Q_D}{A_D}\right)^{(1/a_D)} \quad \textbf{(14a)}$$

and $$L_B = \left(\frac{Q_B}{A_B}\right)^{(1/a_B)} \quad \textbf{(14b)}$$

A given quantity of *labour power* can be employed alternatively in deer or beaver hunting:

$$L = L_D + L_B \quad \textbf{(15)}$$

(14a) and **(14b)** substituted in **(15)** gives

$$L = \left(\frac{Q_D}{A_D}\right)^{(1/a_D)} + \left(\frac{Q_B}{A_B}\right)^{(1/a_B)} \quad \textbf{(16)}$$

and this resolved for $\mathbf{Q_D}$ is

$$Q_D = \left[A_D^{1/a_D} L - \frac{A_D^{1/a_D}}{A_B^{1/a_B}} Q_B^{1/a_B} \right]^{a_D} \quad \textbf{(17)}$$

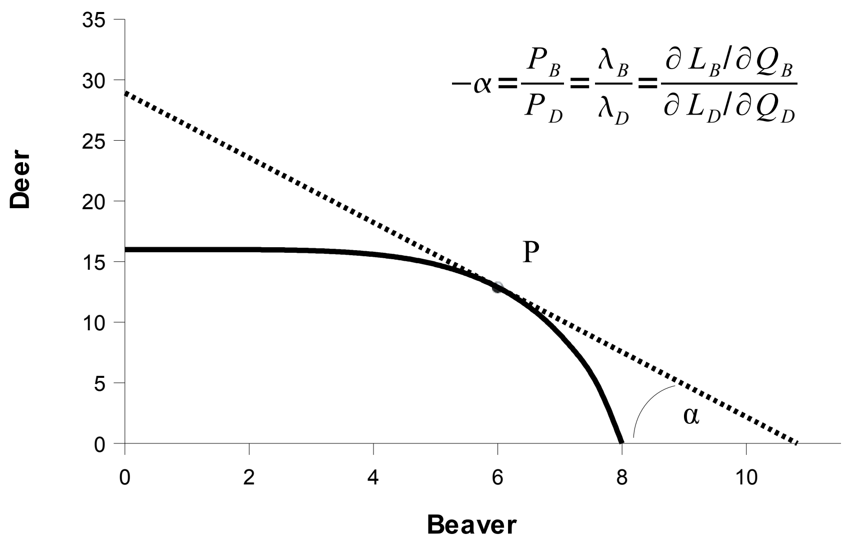

Figure 10: Production Possibility Frontier for Deer and Beaver Hunting

The negative of the slope of *the Production Possibility Frontier*, known as *the marginal rate of transformation*, (MRT), represents the ratio of prices in the sense of opportunity cost as discussed above.

$$MRT = -\frac{dQ_D}{dQ_B} = a_D \left[A_D^{1/a_D} L - \frac{A_D^{1/a_D}}{A_B^{1/a_B}} Q_B^{1/a_B} \right]^{a_D - 1} \left[-\frac{1}{a_B} * \frac{A_D^{1/a_D}}{A_B^{1/a_B}} Q_B^{1/a_B - 1} \right] \qquad \textbf{(18)}$$

But this slope is also equal to the ratio of *marginal cost* and also equal to the *ratio of labour values*. This holds for all points of *the Production Possibility Frontier*. The point selected by the *demand conditions* represents the equilibrium price ratio and also the ratio of labour values which are the expressions of *the socially necessary labour* for the production of the corresponding commodities.

All these conditions are valid only under perfect competition. This state of affairs is '*Pareto efficient*'. In a *Pareto efficient equilibrium* prices are proportional to labour values as we have stated in equation **(10)** above.

$$MRT = \frac{P_B}{P_D} = \frac{\lambda_B}{\lambda_D} = \frac{\frac{\partial L_B}{\partial Q_B}}{\frac{\partial L_D}{\partial Q_D}} \qquad \textbf{(10)}$$

What distinguishes the general case of variable marginal labour values from the special case of Adam Smith's example of constant marginal labour values is the occurrence of *surplus labour*. As is shown very clearly in **Figures 7a** and **7b**, there is a difference between marginal and average labour values. This difference between the *marginal labour value*, ***λ***, and average labour value, ***v***, is *surplus labour* (per unit of output), $s = \frac{L_s}{Q}$ **:**

$$s = \lambda - v \qquad \textbf{(19)}$$

or more explicitly

$$\frac{L_S}{Q} = \frac{\partial L}{\partial Q} - \frac{L}{Q} \qquad \textbf{(20)}$$

