Existence of Optimal Strategies in Linear Multisector Models with several consumption goods

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Abstract. In this paper we give a sufficient and almost necessary condition for the existence of optimal strategies in linear multisector models when time is continuous.

Key words: Endogenous growth, optimal control with mixed constraints, von Neumann growth model.

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1 Introduction

In 1981 Magill [10] provided a result concerning the existence of optimal strategies in linear multisector models when time is continuous and the preferences of the representative agent are characterized by two parameters: the rate of time discount ρ and the constant elasticity of substitution $\sigma > 0$, within a more general formulation in which technology is not necessarily linear. He proved indeed ([10], Theorem 9.15, p. 703) that in a von Neumann technology with constant returns if

$$\Gamma_0 > \frac{\Gamma_0 - \rho}{\sigma},$$

then an optimal strategy exists, where Γ_0 is the maximum rate of growth (Magill [10] did not provide any non-existence results; but see [11]). Magill, however, used an assumption on "regularity" ([10], Assumption T.2, p. 703) justified on the basis of the Gale [8] indecomposability assumption implying that the upper bound of the uniform over time rates of reproduction of any commodity equals the maximum rate of growth.

In a more recent paper Freni et alii [6] analyzed more deeply the existence of optimal strategies in linear multisector models when time is continuous and proved in the assumption that only one commodity is consumed that if

$$\Gamma_1 > \frac{\Gamma_1 - \rho}{\sigma},$$

then an optimal strategy exists, whereas if

$$\Gamma_1 < \frac{\Gamma_1 - \rho}{\sigma},$$

then no optimal strategy exists, where Γ_1 is the upper bound of the uniform over time rate of reproduction of commodity 1, which is the only commodity which is consumed. Freni et alii [6] considered also the case in which

$$\Gamma_1 = \frac{\Gamma_1 - \rho}{\sigma},$$

and provided further results of existence on non existence in dependence of size of σ. Therefore what matters is not the maximum rate of growth, but the upper bound of the uniform over time rates of reproduction of the consumption good *if only one commodity is consumed.* In this paper we want to generalize that result to the case in which several consumption goods exist. More precisely we will prove that if

$$\Gamma_\nu > \frac{\Gamma_\nu - \rho}{\sigma},$$

then an optimal strategy exists, whereas if

$$\Gamma_\nu < \frac{\Gamma_\nu - \rho}{\sigma},$$

then no optimal strategy exists, where Γ_ν is an average of the upper bounds of the uniform over time rates of reproduction of consumption commodities. Such an average is defined by the instantaneous utility function only and is totally independent from technology whereas the upper bound of the uniform over time rate of reproduction of a commodity depends on technology only and is independent on the preferences of consumers.

The plan of the paper is the following: first we describe the model in Section 2, discussing also the main assumptions. In Section 3 we give the main results and a couple of examples to show the complexity of the limiting cases; Section 4 is devoted to proving the main results.

2 The Model

There are $n \geq 1$ commodities, and k of them are consumed, say commodities $1, \ldots, k$. Preferences with respect to consumption over time are such that they can be described by a single intertemporal utility function U_σ, which is the usual C.E.S. (Constant Elasticity of Substitution) function: for a given consumption path $\mathbf{c} : [0, +\infty) \to \mathbb{R}^k$, ($\mathbf{c}_t \geq \mathbf{0}$ a.e.), we set

$$U_\sigma (\mathbf{c}(\cdot)) = \int_0^{+\infty} e^{-\rho t} u_\sigma (\nu(\mathbf{c}(t)))\, dt \tag{1}$$

where $\rho \in \mathbb{R}$ is the rate of time discount of the representative agent, the instantaneous utility function $u_\sigma : [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ depends on a single parameter $\sigma > 0$ (the elasticity of substitution) and is given by

$$u_\sigma (\nu) = \frac{\nu^{1-\sigma} - 1}{1-\sigma} \qquad for \quad \sigma > 0,\ \sigma \neq 1$$

$$u_1 (\nu) = \log \nu \qquad for \quad \sigma = 1$$

(with the agreement that $u_\sigma(0) = -\infty$ for $\sigma \geq 1$), and $\nu : \mathbb{R}^k_+ \to \mathbb{R}$ is continuous, increasing on every component, concave, homogeneous of degree 1. Possible examples of function ν are the following.

$$\nu(\mathbf{c}) = c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_k^{\alpha_k}, \qquad \alpha_i \in (0,1),\ i = 1, \ldots k, \quad \sum_{i=1}^k \alpha_i = 1 \tag{2}$$

$$\nu(\mathbf{c}) = \min\{\alpha_1 c_1, \alpha_2 c_2, \ldots, \alpha_k c_k\}, \qquad \alpha_i \in (0,1),\ i = 1, \ldots k, \quad \sum_{i=1}^k \alpha_i = 1 \tag{3}$$

$$\nu(\mathbf{c}) = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k, \qquad \alpha_i \in (0,1),\ i = 1, \ldots k, \quad \sum_{i=1}^k \alpha_i = 1. \tag{4}$$

In the first case preferences are Cobb-Douglas, in the second consumed commodities are perfect complements, in the third consumed commodities are perfect substitutes.

For the sake of simplicity we will drop the additive constant $-(1-\sigma)^{-1}$ in the following since this will not affect the optimal paths.

Technology is fully described by a pair of nonnegative matrices (the $m \times n$ material input matrix $\mathbf{A}$ and the $m \times n$ material output matrix $\mathbf{B}$, $m \geq 0$) and by a uniform rate of depreciation $\delta_{\mathbf{x}}$ of capital goods used for production. The rate of depreciation for goods not employed in production is $\delta_{\mathbf{z}}$. If $m = 0$, we say that matrices $\mathbf{A}$ and $\mathbf{B}$ are void. In this degenerate case production does not hold and all capital goods decay at rate $\delta_{\mathbf{z}}$ the model reduces to the standard one-dimensional AK model with $A = -\delta_{\mathbf{z}} \leq 0$.

The amounts of commodities available as capital at time t are defined by the vector $\mathbf{s}_t$. They may be either used for production (if $m > 0$) or disposed of. That is

$$\mathbf{s}_t^T = \mathbf{x}_t^T \mathbf{A} + \mathbf{z}_t^T,$$

where $\mathbf{x} \geq \mathbf{0}$ denotes the vector of the intensities of operation and $\mathbf{z} \geq \mathbf{0}$ the vector of the amounts of goods which are disposed of. Production consists in combining the productive services from the stocks to generate flows that add to the existing stocks. Decay and consumption, on the other hand, drain away the stocks:

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T [\mathbf{B} - \delta_{\mathbf{x}} \mathbf{A}] - \delta_{\mathbf{z}} \mathbf{z}_t^T - \hat{\mathbf{c}}_t^T; \qquad \hat{\mathbf{c}}_t \geq 0 \qquad \mathbf{s}_0 = \bar{\mathbf{s}}$$

where $\hat{\mathbf{c}}_t$ is the $n \times 1$ vector obtained from the $k \times 1$ consumption vector $\mathbf{c}_t$ and adding a zero component for each pure capital good at the places $k+1, \ldots, n$. By eliminating the variable $\mathbf{z}$ and setting $\delta = -\delta_{\mathbf{z}} + \delta_{\mathbf{x}}$, we obtain

$$\dot{\mathbf{s}}_t^T = \mathbf{x}_t^T [\mathbf{B} - \delta \mathbf{A}] - \delta_{\mathbf{z}} \mathbf{s}_t^T - \hat{\mathbf{c}}_t^T; \tag{5}$$

with the initial condition

$$\mathbf{s}_0 = \bar{\mathbf{s}} \geq \mathbf{0} \tag{6}$$

and the constraints

$$\mathbf{x}_t \geq \mathbf{0}, \qquad \mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}, \qquad c_t \geq 0. \tag{7}$$

If we add also the constraint

$$\mathbf{x}_t^T \mathbf{B} \geq \hat{\mathbf{c}}_t^T \tag{8}$$

the proof of existence here provided would be simplified since the constraint (8) would imply that the set of admissible control strategies is relatively compact in the space of integrable functions with a suitable weight. As a consequence a simpler procedure to prove existence could be used (see Remark 4.4 after the proof of Lemma 4.2). The economic interpretation of the constraint (8) is the following: commodities which in principle can be used both as consumption and as capital (the first k commodities in our case) cannot be converted to consumption once they are installed as capital. One of the aims of this paper is to show that a constraint of this type is not needed.[1]

[1] On the contrary, constraints of this type are used by Magill [10], Becker *et alii* [3], and Balder [1]. In [10], Definition 4.1 and Assumption 1, p. 686 (then in Section 9, Definition 9.5 and subsequent results) allow to get the existence of what Magill calls an expansion function (Definition 5.1 and Assumption 3, p.687, [10]) which is a key assumption for proving the existence theorem. In [3], Section 4.3, the same setting of [10], Section 9, is used. This allows to prove that the Technology Conditions (i) and (ii), p. 81 are verified and again this is a key point to prove the existence theorem. In [1] we find the Growth Condition 2.4 (p. 424) to be essential for the proof of existence (together with the compactness of $A(0)$).

Our problem is then to maximize the intertemporal utility (1) over all production-consumption strategies $(\mathbf{x}, \mathbf{c})$ that satisfy the constraints (5), (6) and (7). This is an optimal control problem where $\mathbf{s}$ is the state variable and $\mathbf{x}$ and $\mathbf{c}$ are the control variables. We now describe this problem more formally.

A production-consumption strategy $(\mathbf{x}, \mathbf{c})$ is defined as a measurable and locally integrable function of $t : \mathbb{R}^+ \rightarrow \mathbb{R}^m \times \mathbb{R}^k$ (we will denote by $L^1_{\text{loc}}\left(0, +\infty; \mathbb{R}^{m+k}\right)$ the set of such functions). Then the differential equation (5) has a unique solution : $\mathbb{R}^+ \mapsto \mathbb{R}^n$ which is absolutely continuous (we will denote by $W^{1,1}_{\text{loc}}\left(0, +\infty; \mathbb{R}^n\right)$ the set of such functions). Such a solution clearly depends on the initial datum $\bar{\mathbf{s}}$ and on the production-consumption strategy $(\mathbf{x}, \mathbf{c})$ so it will be denoted by the symbol $\mathbf{s}_{t;\bar{\mathbf{s}},(\mathbf{x},\mathbf{c})}$, omitting the subscript $\bar{\mathbf{s}}, (\mathbf{x}, \mathbf{c})$ when it is clear from the context.

Given an initial endowment $\bar{\mathbf{s}}$ we will say that a strategy $(\mathbf{x}, \mathbf{c})$ is admissible from $\bar{\mathbf{s}}$ if the triple $\left(\mathbf{x}, \mathbf{c}, \mathbf{s}_{t;\bar{\mathbf{s}},(\mathbf{x},\mathbf{c})}\right)$ satisfies the constraints (7) and $U_1(\mathbf{c})$ is well defined[2]. The set of admissible control strategies starting at $\bar{\mathbf{s}}$ will be denoted by $\mathcal{A}(\bar{\mathbf{s}})$. We adopt the following definition of optimal strategies.

Definition 2.1 *A strategy $(\mathbf{x}^*, \mathbf{c}^*) \in \mathcal{A}(\bar{\mathbf{s}})$ will be called optimal if we have $U_\sigma(\mathbf{c}^*) > -\infty$ and*

$$+\infty > U_\sigma(\mathbf{c}^*) \geq U_\sigma(\mathbf{c})$$

for every admissible control pair $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$.

We now comment on a set of assumptions that will be used throughout the paper.

Assumption 2.2 *Each row of matrix* $\mathbf{A}$ *is semipositive.*

This assumption means that no commodity can be produced without using some commodity as an input.

Assumption 2.3 *Each row of matrix* $\mathbf{B}$ *is semipositive.*

This assumption means that each process produces something: i.e. that pure destruction processes are not dealt with as production processes.

Assumption 2.4 *The initial datum $\bar{\mathbf{s}} \geq \mathbf{0}$ and the matrices $\mathbf{A}$ and $\mathbf{B}$ are such that there is an admissible strategy $(\mathbf{x}^*, \mathbf{c}^*) \in \mathcal{A}(\bar{\mathbf{s}})$ and a time $t^* > 0$ such that $\nu\left(\tilde{\mathbf{s}}_{t^*;\bar{\mathbf{s}},(\mathbf{x}^*,\mathbf{c}^*)}\right) > 0$, where $\tilde{\mathbf{s}}$ is the subvector of vector $\mathbf{s}$ consisting of the first k elements.*

[2]The condition on $U_1(\mathbf{c})$ is relevant only when $\sigma = 1$. Note that for $\sigma \in (0,1)$ the function $t \rightarrow e^{-\rho t} u_\sigma(\nu(\mathbf{c}_t))$ is always nonnegative so it is always semiintegrable (with the integral eventually $+\infty$). On the other hand for $\sigma > 1$ the function $t \rightarrow e^{-\rho t} u_\sigma(\nu(\mathbf{c}_t))$ is always negative (and may be $-\infty$ when $\nu(\mathbf{c})_t = 0$) and again it is always semiintegrable (with the integral eventually $-\infty$). This means that the intertemporal utility U_σ is always well defined for $\sigma \neq 1$. For $\sigma = 1$ the function $t \rightarrow e^{-\rho t} u_\sigma(\nu(\mathbf{c}_t))$ may change sign so it may be not semiintegrable on $[0, +\infty)$. This is the reason why we need to require that $U_1(\mathbf{c})$ is well defined to define the admissibility of $\mathbf{c}$.

If this assumption does not hold, then every admissible strategy starting from $\bar{\mathbf{s}}$ must have that $\nu(\mathbf{c}_t) = 0$ a.e. This case is not an interesting case to be investigated.

Assumption 2.5 *The initial datum* $\bar{\mathbf{s}} \geq \mathbf{0}$ *and the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are such that there is an admissible strategy* $(\mathbf{x}^*, \mathbf{c}^*) \in \mathcal{A}(\bar{\mathbf{s}})$ *and a time* t^* *such that* $\mathbf{s}_{t^*;\bar{\mathbf{s}},(\mathbf{x}^*,\mathbf{c}^*)}$ *is positive.*

Assumption 2.5 implies that all commodities are available at any time $t > 0$ and, in particular, implies that Assumption 2.4 is satisfied. Moreover Assumptions 2.4 and 2.5 could be stated in terms of the zero components of the initial datum $\bar{\mathbf{s}}$ and of the structure of the matrices $\mathbf{A}$ and $\mathbf{B}$, see on this Appendix D of [7].

It is also obvious that if Assumption 2.5 holds, then Assumption 2.4 holds too. Nevertheless it can be shown that Assumption 2.5 is not really restrictive, provided that Assumption 2.4 holds, in the sense that when it does not hold, matrices $\mathbf{A}$ and $\mathbf{B}$, vector $\bar{\mathbf{s}}$, and consumption goods can be redefined in order to obtain an equivalent model in which Assumption 2.5 holds. Assume, in fact, that Assumption 2.5 does not hold. Then there is a commodity j which is not available at any time $t \geq 0$ ($\mathbf{s}_t^T \mathbf{e}_j = 0$ for every $t \geq 0$). In this case any production process i in which commodity j is employed ($a_{ij} > 0$) cannot be used. The model is then equivalent to one in which matrices $\mathbf{B}$ and $\mathbf{A}$ and vector $\mathbf{s}$, in the state equation (5), are substituted with matrices $\mathbf{D}$ and $\mathbf{C}$ and vector $\mathbf{s}'$, respectively, where matrix $\mathbf{C}$ is obtained from $\mathbf{A}$ by deleting the j-th column and all rows which on the j-th column have a positive element, matrix $\mathbf{D}$ is obtained from matrix $\mathbf{B}$ by deleting the corresponding rows and the j-th column, and vector $\mathbf{s}'$ is obtained from vector $\mathbf{s}$ by deleting the j-th element. (If commodity j is a consumption good, it is also deleted by the list of consumption goods.) Note that if in the new equivalent model the Assumption 2.5 does not hold and matrices $\mathbf{C}$ and $\mathbf{D}$ are not void, the argument can be iterated. If matrices $\mathbf{C}$ and $\mathbf{D}$ are void, then an equivalent model satisfying Assumption 2.5 is obtained by deleting the nought elements of vector $\mathbf{s}'$. In any case the algorithm is able to determine an equivalent model in which Assumption 2.5 does hold. We will refer to the equivalent model found in this way as the *truncated model* and to the corresponding technology as the *truncated technology*, which then depends on $\bar{\mathbf{s}}$. It can easily be proved that if Assumptions 2.2, 2.3, 2.4 hold in the original technology, then they hold in the truncated technology too (see Appendix D of [7]). Except when it is not mentioned explicitly, all the following assumptions refer to the truncated technology.