Of course, *total surplus labour*, **L_S**, is also a function of output

$$L_S = [\lambda - v]Q = \left[\frac{\partial L}{\partial Q} - \frac{L}{Q}\right]Q \qquad \textbf{(21)}$$

And total profits, **π,** is the value of *total surplus labour* in terms of money,

$$\pi = w\,L_S \qquad \textbf{(22)}$$

(22) combined with **(21)** gives

$$\pi = w\,L_S = w[\lambda - v]Q = w\left[\frac{\partial L}{\partial Q} - \frac{L}{Q}\right]Q \qquad \textbf{(23)}$$

At this point, it may be appropriate to consider the Marxian notion of *constant capital*. Constant capital consists of commodities applied to the production of *surplus value* (commodities, bought to make a profit). We have shown above, that in equilibrium, the value of a commodity is equal to the *marginal labour value* embodied in it. But this labour value contains also *surplus labour* apart from all the *value of labour power* used up for its production. And this surplus labour is a function of output and therefore of demand. We see here that demand has an impact on *distribution* and the *value of constant capital*. But this does not mean, that the impact of demand invalidates the *labour theory of value*. The effect of changes of the *value of capital* is discussed in *capital theory* under the name of *Wicksell effects*. Unfortunately, this discussion is as confused as the discussion on the labour theory of value. We shall come back to this in the following section proving the results for the case of production including capital.

V. Labour Values and the Profit Maximizing Firm

The working processes of production are determined by the profit maximizing firm. It is the firm which realizes the law of value. We distinguish between the firm under perfectly competitive

conditions and the monopolistic firm. The competitive firm faces an environment where the price of the product, **p**, as well as the prices of the inputs[6], **w** and **r**, are taken as given and the firm has no power to influence these prices. Its output is negligible in relation to overall output of the industry and its demand for the factors of production is also not sufficient to influence the prices of these factors.

In order to maximize profits the firm equalizes its *marginal cost* to the price. This results from the maximization of *the profit function*.

$$\pi = R(Q) - C(Q) \qquad \textbf{(24)}$$

where **π** – profits, **R** – revenue, **C** – cost

Revenue is **R = pQ**, and under perfect competition **p** is constant. Therefore:

$$\frac{dR}{dQ} = p \qquad \textbf{(25)}$$

Cost consists of the cost of the inputs. In general, some cost are fixed and some cost depend on the level of output. The following expression distinguishes the variable inputs labour power, **L**, in terms of working hours, and capital, **K**, in terms of money, and an amount of fixed cost, $\mathbf{C_F}$ in terms of money. *The cost equation* is:

$$C = wL + (1+r)K + C_F \qquad \textbf{(26)}$$

where **w** - the wage rate, **r** - the price of the services of capital (the interest rate)

The first order condition for maximizing profits is:

$$\frac{d\pi}{dQ} = \frac{dR}{dQ} - \frac{dC}{dQ} = 0 \qquad \textbf{(27)}$$

and therefore

$$\frac{dR}{dQ} = \frac{dC}{dQ} \qquad \textbf{(28)}$$

6 When we consider **r** as a price of the input *capital* this does not mean the price of the capital good as a commodity but the price for using the value of the capital good for the length of the production period. Usually one speaks of the *price of the services of capital*. We do not discuss the determination of **r**; this will be the subject of another paper dealing with Kantorovich's *norm of effectiveness*.

As from **(25)** at the point of maximum profits, price, **p**, is equal to **dR/dQ** and substituted in **(28)**

$$p = \frac{dC}{dQ} \qquad \textbf{(29)}$$

For the competitive firm, generally, marginal cost is an increasing function of output. The firm expands its output until marginal cost equals the price.

In order to calculate marginal cost, **dC/dQ**, we have to transform the cost equation **(26)** into a function of output only, the resulting function is called *the classical cost function,* **C =** ***f*****(Q)**.