Let us define

$$\mathcal{G}_0 := \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \mathbf{0}\right\}, \qquad \Gamma_0 = \max \mathcal{G}_0$$

and, for $i = 1, \ldots, k$,

$$\mathcal{G}_i := \left\{\gamma | \exists \mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\gamma + \delta_{\mathbf{x}}) \mathbf{A}\right] \geq \mathbf{e}_i^T\right\}, \qquad \Gamma_i = \sup \mathcal{G}_i$$

Γ_0 is clearly the maximum among the uniform over time rates of growth feasible for this economy and corresponds to what von Neumann [14] found both as growth rate and as rate of profit. Γ_i is the upper bound of the uniform over time rates of reproduction of the $i-$th consumption good. Obviously $\Gamma_i \leq \Gamma_0$ for every $i = 1, \ldots, k$.

Magill [10], Assumption T.2, p. 703, assumed that if $\mathbf{x} \in \mathcal{X}$, then $\mathbf{x}^T\mathbf{B} > \mathbf{0}^T$, where

$$\mathcal{X} = \left\{\mathbf{x}|\mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}, \mathbf{x}^T\left[\mathbf{B} - \left(\Gamma_0 + \delta_{\mathbf{x}}\right)\mathbf{A}\right] \geq \mathbf{0}^T\right\}$$

It is easily checked that, under this assumption, if $\mathbf{x} \in \mathcal{X}$ and $\mathbf{x}^T\mathbf{A}\mathbf{e}_i = 0$, then there is $\alpha > 0$ such that

$$\mathbf{x}^T\left[\mathbf{B} - \left(\Gamma_0 + \delta_{\mathbf{x}}\right)\mathbf{A}\right] \geq \alpha\mathbf{e}_i^T$$

whereas if $\mathbf{x} \in \mathcal{X}$ and $\mathbf{x}^T\mathbf{A}\mathbf{e}_i > 0$, then for any $\varepsilon > 0$ there is $\alpha > 0$ such that

$$\mathbf{x}^T\left[\mathbf{B} - \left(\Gamma_0 - \varepsilon + \delta_{\mathbf{x}}\right)\mathbf{A}\right] \geq \alpha\mathbf{e}_i^T$$

In any case

$$\sup\left\{\gamma|\exists\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}, \mathbf{x}^T\left[\mathbf{B} - \left(\gamma + \delta_{\mathbf{x}}\right)\mathbf{A}\right] \geq \mathbf{e}_i^T\right\} = \Gamma_0$$

that is, the upper bound of the uniform over time rates of reproduction of any commodity equals the maximum rate of growth. In this paper we will not make any assumption on indecomposability.

It is easily proved that the Γ_i's relative to the truncated technology are not greater than the corresponding Γ_i's relative to the original one. If either $\mathbf{B}\mathbf{e}_i = \mathbf{A}\mathbf{e}_i = \mathbf{0}$ or matrices $\mathbf{A}$ and $\mathbf{B}$ are void, then $\Gamma_i = -\infty$. Moreover if $\mathbf{B}\mathbf{e}_i = \mathbf{0}$ and $\mathbf{A}\mathbf{e}_i \neq \mathbf{0}$ then $\Gamma_i = -\delta_{\mathbf{x}}$. Finally, if $\mathbf{B}\mathbf{e}_i \neq \mathbf{0}$ and if commodity j is available at time 0 ($\bar{\mathbf{s}}^T\mathbf{e}_j > 0$) and is essential to the reproduction of the $i-$th consumption good, then $\Gamma_i = -\delta_{\mathbf{x}}$.[3] (see [7] Proposition 4.4).

Assumption 2.6 $\mathbf{B}\mathbf{e}_i \neq \mathbf{0}$ *for each consumption good* i *and* $\delta_{\mathbf{z}} < \delta_{\mathbf{x}}$.

Assumption 2.6 is not necessary, but it helps in simplifying the exposition since it implies that $\Gamma_i > -\delta_{\mathbf{z}}$ for each consumption good i. Moreover, it is not very restrictive. It implies that the use of commodities in production dominates their storing.

We call

$$\Gamma_{max} := \max_{i=1,\dots,k} \Gamma_i, \qquad \Gamma_{min} := \min_{i=1,\dots,k} \Gamma_i > 0$$

Moreover we introduce the following number

$$\Gamma_\nu = \inf\left\{\eta \in \mathbb{R} : \lim_{t\to+\infty} e^{-\eta t}\nu\left(e^{\Gamma_1 t}, e^{\Gamma_2 t}, \dots, e^{\Gamma_k t}\right) = 0\right\}$$

$$= \sup\left\{\eta \in \mathbb{R} : \lim_{t\to+\infty} e^{-\eta t}\nu\left(e^{\Gamma_1 t}, e^{\Gamma_2 t}, \dots, e^{\Gamma_k t}\right) = +\infty\right\}$$

[3] We say that commodity j is essential to the reproduction of the $i-$th consumption good when

$$\left(\mathbf{x} \geq 0, \varepsilon > 0, \mathbf{x}^T\left[\mathbf{B} - \varepsilon\mathbf{A}\right] \geq \mathbf{e}_i\right) \Rightarrow \mathbf{x}^T\mathbf{A}\mathbf{e}_j \neq 0.$$

Since, by the 1−homogeneity and the monotonicity of ν we have

$$e^{\Gamma_{min}t}\nu(1,1,\ldots,1) \le \nu\left(e^{\Gamma_1 t}, e^{\Gamma_2 t}, \ldots, e^{\Gamma_k t}\right) \le e^{\Gamma_{max}t}\nu(1,1,\ldots,1)$$

then it is easy to see that $\Gamma_{min} \le \Gamma_\nu \le \Gamma_{max}$.

Furthermore observe that, calling, for $\eta > 0$, $\bar{\mathbf{s}} \ge 0$ and $(\mathbf{x},\mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$,

$$I_\eta(\mathbf{c}) := \int_0^{+\infty} e^{-\eta r}\nu\left(c_{1r}, c_{2r}, \ldots, c_{kr}\right) dr$$

we have (see section 4: proof of Propositions 4.5 and 4.7)

$$\Gamma_\nu = \inf\left\{\eta > 0 : \sup_{\mathbf{c}\in\mathcal{A}(\bar{\mathbf{s}})} I_\eta(\mathbf{c}) < +\infty\right\}$$

It is also easy to check that, for the three examples given in (2), (3) and (4), we have:

- in the example (2), $\Gamma_\nu = \sum_{i=1}^k \alpha_i \Gamma_i$,
- in the example (3), $\Gamma_\nu = \min\{\Gamma_i,\ i = 1,\ldots,k\}$,
- in the example (4), $\Gamma_\nu = \max\{\Gamma_i,\ i = 1,\ldots,k\}$.

As mentioned in the introduction this paper is mainly devoted to show the role that the following assumption plays for the existence of optimal strategies of the problem under analysis.

Assumption 2.7

$$\Gamma_\nu > \frac{\Gamma_\nu - \rho}{\sigma}$$

The reader should have noticed that we have used the convoluted expression “the upper bound of the uniform over time rates of reproduction of the i−th consumption good” instead of the more straightforward “the upper bound of the rates of reproduction of the i−th consumption good”. This phraseology is used as for particular forms of matrices growth rates of consumption might be found which are higher, but not uniform over time. In [6] we provided the following example, with the details, to clarify this point.

Example 2.8 $k = 1$, $\delta_{\mathbf{x}} = \delta_{\mathbf{z}} \in (0,1)$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is immediately recognized that $\Gamma_1 = 1 - \delta_{\mathbf{x}} > 0$. *Nevertheless consumption can grow at the rate*

$$\frac{\dot{c}}{c} = \Gamma_1 + \frac{\beta}{\alpha + \beta t} > \Gamma_1$$

where α and β are positive constants.