The firm faces its production conditions represented by its production function, **Q =** ***g*****(K, L)**, and tries to minimize cost. The *cost minimization problem* can be expressed in form of the Lagrangian:

$$\pounds = wL + (1+r)K + \mu[Q_0 - g(K, L)] \qquad \textbf{(30)}$$

The first order conditions for *cost minimization* are:

$$\frac{\partial \pounds}{\partial L} = w - \mu \frac{\partial g(K, L)}{\partial L} = 0$$

$$\frac{\partial \pounds}{\partial K} = (1+r) - \mu \frac{\partial g(K, L)}{\partial K} = 0 \qquad \textbf{(31)}$$

$$\frac{\partial \pounds}{\partial \mu} = Q_0 - g(K, L) = 0$$

From this we obtain the following expression for *the optimal factor input combination* as a function of the prices of the inputs:

$$\frac{w}{1+r} = \frac{\partial g(K, L)/\partial L}{\partial g(K, L)/\partial K} \qquad \textbf{(32)}$$

Notice, that the *marginal productivities* are functions of the *factor input combination*. This can be expressed as an implicit function which is called *the expansion path*

$$\frac{w}{1+r} - \frac{\partial g(K, L)/\partial L}{\partial g(K, L)/\partial K} = 0 \qquad \textbf{(33)}$$

The expansion path is the locus of the *optimal factor combinations* for different levels of output on the capital labour plane. The *total differential* of a given level of output is:

$$dQ = \frac{\partial Q}{\partial L} dL + \frac{\partial Q}{\partial K} dK = 0 \qquad \textbf{(34)}$$

which gives:

$$\frac{dK}{dL} = -\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} \qquad \textbf{(35)}$$

Graphically this can be shown as follows:

Figure 11 shows a production function of type Cobb-Douglas in three-dimensional space. The border lines between different colours represent *isoquants*, points with the same level of output.

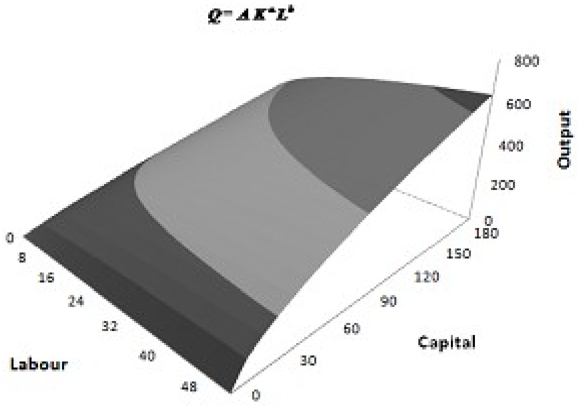

Figure 11: The Production Function

Figure 12 shows the same production function on the capital labour plane. In addition, an iso-cost line and *the expansion path* are presented.

On the capital labour plane an *isoquant*, **(34)**, represents a level of output produced by different factor combinations. The factor costs are represented by an *iso-cost line,* the slope of which is *the*

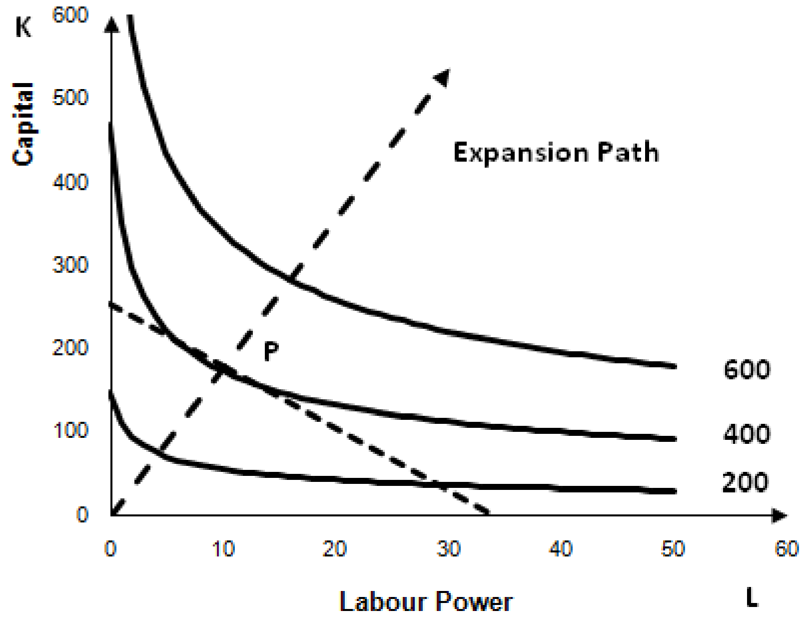

Figure 12: Isoquants and Iso-Cost Line

factor price ratio, ***w/(1+r)****. The iso-cost line* is:

$$K = \frac{\bar{C}}{(1+r)} - \frac{w}{(1+r)} L \qquad \textbf{(36)}$$

The *optimal factor combination* is there where the *iso-cost curve* is tangent to the *isoquant.* Along the *iso-cost line* it is here, where output is highest. All the points of tangency for different levels of cost, (different iso-cost lines which are parallel) represent *the expansion path*.

The expansion path allows us to express the optimal amount of one factor input, **L*** or **K*,** as a function of the other. In our case we have 2 functions, **L* =** ***f*****(K)** and **K* =** ***f*****(L)**. Substituting capital for labour we can express cost as a function of capital only, **C =** ***f*****(K*)**, and substituting labour for capital we can express cost as a function of labour only, **C =** ***f*****(L*)**. Then we derive the inverses of these functions and obtain another 2 functions expressing the optimal amounts of factor inputs as functions of cost, **L* =** ***f*****(C) and K* =** ***f*****(C)**. These expressions can now be substituted into the production function which gives us output in terms of cost only. The inverse of this function is *the classical cost function,* **C =** ***f*****(Q)** and the marginal cost function can be derived from this. The

marginal cost function

$$\frac{dC}{dQ} = f'(Q) \qquad \textbf{(37)}$$

is the *supply function of the firm* under conditions of perfect competition. For a presentation of the derivation of the marginal cost function see (Henderson & Quandt, 1980, p. 83 ff).

In this type of derivation of the *marginal cost curve* there does not seem to be any *labour values* involved. Output is the result of the combined effects of the factor inputs. To attribute the result of the production process to the labourer only seems to contradict the facts of economic analysis. But surely, the only one who produces is the labourer. The other factors of production only increase his productivity. To reveal this, we shall derive *the Labour Value Function*.

VI. The Derivation of the Labour Value Function

The *cost minimization conditions* reveal also that the *Lagrange multiplier*, **μ,** is marginal cost. This can be shown as follows. The total differential of cost is

$$dC = \frac{\partial C}{\partial L} dL + \frac{\partial C}{\partial K} dK \qquad \textbf{(38)}$$

and the total differential of output is

$$dQ = \frac{\partial Q}{\partial L} dL + \frac{\partial Q}{\partial K} dK \qquad \textbf{(39)}$$

From the cost equation **(26)** we can derive the derivatives of cost with respect to labour and capital respectively as:

$$\frac{\partial C}{\partial L} = w$$
$$\frac{\partial C}{\partial K} = (1+r) \qquad \textbf{(40)}$$

and substituted into **(38)** we get

$$dC = w\, dL + (1+r)\, dK \tag{41}$$

From the first order conditions **(31)** we get

$$\begin{aligned} w &= \mu \frac{\partial Q}{\partial L} \\ (1+r) &= \mu \frac{\partial Q}{\partial K} \end{aligned} \tag{42}$$

and these expressions substituted into **(41)** is:

$$dC = \mu \frac{\partial Q}{\partial L} dL + \mu \frac{\partial Q}{\partial K} dK \tag{43}$$

which is equal to

$$dC = \mu \left[\frac{\partial Q}{\partial L} dL + \frac{\partial Q}{\partial K} dK \right] \tag{44}$$

But the term in brackets is **dQ** from **(39)**. Therefore

$$\mu = \frac{dC}{dQ} \tag{45}$$

Looking again at **(42)** we see that

$$w = \mu \frac{\partial Q}{\partial L} \tag{46}$$

Here we have the link to *labour values* as the reciprocal of *the marginal productivity of labour* is the expression for *labour value* as we have defined it above.

$$\mu = w \frac{\partial L}{\partial Q} \tag{47}$$

As $\boldsymbol{\mu}$ is *marginal cost* we see that at *minimum cost*, marginal cost is nothing else but *labour value* expressed in terms of money.

$$\mu = \frac{dC}{dQ} = w \frac{\partial L}{\partial Q} = w \lambda \tag{48}$$

This holds for all *cost minimising factor combinations*, **(K*, L*)**, along the *expansion path*.