3 The main results

The main goal of this paper is to show that in the general context outlined by Assumptions 2.2, 2.3, 2.5, 2.6, we have substantially an if and only if condition for the existence of optimal strategies. In fact in this paper we will prove the following results:

Theorem 3.1 *If Assumptions 2.2, 2.3, 2.5, 2.6 and 2.7 hold, then there is an optimal strategy* $(\mathbf{x}, \mathbf{c})$ *for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$*. Moreover this strategy is unique in the sense that if* $(\hat{\mathbf{x}}, \hat{\mathbf{c}})$ *is another optimal strategy, then* $\nu(\hat{\mathbf{c}}) \equiv \nu(\mathbf{c})$*. If* ν *is strictly concave, then we also have* $\hat{\mathbf{c}} \equiv \mathbf{c}$*.*

Theorem 3.2 *Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. If*

$$\Gamma_\nu < \frac{\Gamma_\nu - \rho}{\sigma}$$

then no optimal strategy exists for problem (P_σ) *starting at* $\bar{\mathbf{s}}$*.*

Theorem 3.3 *Let Assumptions 2.2, 2.3, 2.5, 2.6 hold. Let*

$$\Gamma_\nu = \frac{\Gamma_\nu - \rho}{\sigma}.$$

Then we have the following:

1. *Let* $\sigma = 1$*. If, either* $\Gamma_\nu < 0$*, or* $\Gamma_\nu = 0$*, each* Γ_i *is not a maximum and*
$$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu \left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t} \right) < +\infty, \tag{9}$$
then all strategies have value $-\infty$ *and so no optimal strategy exists for problem* (P_1) *starting at* $\bar{\mathbf{s}}$*. Moreover if* $\Gamma_\nu > 0$ *there exists an admissible strategy with value* $+\infty$ *and so no optimal strategy exists for problem* (P_1) *starting at* $\bar{\mathbf{s}}$*.*

2. *Let* $\sigma \in (0, 1)$*. If each* Γ_i *is a maximum and*
$$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu \left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t} \right) > 0 \tag{10}$$
then there exists an admissible strategy with value $+\infty$ *and so no optimal strategy exists for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$*.*

3. *Let* $\sigma > 1$*. If each* Γ_i *is not a maximum and (9) holds, then all strategies have value* $-\infty$ *and so no optimal strategy exists for problem* (P_σ) *starting at* $\bar{\mathbf{s}}$*.*

The limit cases where $\Gamma_\nu = \frac{\Gamma_\nu - \rho}{\sigma}$ and

1. $\sigma = 1$, $\Gamma_\nu = 0$, and at least one Γ_i is a maximum or (9) does not hold;
2. $\sigma \in (0, 1)$ and at least one Γ_i is not a maximum or (10) does not hold;
3. $\sigma > 1$ and at least one Γ_i is a maximum or (9) does not hold;

are intrinsically more complex than the others. Indeed in such cases we can have existence or nonexistence depending on the value of σ. In the paper [6] (see also [7] for details) we have provided two examples of matrices A and B and scalars ρ, $\delta_{\mathbf{x}}$, $\delta_{\mathbf{z}}$ showing this fact with $k = 1$.

4 Proofs of the main results

In this section, we provide the proofs of the main results stated in section 3. The proofs require a set of preliminary results which we discuss in Subsection 4.1. In Subsection 4.2 we prove the existence and nonexistence results stated above as Theorems 3.1, 3.2 and 3.3.

Throughout this section we will assume that Assumptions 2.2, 2.3, 2.5, 2.6 hold without explicitly mentioning them. We observe that some results hold in the more general framework when Assumption 2.6 does not hold; on this point see also Appendix A and B of [7].

4.1 Preliminary lemmata

The following Lemma provides the basis for estimates of the state and control trajectories.

Lemma 4.1 *Let $i \in \{1, \ldots, k\}$. If Γ_i is not a maximum*

$$\gamma \geq \Gamma_i \Longleftrightarrow \exists \mathbf{v}_F^i \geq \mathbf{0} : \quad (\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A})\,\mathbf{v}_F^i \leq \mathbf{0}, \quad \mathbf{e}_i^T \mathbf{v}_F^i = 1. \tag{11}$$

If Γ_i is a maximum

$$\gamma > \Gamma_i \Longleftrightarrow \exists \mathbf{v}_F^i \geq \mathbf{0} : \quad (\mathbf{B} - (\gamma + \delta_{\mathbf{x}})\,\mathbf{A})\,\mathbf{v}_F^i \leq \mathbf{0}, \quad \mathbf{e}_i^T \mathbf{v}_F^i = 1, \tag{12}$$

Moreover

$$\exists \mathbf{v}_S^i \geq \mathbf{0} : \quad (\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A})\,\mathbf{v}_S^i \leq \mathbf{0}, \mathbf{v}_S^i \neq \mathbf{0}, \tag{13}$$

and

$$\mathbf{e}_i^T \mathbf{v}_S^i = \mathbf{y}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] \mathbf{v}_S^i = 0, \tag{14}$$

where

$$\mathbf{y} \in \left\{ \mathbf{x} | \mathbf{x} \geq \mathbf{0}, \mathbf{x}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] \geq \mathbf{e}_i^T \right\}.$$

Proof. Statements (11) and (12) are obvious applications of the Farkas Lemma (see for instance Gale's theorem for linear inequalities; [9] or [12], pp. 33-34). Assume now that statement (13) does not hold and obtain, once again from the Farkas Lemma (see for instance Motzkin's theorem of the alternative; [13] or [12], pp. 28-29), that

$$\exists \mathbf{w} \geq \mathbf{0} : \mathbf{w}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] > \mathbf{0}^T.$$

Hence there is $\phi > 0$ so large and $\eta > 0$ so small that

$$\phi \mathbf{w}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] \geq \mathbf{e}_i^T + \eta \phi \mathbf{w}^T \mathbf{A}$$

Hence a contradiction since $\Gamma_i = \sup \mathcal{G}_i$. By remarking that

$$0 \geq \mathbf{y}^T \left[\mathbf{B} - (\Gamma_i + \delta_{\mathbf{x}})\,\mathbf{A}\right] \mathbf{v}_S^i \geq \mathbf{e}_i^T \mathbf{v}_S^i \geq 0$$

the proof is completed. ∎

The next lemma and the subsequent corollary give various estimates for the state and control variables that will be the basis for the proof of existence and nonexistence. Note that for the case $\sigma \in (0,1)$ we are interested in an estimate from above of the integral $\int_0^t e^{-\rho s}\nu(\mathbf{c}_s)^{1-\sigma}ds$ giving finiteness of the value function for $\rho - \Gamma_\nu(1-\sigma) > 0$ (so we need terms that remain bounded when $t \to +\infty$), while for the case $\sigma \in (1,+\infty)$ we are interested in an estimate from below of the same integral to show that the value function is equal to $-\infty$ when $\rho - \Gamma_\nu(1-\sigma) < 0$, (so we need terms that explode when $t \to +\infty$). These different targets require the use of different estimates with different methods of proof. Of course, both methods can be applied to both cases, albeit yielding estimates that are not useful for our target. In order to simplify notation we will set, for $\varepsilon \geq 0$,

$$\overline{\Gamma}_i := \max\{-\delta_{\mathbf{z}}, \Gamma_i\} \tag{15}$$

$$\overline{\Gamma}_{i,\varepsilon} := \max\{-\delta_{\mathbf{z}}, \Gamma_i + \varepsilon\} \tag{16}$$

$$a_{i,\varepsilon} = \rho - \overline{\Gamma}_{i,\varepsilon}(1-\sigma) \tag{17}$$

$$a_{\nu,\varepsilon} = \rho - (\Gamma_\nu + \varepsilon)(1-\sigma). \tag{18}$$

Obviously, if Assumption 2.6 holds, $\overline{\Gamma}_i = \Gamma_i$ and $\overline{\Gamma}_{i,\varepsilon} := \Gamma_i + \varepsilon$.