For a given amount of capital, **K***, there is an optimal amount of labour power, **L*,** with which a

certain output, **Q***, can be produced with minimum cost. The *labour value* of a unit of that output is *the marginal labour value,* **δL/δQ,** calculated on the basis of the *optimal factor combination*, (K*, L*).

We shall use an example presenting the derivation of *the Labour Value Function*:

Let the production function be

$$Q = A K^a L^b; \text{ and } a, b < 1; a + b < 1 \qquad \textbf{(49)}$$

and the cost function

$$C = wL + (1+r)K \qquad \textbf{(50)}$$

From the first order conditions for cost minimization we have as in **(31)**

$$w = \mu \frac{\partial Q}{\partial L}; \quad (1+r) = \mu \frac{\partial Q}{\partial K} \qquad \textbf{(51)}$$

and as the marginal productivities of labour and capital for the production function **(49)** are

$$\frac{\partial Q}{\partial L} = b \frac{Q}{L}$$
$$\frac{\partial Q}{\partial K} = a \frac{Q}{K} \qquad \textbf{(52)}$$

we can derive the optimal factor combination for a given factor price ratio as in **(32)**

$$\frac{w}{(1+r)} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{b}{a} \frac{K}{L} \qquad \textbf{(53)}$$

This gives us the optimal quantity of labour power, **L***, as a function of **K** and **K*** as a function of **L**.

$$L^* = \frac{(1+r)}{w} \frac{b}{a} K$$
$$K^* = \frac{w}{(1+r)} \frac{a}{b} L \qquad \textbf{(54)}$$

Expressing **K** in the production function **(49)** as a function of **L** we get

$$Q = A\left[\frac{w}{(1+r)}\frac{a}{b}L\right]^{a} L^{b} \qquad \textbf{(55)}$$

The inverse of this function is *the demand for labour power* as a function of output

$$L^{*} = \left[\frac{1}{A}\left[\frac{w}{(1+r)}\frac{a}{b}\right]^{-a} Q\right]^{1/(a+b)} \qquad \textbf{(56)}$$

Now we express *marginal labour value* as the reciprocal of *the marginal productivity of labour* from **(52)** for an optimal amount of labour power, **L***, that is for a point on the expansion path.

$$\frac{\partial L}{\partial Q} = \frac{1}{b}\frac{L^{*}}{Q} \qquad \textbf{(57)}$$

and substitute labour by the expression of labour demand from **(56)** which gives us labour value as a function of output only, which is *the Labour Value Function*.

$$\lambda = \frac{\partial L}{\partial Q} = \frac{1}{b}\left(\frac{1}{A}\right)^{\frac{1}{a+b}}\left[\frac{w}{(1+r)}\frac{a}{b}\right]^{\frac{-a}{a+b}} Q^{\frac{1}{a+b}-1} \qquad \textbf{(58)}$$

Expressing the complicated constant term as A* the function becomes

$$\lambda = \frac{\partial L}{\partial Q} = A^{*} Q^{\frac{1}{a+b}-1} \qquad \textbf{(59)}$$

$$A^{*} = \frac{1}{b}\left(\frac{1}{A}\right)^{\frac{1}{a+b}}\left[\frac{w}{(1+r)}\frac{a}{b}\right]^{\frac{-a}{a+b}}$$

Here one can see clearly that all what matters are the input prices, **w** and **r**, and the shares in the revenue of these inputs, **a** and **b** as well as the quantity of output **Q.**

This *Labour Value Function* multiplied with the wage rate is the *marginal cost function*.

$$\frac{dC}{dQ} = w\lambda = w\frac{\partial L}{\partial Q} = w A^{*} Q^{\frac{1}{a+b}-1} \qquad \textbf{(60)}$$

The *Labour Value Function* is the *supply function* in terms of *labour values* of a profit maximizing firm under conditions of perfect competition.