Lemma 4.2 *Let $i \in \{1,\dots,k\}$ and $\sigma > 0$. Fix $\varepsilon > 0$ when Γ_i is a maximum and $\varepsilon = 0$ when Γ_i is not maximum; call $\mathbf{v}^i_{F,\varepsilon}$ the vector given by Lemma 4.1. For every $0 \leq t < +\infty$, $\overline{\mathbf{s}} \in \mathbb{R}^n$, $\overline{\mathbf{s}} \geq 0$ we have, for every admissible control strategy $(\mathbf{x},\mathbf{c}) \in \mathcal{A}(\overline{\mathbf{s}})$,*

$$\mathbf{s}_t^T \mathbf{v}^i_{F,\varepsilon} \leq e^{\overline{\Gamma}_{i,\varepsilon} t}\overline{\mathbf{s}}^T\mathbf{v}^i_{F,\varepsilon}, \tag{19}$$

and, for $\eta \in \mathbb{R}$

$$\begin{aligned} \int_0^t e^{-\eta s}\mathbf{s}_s^T\mathbf{v}^i_{F,\varepsilon}ds &\leq \overline{\mathbf{s}}^T\mathbf{v}^i_{F,\varepsilon}\frac{e^{\left(\overline{\Gamma}_{i,\varepsilon}-\eta\right)t}-1}{\overline{\Gamma}_{i,\varepsilon}-\eta}; \qquad \eta \neq \overline{\Gamma}_{i,\varepsilon}; \\ \int_0^t e^{-\eta s}\mathbf{s}_s^T\mathbf{v}^i_{F,\varepsilon}ds &\leq \overline{\mathbf{s}}^T\mathbf{v}^i_{F,\varepsilon}t; \qquad \eta = \overline{\Gamma}_{i,\varepsilon}; \end{aligned} \tag{20}$$

and, setting $I_{i,\varepsilon}(t) := \int_0^t e^{-\overline{\Gamma}_{i,\varepsilon}s}\hat{\mathbf{c}}_s^T\mathbf{v}^i_{F,\varepsilon}ds$,

$$I_{i,\varepsilon}(t) + e^{-\overline{\Gamma}_{i,\varepsilon}t}\mathbf{x}_t^T\mathbf{A}\mathbf{v}^i_{F,\varepsilon} \leq \overline{\mathbf{s}}^T\mathbf{v}^i_{F,\varepsilon}. \tag{21}$$

Moreover, for $0 \leq \tau \leq t < +\infty$, and $\eta \in \mathbb{R}$,

$$\mathbf{x}_t^T\mathbf{A}\mathbf{v}^i_{F,\varepsilon}e^{-\eta t} + \int_\tau^t e^{-\eta s}\hat{\mathbf{c}}_s^T\mathbf{v}^i_{F,\varepsilon}ds \leq e^{-\eta\tau}\overline{\mathbf{s}}_\tau^T\mathbf{v}^i_{F,\varepsilon}e^{\left(\overline{\Gamma}_{i,\varepsilon}-\eta\right)^+(t-\tau)}. \tag{22}$$

Finally there exists a constant $\lambda > 0$ (depending only on the matrices $\mathbf{A}$ and $\mathbf{B}$) such that, for every $t \geq 0$

$$\left|\left|\mathbf{x}_t^T\mathbf{A}\right|\right| \leq ||\mathbf{s}_t|| \leq e^{\lambda t}\,||\overline{\mathbf{s}}||\,, \qquad ||\mathbf{x}_t|| \leq Ce^{\lambda t}\,||\overline{\mathbf{s}}|| \tag{23}$$

for suitable real number $C > 0$ (depending only on the matrix $\mathbf{A}$).

Proof. We prove the five inequalities (19)–(23) in order of presentation. We give only a sketch. To avoid heavy notation we will write $\mathbf{v}_F^i$ for $\mathbf{v}_{F,\varepsilon}^i$ along this proof.

(1) Let $i \in \{1, \dots, k\}$. First we observe that, by multiplying the state equation (5) by $\mathbf{v}_F^i$ we obtain

$$\begin{aligned} \dot{\mathbf{s}}_t^T \mathbf{v}_F^i &= -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i + \mathbf{x}_t^T [\mathbf{B} - \delta \mathbf{A}] \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \quad t \in (0, +\infty), \\ \mathbf{s}_0^T \mathbf{v}_F^i &= \overline{\mathbf{s}}^T \mathbf{v}_F^i \geq 0. \end{aligned}$$

Now for every $\mathbf{x}$ and ε,

$$\mathbf{x}^T [\mathbf{B} - \delta \mathbf{A}] = \mathbf{x}^T [\mathbf{B} - (\Gamma_i + \varepsilon + \delta_{\mathbf{x}}) \mathbf{A}] + (\Gamma_i + \varepsilon + \delta_{\mathbf{z}}) \mathbf{x}^T \mathbf{A}$$

Moreover for $\mathbf{x} \geq 0$ we have by (11) and (12) $\mathbf{x}^T [\mathbf{B} - (\Gamma_i + \varepsilon + \delta_{\mathbf{x}}) \mathbf{A}] \mathbf{v}_F^i \leq 0$ with the agreement that $\varepsilon = 0$ when Γ_i is not a maximum. Then

$$\begin{aligned} \dot{\mathbf{s}}_t^T \mathbf{v}_F^i &= -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i + \mathbf{x}_t^T [\mathbf{B} - (\Gamma_i + \varepsilon + \delta_{\mathbf{x}}) \mathbf{A}] \mathbf{v}_F^i + (\Gamma_i + \varepsilon + \delta_{\mathbf{z}}) \mathbf{x}_t^T \mathbf{A} \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \\ &\leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i + (\Gamma_i + \varepsilon + \delta_{\mathbf{z}}) \mathbf{x}^T \mathbf{A} \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \end{aligned}$$

If $\Gamma_i + \varepsilon \geq -\delta_{\mathbf{z}}$, from the constraint $\mathbf{s}_t^T \geq \mathbf{x}_t^T \mathbf{A}$ and from the non-negativity of $\hat{\mathbf{c}}_t$, we get

$$\dot{\mathbf{s}}_t^T \mathbf{v}_F^i \leq (\Gamma_i + \varepsilon) \mathbf{s}_t^T \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \leq (\Gamma_i + \varepsilon) \mathbf{s}_t^T \mathbf{v}_F^i \qquad t \in (0, +\infty), \tag{24}$$

and so, by integrating on $[0, t]$ and using the Gronwall lemma (see e.g. [2, p. 218]) we get the first claim (19).

Take now $\Gamma_i + \varepsilon < -\delta_{\mathbf{z}}$ in this case we have $(\Gamma_i + \varepsilon + \delta_{\mathbf{z}}) \mathbf{x}^T \mathbf{A} \mathbf{v}_F^i \leq 0$ which gives

$$\dot{\mathbf{s}}_t^T \mathbf{v}_F^i \leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i - \hat{\mathbf{c}}_t^T \mathbf{v}_F^i \leq -\delta_{\mathbf{z}} \mathbf{s}_t^T \mathbf{v}_F^i, \qquad t \in (0, +\infty), \tag{25}$$

and so the claim (in this case we clearly can take $\varepsilon = 0$).

(2) Inequalities (20) are proved by multiplying the inequality (19) by $e^{-\eta s}$ and integrating on $[0, t]$.

(3) From (24) (taking $\varepsilon = 0$ when allowed)

$$\dot{\mathbf{s}}_s^T \mathbf{v}_F \leq \overline{\Gamma}_{i,\varepsilon} \mathbf{s}_s^T \mathbf{v}_F^i - \hat{\mathbf{c}}_s^T \mathbf{v}_F^i \qquad \forall s \in [0, t] \tag{26}$$

so that, by the comparison theorem for ODE's

$$\mathbf{s}_t^T \mathbf{v}_F^i \leq \overline{\mathbf{s}}^T \mathbf{v}_F^i e^{\overline{\Gamma}_{i,\varepsilon} t} - \int_0^t e^{\overline{\Gamma}_{i,\varepsilon}(t-s)} \hat{\mathbf{c}}_s^T \mathbf{v}_F^i ds$$

From the inequality $\mathbf{x}_t^T \mathbf{A} \mathbf{v}_F^i \leq \mathbf{s}_t^T \mathbf{v}_F^i$ we get inequality (21) by rearranging the terms.