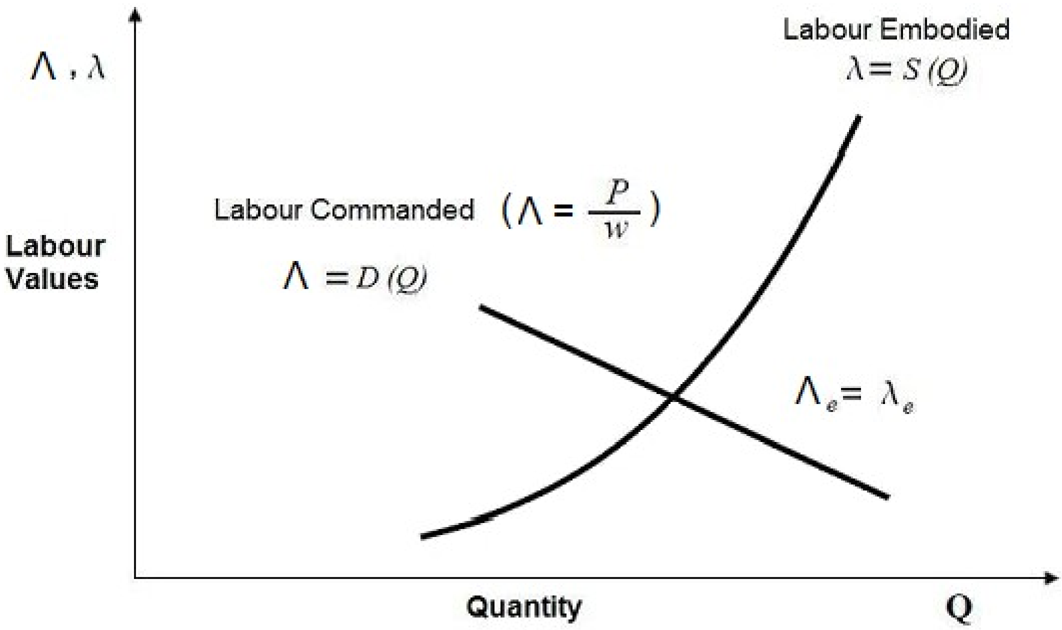

Figure 13: Demand and Supply of a Commodity in Terms of Labour Values

These labour values multiplied with the wage rate give *the marginal cost function*. The *labour value* which corresponds to the equilibrium point where the *Function of Labour Commanded* (the demand function in terms of labour values) cuts the *Labour Value Function* is *the socially necessary labour* to produce the commodity, λ^*. In fact, it is the reciprocal of the real wage.

$$\lambda^* = \frac{p}{w} \qquad \textbf{(61)}$$

The *Labour Value Function* resolves a long-lasting dispute about *labour values* and changes in the *factor price ratio*. In fact, the function reveals that labour values are not only dependent on the level of output but also on the *factor price ratio*. In the analysis above this ratio is treated as a constant. But changes in the *factor price ratio* result in changes of the *labour values*. This is rather natural because prices are proportional to labour values, they reflect labour values. A change in the *factor price ratio* indicates changing cost of the use of inputs in terms of labour values and this must lead to a different *optimal factor input combination*. Therefore a requirement that *labour values* must be independent of the *factor price ratio* is simply unreasonable.

On the other hand, *the standard unit of measurement of labour value* should not and does not depend neither on *the factor price ratio* which reflect the dynamics of an economic system nor on *demand conditions.* It is the time of labour employed in the very special production process which uses *labour power* only and which has *constant returns to scale*, that is, where the productivity of labour does not depend on the level of output but is constant. Quantities of *labour values* are always expressed in terms of this *standard labour value*.

VII. The Standard Unit of Measurement of Labour Value

The production process which offers a definition of the *standard unit of measurement of labour value* is a process which has the following production function:

$$Q_i = A_i L \qquad \textbf{(62)}$$

where $\mathbf{Q_i}$ – amount of output of commodity **i**, $\mathbf{A_i}$ – average labour productivity which is a constant and therefore equal to the marginal labour productivity. There is no *surplus labour* in this production process. In this production process *the value of a unit of labour power* is equal to a *unit of labour value*, for example 1 hour of work in this process is 1 unit of labour power as well as 1 unit of labour value. This is *the standard unit of measurement of labour value*.