(4) For simplicity we take the case $\tau = 0$. Inequality (22) easily follows by multiplying both sides of (26) by $e^{-\eta s}$ and then integrating. Indeed we have

$$0 \le e^{-\eta s}\hat{\mathbf{c}}_s^T \mathbf{v}_F^i \le e^{-\eta s}\left[\overline{\Gamma}_{i,\varepsilon}\mathbf{s}_s^T \mathbf{v}_F^i - \dot{\mathbf{s}}_s^T \mathbf{v}_F^i\right] \qquad \forall s \in [0,t]$$

Now we integrate the above expression, then we integrate by parts and use that $\mathbf{x}_t^T \mathbf{A}\mathbf{v}_F^i \le \mathbf{s}_t^T \mathbf{v}_F^i$:

$$\begin{aligned}
\int_0^t e^{-\eta s}\hat{\mathbf{c}}_s^T \mathbf{v}_F^i ds &\le \int_0^t e^{-\eta s}\left[\overline{\Gamma}_{i,\varepsilon}\mathbf{s}_s^T \mathbf{v}_F^i - \dot{\mathbf{s}}_s^T \mathbf{v}_F^i\right] ds \\
&= \int_0^t e^{-\eta s}\overline{\Gamma}_{i,\varepsilon}\mathbf{s}_s^T \mathbf{v}_F^i ds - e^{-\eta t}\mathbf{s}_t^T \mathbf{v}_F^i + \overline{\mathbf{s}}^T \mathbf{v}_F^i - \eta \int_0^t e^{-\eta s}\mathbf{s}_s^T \mathbf{v}_F^i ds \\
&\le \overline{\mathbf{s}}^T \mathbf{v}_F^i - e^{-\eta t}\mathbf{x}_t^T \mathbf{A}\mathbf{v}_F^i + \left(\overline{\Gamma}_{i,\varepsilon} - \eta\right)\int_0^t e^{-\eta s}\mathbf{s}_s^T \mathbf{v}_F^i ds
\end{aligned}$$

Now, if $\eta \ge \overline{\Gamma}_{i,\varepsilon}$ the above inequality gives the claim immediately. If $\eta < \overline{\Gamma}_{i,\varepsilon}$ we get the claim by using (20).

(5) The inequality (23) comes as follows. By Assumption 2.2 for every i there exists j such that $a_{ij} > 0$ so that

$$\mathbf{x}^T \mathbf{e}_i \le a_{ij}^{-1}\mathbf{s}^T \mathbf{e}_j$$

and we can find a nonnegative matrix $n \times m$ $\mathbf{C}$ with exactly one positive element for every column such that $\mathbf{x}^T \le \mathbf{s}^T \mathbf{C}$. Consequently we have, for $\mathbf{x}^T \mathbf{A} \le \mathbf{s}^T$,

$$\mathbf{x}^T \mathbf{B} \le \mathbf{s}^T \mathbf{C}\mathbf{B}.$$

Now the matrix $\mathbf{D} = \mathbf{C}\mathbf{B}$ is $n \times n$ and has only positive elements. From the state equation (5) it follows that for every admissible strategy we have

$$\dot{\mathbf{s}}_t^T \le \mathbf{s}_t^T \mathbf{D} - \delta_{\mathbf{z}}\mathbf{s}_t^T - \hat{\mathbf{c}}_t^T.$$

Since the control $\hat{\mathbf{c}}$ is positive and $\mathbf{D}$ has only positive elements one gets

$$\mathbf{s}_t^T \le \overline{\mathbf{s}}^T e^{t[\mathbf{D} - \delta_{\mathbf{z}}\mathbf{I}]}$$

so the claim easily follows taking any $\lambda > \max\{Re\mu, \mu \text{ eigenvalue of } \mathbf{D}\} - \delta_{\mathbf{z}}$ ∎

For $\eta \in \mathbb{R}$ define, for every $0 \le s < +\infty$, $\overline{\mathbf{s}} \in \mathbb{R}^n$, $\overline{\mathbf{s}} \ge 0$ and $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\overline{\mathbf{s}})$, the quantity

$$I_\eta(s) := \int_0^s e^{-\eta r}\nu\left(c_{1r}, c_{2r}, \ldots, c_{kr}\right) dr. \tag{27}$$

The following estimates hold.

Lemma 4.3 *Let $t \geq 0$, $\bar{\mathbf{s}} \in \mathbb{R}^n$, $\bar{\mathbf{s}} \geq 0$ and $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$. We have, for $\sigma \in (0,1)$,*

$$\int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \leq t^{\sigma} \left[I_{\frac{\rho}{1-\sigma}}(t)\right]^{1-\sigma} \tag{28}$$

while, for $\sigma \in (1, +\infty)$,

$$\int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \geq t^{\sigma} \left[I_{\frac{\rho}{1-\sigma}}(t)\right]^{1-\sigma}. \tag{29}$$

Moreover let $\eta \in \mathbb{R}$. Then, for $\sigma = 1$ we have

$$\begin{aligned}\int_0^t e^{-\rho s} \log\left[\nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)\right] ds \;\; &\leq \;\; t e^{-\rho t}\left[\frac{\eta}{2} t + \log\left(\frac{I_\eta(t)}{t}\right)\right] \\ &\quad + [\rho]^+ \int_0^t s e^{-\rho s}\left[\frac{\eta}{2} s + \log\left(\frac{I_\eta(s)}{s}\right)\right] ds\end{aligned} \tag{30}$$

while, for $\sigma \in (0,1)$,

$$\begin{aligned}&\int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \\ \leq \;\; & I_\eta(t)^{1-\sigma} t^{\sigma} e^{-(\rho-\eta(1-\sigma))t} + [\rho - \eta(1-\sigma)]^+ \int_0^t I_\eta(s)^{1-\sigma} s^{\sigma} e^{-(\rho-\eta(1-\sigma))s} ds\end{aligned} \tag{31}$$

and, for $\sigma > 1$, and $\rho > \eta(1-\sigma)$,

$$\begin{aligned}&\int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \\ \geq \;\; & I_\eta(t)^{1-\sigma} t^{\sigma} e^{-(\rho-\eta(1-\sigma))t} + (\rho - \eta(1-\sigma)) \int_0^t I_\eta(s)^{1-\sigma} s^{\sigma} e^{-(\rho-\eta(1-\sigma))s} ds\end{aligned} \tag{32}$$

Proof.

(1) Concerning inequality (31) we take $\eta \in \mathbb{R}$. Setting

$$h_\eta(s) := \int_0^s \left[e^{-\eta r} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)\right]^{1-\sigma} dr$$

we have, by Jensen's inequality, for $\sigma \in (0,1)$

$$h_\eta(s) \leq s\left[\frac{1}{s}\int_0^s e^{-\eta r} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right) dr\right]^{1-\sigma} = s^{\sigma} I_\eta(s)^{1-\sigma} \tag{33}$$

Now integrating by parts we obtain

$$\begin{aligned}\int_0^t e^{-\rho s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds \;\; &= \;\; \int_0^t e^{-(\rho-\eta(1-\sigma))s}\left[e^{-\eta s} \nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)\right]^{1-\sigma} ds \\ = e^{-(\rho-\eta(1-\sigma))t} h_\eta(t) \;\; &+ \;\; \int_0^t \left(\rho - \eta\left(1-\sigma\right)\right) e^{-(\rho-\eta(1-\sigma))s} h_\eta(s) ds \qquad (34) \\ \leq e^{-(\rho-\eta(1-\sigma))t} s^{\sigma} I_\eta(t)^{1-\sigma} \;\; &+ \;\; \left[\rho - \eta\left(1-\sigma\right)\right]^+ \int_0^t e^{-(\rho-\eta(1-\sigma))s} s^{\sigma} I_\eta(s)^{1-\sigma} ds.\end{aligned}$$

which gives the claim. Inequality (28) follows observing that

$$\int_0^t e^{-\rho s}\nu\left(c_{1s}, c_{2s}, \ldots, c_{ks}\right)^{1-\sigma} ds = h_{\frac{\rho}{1-\sigma}}(t) \tag{35}$$

and using inequality (33).

(2) To prove inequality (29) dealing with the case when $\sigma \in (1, +\infty)$ we still observe that (35) holds and then apply the Jensen inequality. Since the power function $x \to x^{1-\sigma}$ is convex we get the inequality (33) with $\geq$ and so the claim.

Similarly inequality (32) follows integrating by part exactly as for proving (31) and then applying the reversed Jensen inequality.

(3) Inequality (30) follows by similar arguments. In fact, calling, for $\eta \in \mathbb{R}$

$$h(s) := \int_0^s \log \nu(\mathbf{c}_r) dr = \int_0^s \eta r dr + \int_0^s \log\left(e^{-\eta r}\nu(\mathbf{c}_r)\right) dr$$

we have, because of Jensen's inequality

$$h(s) \leq \eta\frac{s^2}{2} + s\log\left[\frac{I_\eta(s)}{s}\right]. \tag{36}$$

Now, integrating by parts as in (34), we obtain

$$\int_0^t e^{-\rho s}\log \nu(\mathbf{c}_s) ds = e^{-\rho t}h(t) + \int_0^t \rho e^{-\rho s}h(s) ds. \tag{37}$$

which, together with (36) and (21), gives the claim.

∎

Remark 4.4 *We observe that, if the constraint (8) is assumed to hold then the proof of the above lemma would be simpler. Indeed all estimates on the integrals containing the consumption strategy (21)–(30) would be immediately true since, thanks to (19) and (23) we would have an estimate of the type $c_t \leq Ce^{\lambda t}\,||\bar{\mathbf{s}}||$.*

Moreover an estimate of this kind would allow us to prove the existence result more simply, using the technique of proof of the existence Theorem 2.8 of [1] (see also [3, 10]), based on the compactness of the derivatives of the stock (Theorem 4.2) in the space of absolutely continuous functions (which, in our model, would be equivalent to the compactness of the set of admissible strategies in a suitable weighted space of integrable functions).

Since we do not have this property we employ a different technique that exploits the structure of our problem. In the case when $\sigma \in (0,1)$, we change variables to get compactness in the new variable and then we go back to the old variable; in the other cases we use a result that strongly exploits the structure of the problem, in particular the monotonicity of the functions involved. ∎

4.2 Proof of existence and nonexistence theorems

We now prove the above Theorem 3.1 about existence and Theorems 3.2, 3.3 about nonexistence of optimal strategies. The proof of nonexistence consists in providing suitable estimates for the value of admissible strategies; the proof of existence requires a “dual” version of such estimates and then uses compactness arguments. Due to the complexity of the problem (that combines the difficulties of solving inequalities for positive matrices with the dynamic optimization problem), to our knowledge the results given in the literature cannot apply to this case (see [4] and [16] for similar results). For this reason we give a complete proof.

The structure of the proof is a little complex since various cases need to be analyzed. To be precise, for existence we need to prove that:

1 the admissible strategies always have the value $< +\infty$ (this is obvious for $\sigma > 1$ as $u_\sigma \geq 0$, but nontrivial for $\sigma \leq 1$)

2 at least one admissible strategy has the value $> -\infty$ (this is obvious for $\sigma < 1$ as $u_\sigma > 0$, but nontrivial for $\sigma \geq 1$);

3 suitable compactness arguments can be applied.

For the nonexistence proof we need to prove that

1′ in the case when $\sigma < 1$ (or $\sigma = 1$ and $\Gamma_\nu > 0$) either at least one admissible strategy has the value $= +\infty$ or there exists a sequence of admissible strategies with values converging to $+\infty$;

2′ in the case $\sigma > 1$ (or $\sigma = 1$ and $\Gamma_\nu \leq 0$), all admissible strategies have the value $= -\infty$.

The techniques needed to prove points **1** and **2′** are very similar. Moreover, the techniques needed to prove points **2** and **1′** are also very similar. So we give first Proposition 4.5 where points **1** and **2′** are dealt with. Then in Proposition 4.7 points **2** and **1′** are treated. These two Propositions prove the nonexistence Theorems 3.2, 3.3 and provide elements for the proof of Theorem 3.1. In order to complete the proof of the existence Theorem 3.1 we still have to tackle point **3** which is the aim of Proposition 4.9. In the statement below we denote by Γ_E the Euler Gamma function.

Proposition 4.5 *Given any* $\bar{\mathbf{s}} \geq \mathbf{0}$*, satisfying Assumption 2.5 the following hold.*

1. *Let* $\sigma \in (0,1)$*. Fix* $\varepsilon > 0$ *(*$\varepsilon = 0$ *when* Γ_i *is not a maximum for every* $i = 1, \ldots, k$*) such that* $\frac{\rho}{1-\sigma} > \Gamma_\nu + \varepsilon$*. Then for any* $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ *and* η *such that* $\frac{\rho}{1-\sigma} > \eta > \Gamma_\nu + \varepsilon$ *we have*

$$0 \leq U_\sigma(\mathbf{c}) \leq \frac{\rho - \eta(1-\sigma)}{1-\sigma} \cdot \frac{\Gamma_E(1+\sigma)}{(\rho - \eta(1-\sigma))^{1+\sigma}} \left[C_\varepsilon \sum_{i=1}^{k} \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon} \right]^{1-\sigma} < +\infty \qquad (38)$$

for a suitable C_ε *independent of the initial datum and of the control strategy.*

2. *Let $\sigma = 1$ (in this case for every ε we have $a_{\nu,\varepsilon} = \rho$). If $\rho > 0$ and $\eta > \Gamma_\nu + \varepsilon$ then for every $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ we have*

$$U_\sigma(\mathbf{c}) \leq \rho \int_0^{+\infty} e^{-\rho s} s \left[\frac{\eta}{2} s + \log \left(\frac{C_\varepsilon^1 \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}_{F,\varepsilon}^i}{s} \right) \right] ds < +\infty. \tag{39}$$

for a suitable C_ε^1 independent of the initial datum and the control strategy. If $\rho \leq 0$ and $\Gamma_\nu < 0$ then $U_\sigma(\mathbf{c}) = -\infty$ for every $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$. The same if $\rho \leq 0$, $\Gamma_\nu = 0$,

$$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu \left(e^{\bar{\Gamma}_1 t}, \ldots, e^{\bar{\Gamma}_k t} \right) < +\infty \tag{40}$$

and Γ_i is not a maximum for every $i = 1, \ldots k$.

3. *If $\sigma > 1$, then*

$$U_\sigma(\mathbf{c}) \leq 0.$$

Moreover if $a_{\nu,0} < 0$ then $U_\sigma(c) = -\infty$ for every $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$. The same holds if $a_{\nu,0} = 0$, (40) holds and Γ_i is not a maximum for every $i = 1, \ldots k$.

Proof.

(0) We first prove a key estimate for $I_\eta(t)$. Setting, for $i = 1, \ldots, k$, $\varepsilon > 0$ ($\varepsilon = 0$ if each Γ_i is not a maximum), $s \geq 0$,

$$\omega_{i,\varepsilon,s} := e^{-\bar{\Gamma}_{i,\varepsilon} s} c_{is}, \qquad \omega_{max,\varepsilon,s} := \max\{\omega_{i,\varepsilon,s},\ i = 1, \ldots, k\},$$

we have, using (21) and the fact that $\mathbf{e}_i^T \mathbf{v}_{F,\varepsilon}^i = 1$,

$$\int_0^t \omega_{max,\varepsilon,s} ds \leq \sum_{i=1}^k \int_0^t \omega_{i,\varepsilon,s} ds \leq \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}_{F,\varepsilon}^i, \qquad \forall t \geq 0. \tag{41}$$

Now, for $\eta \in \mathbb{R}$, we have

$$\begin{aligned} I_\eta(t) &= \int_0^t e^{-\eta s} \nu \left(c_{1s}, c_{2s}, \ldots, c_{ks} \right) ds \\ &= \int_0^t e^{-(\eta - \Gamma_\nu)s} e^{-\Gamma_\nu s} \nu \left(e^{\bar{\Gamma}_{1,\varepsilon} s} \omega_{1,\varepsilon,s}, e^{\bar{\Gamma}_{2,\varepsilon} s} \omega_{2,\varepsilon,s}, \ldots, e^{\bar{\Gamma}_{k,\varepsilon} s} \omega_{k,\varepsilon,s} \right) ds \\ &\leq \int_0^t e^{-(\eta - (\Gamma_\nu + \varepsilon))s} e^{-\Gamma_\nu s} \nu \left(e^{\bar{\Gamma}_1 s} \omega_{1,\varepsilon,s}, e^{\bar{\Gamma}_2 s} \omega_{2,\varepsilon,s}, \ldots, e^{\bar{\Gamma}_k s} \omega_{k,\varepsilon,s} \right) ds \\ &\leq \int_0^t e^{-(\eta - (\Gamma_\nu + \varepsilon))s} \omega_{max,\varepsilon,s} e^{-\Gamma_\nu s} \nu \left(e^{\bar{\Gamma}_1 s}, \ldots, e^{\bar{\Gamma}_k s} \right) ds \end{aligned} \tag{42}$$