The total amount of labour value (hours worked) in the standard production process is equal to the labour value (per unit of output), $\boldsymbol{\lambda_i}$ times output. This is equal to average labour value, $\mathbf{1/A_i}$**,** times output.

$$L = \lambda_i Q_i = \frac{1}{A_i} Q_i \qquad \textbf{(63)}$$

In a perfectly organized economy where all factors of production are fully used and optimally allocated to the production processes of the economy, which is usually termed perfect competition, prices are the monetary expressions of labour values.

$$P_i Q_i = w \lambda_i^* Q_i = w \frac{\partial L}{\partial Q_i} Q_i \qquad \textbf{(64)}$$

The total labour value represented by some quantity **Q** of commodity **i** is:

$$\lambda_i^* Q_i = \frac{P_i Q_i}{w} = \frac{\partial L}{\partial Q_i} Q_i \qquad \textbf{(65)}$$

When we say that the production of a quantity of commodity **i**, $\mathbf{Q_i}$, has cost $\boldsymbol{\lambda^*_i Q_i}$, *in terms of labour,* this refers to *the standard unit of labour value* which is labour time producing commodity according to production function of type **(62).** This does not mean that there have been $\boldsymbol{\lambda^*_i Q_i}$ hours of work involved, but usually less. The difference is surplus labour, $\mathbf{L_S}$, as shown already above in equations **(19)** and **(20)**. This also means that the hours worked are being weighted according to their capital intensity.

$$L_S = [\lambda - v]Q = \left[\frac{\partial L}{\partial Q} - \frac{L}{Q}\right]Q \qquad \textbf{(20)}$$

Here we see clearly the difference between our definition of labour values $\lambda = \frac{\partial L}{\partial Q}$ and the Morishima-Pasinetti definition, $v = \frac{L}{Q}$, which represents only all labour power, directly and indirectly used up.

Now we can approach the difficult question of the *value of constant capital* again. *Constant capital* is an amount of money invested in the creation of *surplus value*, that is, money has been spent on commodities and these commodities are used in production processes to make a profit. Basically, the value of constant capital is just the value of the commodities it consists of.

We have established that the price of a commodity divided by the wage rate equals its labour value **(61)**. The labour value can be divided into the *value of labour power* used up in the production, this value is represented as $\boldsymbol{v}$, the *vector of vertically integrated labour coefficients* and *surplus labour*, $\boldsymbol{s}$.

$$\lambda = s + v \qquad \textbf{(66)}$$

Its relation to price is

$$p = w\lambda = ws + wv \qquad \textbf{(67)}$$

Now we regard the production process as the consumption of *labour power*, $\mathbf{L_i}$, and some *constant capital*, $\boldsymbol{c_i}$, which is a sum of commodities, the capital goods, to create a new commodity.

$$Q_i \leftarrow L_i + c_i; \quad c_i = \sum x_{ij} \qquad \textbf{(68)}$$

In monetary terms this is the cost of production equation

$$p_i Q_i = \pi + wL_i + \sum p_j x_{ij} \qquad \textbf{(69)}$$

The value of output is the sum of profits, the wages for the direct labour power employed, as well as the value of the commodities which constitute constant capital. Expressing profits as surplus labour in monetary terms ($\boldsymbol{\pi} = \boldsymbol{ws}$)

$$p_i Q_i = ws_i + wL_i + \sum p_j x_{ij} \qquad \textbf{(70)}$$

This holds not only for the commodity **i**, but also for all the other commodities constituting *constant capital*, the $\mathbf{x_{ij}}$. We know that their labour value depends on the *factor price ratio* as it is expressed in their *Labour Value Functions*. When the factor price ratio changes so changes the *optimal factor input combination* as well as the *surplus labour*. So the value of constant capital changes. However, the commodities, constituting *constant capital*, are evaluated always at actual market prices and so their value changes even when they have been produced before the change of factor prices has occurred. This shows that the socially necessary labour is determinant of the value of a commodity and not what amount of labour has actually been expended on its production.

There has been a big debate, if it is possible to relate *factor price ratios* unequivocally to *optimal factor combinations*. It has been proven that for the case of Leontief production functions using also *constant capital* where there are no possibilities of substitution, re-switching of factor combinations can occur so that a certain factor combination is optimal at two different levels of the factor price ratio. This has been considered as a great defect of neoclassical economic theory. But although this has been shown to be a valid criticism for the special case of *fixed coefficient production functions* there has not been presented an example for a production system which allows for substitution amongst the factors of production.

On the other hand, it has been proven that for the case of the static Leontief model the 'pure' labour theory of value holds, which means that there is no *surplus labour* and therefore the rate of interest, **r,** is zero. This reflects precisely Schumpeter's position that in a stationary economy the interest rate must be zero. And this implies that in a stationary economy there is no factor price ratio. So the re-switching debate is based on the Sraffian system which is basically flawed as one cannot assume a static, stationary economy without marginal changes and at the same time introduce a positive rate of profit which occurs only under dynamic conditions.

VIII. Some Observations on Demand

Finally, we want to make some observations on labour values and demand in a perfect economy. In **Figure 9** we have already introduced a demand function in terms of labour values[7]

$$\Lambda = f(Q) \ ; \ \Lambda = \frac{\partial L}{\partial Q} \qquad \textbf{(71)}$$

7 See the comments and the footnote to **Figure 9** above. There will be another paper "Labour Values and the Theory of Consumer Behaviour"

In such a demand function $\mathbf{\Lambda}$ represents *labour commanded* , $\Lambda = \frac{p}{w}$, which means that amount of labour power which can be bought by a sum of money. This has to be distinguished from the labour values, $\boldsymbol{\lambda}$, in the supply function $\lambda = f(Q)$, which represent labour embodied. Under perfect competition prices are nothing else but monetary expressions of embodied labour value; all revenue is also nothing but labour value and so is demand as expressed on the markets; labour commanded is equal to labour embodied, ($\mathbf{\Lambda} = \boldsymbol{\lambda}$). But this does not mean that all revenue has been gained through labour. The owners of capital may not have worked at all, receiving nevertheless profits.

These profits are representing labour values only in a perfect economy; if there are mark-ups due to monopoly power this is no longer the case. So, considering demand, we have to distinguish between at least 3 different situations.

If the economy is not perfect, there is monopoly power and there are mark-ups increasing the prices above their embodied labour values, ($\mathbf{\Lambda} > \boldsymbol{\lambda}$ or $\mathbf{p} = \mathbf{w\Lambda} > \mathbf{w}\boldsymbol{\lambda}$), and consequently the revenue derived exceeds the monetary value of embodied labour values, and the demand based upon this revenue in terms of labour commanded, $\mathbf{\Lambda}$, is greater than the (embodied) labour values created in production ($\mathbf{\Lambda} > \boldsymbol{\lambda}$).[8]

Under conditions of perfect competition, **(**$\mathbf{\Lambda} = \boldsymbol{\lambda}$**),** one needs to distinguish between 2 cases. First, there is no relation between the property of capital and work effort or the amount of labour power provided for production. This means, there is no relation between earnings from labour and income from profits for the individuals. Under these conditions demand does not express purely the wants of the labourers although all demand represents embodied labour values.

8 This shall be dealt with in another article on imperfect competition.

There exists also the very special case where income from profits is proportionally related to earnings from labour. In this very special case - as in the workers cooperatives in Paris in 1848 – there is no exploitation at the work place and all demand represents the wants of the labourers in proportion to their work effort. One may well consider this very special case as a perfect economy in the narrow sense. An economy of this type may serve as a reference system for the analysis of consumer behaviour along the lines of historical materialism. A task which is well beyond the scope of this paper.

IX. Conclusions

This analysis has tried to develop a theory of production based on labour value analysis. The demand on the traditional Marxist is very high indeed, the commonly accepted definition of labour value has been rejected on the ground that it does not reflect production conditions properly and has been replaced by a concept which appears very remote from Classical Political Economy and Marx. However, this remoteness is only apparent. On the contrary, in the process of exploring it, its intrinsic Marxian character becomes evident.

There is no doubt that the analysis presented here offers a convincing solution to the problems raised by the analysis of labour values in the first volume of *Capital*, We have shown that in a perfect economy there is no *transformation problem of values into prices*. Prices are the monetary expressions of labour values. In fact, pure economic theory can totally dispose of prices and be developed properly in terms of labour values only! This revolutionary process in Marxism has to be developed further and in a productive sense if Marxism is to become again a progressive and cutting

edge approach to the social sciences.

In the follow-up part II of this paper we shall be leaving the perfect world. In the real world we live in, there exist indeed serious problems of reaching an efficient organization of production processes. Considering monopoly capitalism, the failures of the markets are becoming more and more apparent and this on a global scale, in particular concerning the proper management of the eco-systems and natural resources. The solutions of these problems seem to be well above the scope of capitalistic organization, its costs are increasing permanently, the real limitation being wage labour.

Université Paris Ouest, 19.12.2009

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