If $\eta > \Gamma_\nu + \varepsilon$ then, by the definition of Γ_ν, we have that the function

$$t \to e^{-(\eta - (\Gamma_\nu + \varepsilon))s} e^{-\Gamma_\nu s} \nu \left(e^{\bar{\Gamma}_1 s}, \ldots, e^{\bar{\Gamma}_k s} \right)$$

is bounded on $[0,+\infty)$ (say by a constant C_ε independent of the initial datum and on the control strategy) and so, by putting (41) into (42) we get

$$I_\eta(t) \le C_\varepsilon \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon}, \qquad \forall t \ge 0. \tag{43}$$

Similarly, if

$$\lim_{t\to+\infty} e^{-\Gamma_\nu t}\nu\left(e^{\bar\Gamma_1 t},\dots,e^{\bar\Gamma_k t}\right) < +\infty$$

and Γ_i is not a maximum for every $i=1,\dots k$, then we can choose $\varepsilon = 0$ and $\eta = \Gamma_\nu$ in (42) and still get (43) with $\varepsilon = 0$, $\eta = \Gamma_\nu$.

(1) Now we prove estimate (38) using (31). Take $\eta < \frac{\rho}{1-\sigma}$, put the estimate (43) into (31) and let $t \to +\infty$. We get

$$U_\sigma(\mathbf{c}) \le \frac{\rho - \eta(1-\sigma)}{1-\sigma}\left[C_\varepsilon \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon}\right]^{1-\sigma} \int_0^{+\infty} s^\sigma e^{-(\rho-\eta(1-\sigma))s} ds$$

$$\le \frac{\rho - \eta(1-\sigma)}{1-\sigma} \cdot \frac{\Gamma_E(1+\sigma)}{(\rho-\eta(1-\sigma))^{1+\sigma}}\left[C_\varepsilon \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon}\right]^{1-\sigma}$$

so the claim (38) follows.

(2) If $\sigma = 1$ and $\rho > 0$ we get (39) taking $\eta > \Gamma_\nu + \varepsilon$, putting (43) into (30) and letting $t \to +\infty$.

If $\rho = 0$, then from (30) and (43) we get, for $\eta > \Gamma_\nu + \varepsilon$

$$\int_0^t \log \nu(\mathbf{c}_s) ds \le t\left[t\frac{\eta}{2} + \log\left(\frac{C_\varepsilon \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon}}{t}\right)\right]$$

so if $\Gamma_\nu < 0$ we take $\varepsilon > 0$, $\eta < 0$ such that $\eta > \Gamma_\nu + \varepsilon$. Then we get, in the limit for $t \to +\infty$, that $U_1(\mathbf{c}) = -\infty$.

Let finally $\Gamma_\nu = 0$,

$$\lim_{t\to+\infty} e^{-\Gamma_\nu t}\nu\left(e^{\bar\Gamma_1 t},\dots,e^{\bar\Gamma_k t}\right) < +\infty$$

and Γ_i is not a maximum for every $i = 1,\dots k$. Then by part (0) of this proof we can take $\eta = \Gamma_\nu = 0$ and $\varepsilon = 0$ in (43) so the estimate (30) becomes

$$\int_0^t \log \nu(\mathbf{c}_s) ds \le t \log\left(\frac{C_0 \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,0}}{t}\right)$$

so, in the limit for $t \to +\infty$, we still get that $U_1(\mathbf{c}) = -\infty$. If $\rho < 0$ and $\Gamma_\nu < 0$ (or $\Gamma_\nu = 0$ when possible) then

$$\int_0^t e^{-\rho s} \log \nu(\mathbf{c}_s) ds = \int_0^t e^{-\rho s}\left[\log \nu(\mathbf{c}_s)\right]^+ ds + \int_0^t e^{-\rho s}\left[\log \nu(\mathbf{c}_s)\right]^- ds.$$

Now, thanks to the nonpositivity of the negative part and to the fact that $e^{-\rho s} \geq 1$,

$$\int_0^t e^{-\rho s} \left[\log \nu(\mathbf{c}_s)\right]^- ds \leq \int_0^t \left[\log \nu(\mathbf{c}_s)\right]^- ds.$$

Since the right hand side goes to $-\infty$ as $t \to +\infty$ (thanks to the case $\rho = 0$) we have $\int_0^{+\infty} e^{-\rho s} \left[\log \nu(\mathbf{c}_s)\right]^- ds = -\infty$. By admissibility this implies that the integral of the positive part is finite and so $U_1(\mathbf{c}) = -\infty$.

(3) When $\sigma > 1$ it is obvious that $U_\sigma(\mathbf{c}) \leq 0$ by construction. Moreover using (29) and (43) with $\eta = \frac{\rho}{1-\sigma} > \Gamma_\nu + \varepsilon$ we get

$$\frac{1}{1-\sigma} \int_0^t e^{-\rho s} \nu(c_{1s}, c_{2s}, \ldots, c_{ks})^{1-\sigma} ds \leq \frac{t^\sigma}{1-\sigma} \left(C_\varepsilon \sum_{i=1}^k \bar{\mathbf{s}}^T \mathbf{v}^i_{F,\varepsilon} \right)^{1-\sigma}$$

and letting $t \to +\infty$ we get $U_\sigma(c) = -\infty$ for every admissible strategy. Finally if $a_{\nu,0} = 0$, (40) holds and Γ_i is not a maximum for every $i = 1, \ldots k$, we know from part (0) of this proof that (43) still holds so we can still let $t \to +\infty$ and get $U_\sigma(c) = -\infty$ for every admissible strategy. ∎

Remark 4.6 *The above result shows in particular that, when $a_{\nu,0} > 0$ and $\sigma \in (0,1)$, the intertemporal utility functional $U_\sigma(\mathbf{c})$ is finite and uniformly bounded for every admissible production-consumption strategy (while for $\sigma \geq 1$ it is only bounded from above). In the cases when*

1. $\sigma = 1$, $\rho \leq 0$, $\Gamma_\nu < 0$;
2. $\sigma = 1$, $\rho \leq 0$ *and* $\Gamma_\nu = 0$, *each* Γ_i *is not a maximum and (40) holds;*
3. $\sigma > 1$, $a_{\nu,0} < 0$;
4. $\sigma > 1$, $a_{\nu,0} = 0$, *each* Γ_i *is not a maximum and (40) holds;*

Proposition 4.5 shows that there are no optimal strategies in the sense of Definition 2.1 since all strategies have utility $-\infty$. ∎

Proposition 4.7 *Let* $\bar{\mathbf{s}} \geq \mathbf{0}$.

1. *Let* $\sigma \in (0,1)$ *and, either* $a_{\nu,0} < 0$, *or* $a_{\nu,0} = 0$, *each* Γ_i *is a maximum for* $i = 1, \ldots, k$ *and*
$$\lim_{t \to +\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right) > 0. \tag{44}$$
Then there exists an admissible strategy $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ *such that* $U_\sigma(\mathbf{c}) = +\infty$.

2. *Let* $\sigma = 1$, $a_{\nu,0} \leq 0$, $\Gamma_\nu > 0$. *Then there exists an admissible strategy* $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ *such that* $U_\sigma(\mathbf{c}) = +\infty$.

3. *Let* $\sigma \geq 1$ *and* $a_{\nu,0} > 0$, *then there exists an admissible strategy* $(\mathbf{x}, \mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$ *with* $U_\sigma(\mathbf{c}) > -\infty$.

Proof. We prove the three points separately.

Proof of 1. Consider first the case when $\sigma \in (0, 1)$ and $a_{\nu,0} < 0$. First let the system evolve to reach a state $\mathbf{s}_0 > \mathbf{0}$ (this is possible since Assumption 2.5 holds). This means that we can take from the beginning $\bar{\mathbf{s}} > \mathbf{0}$. At this point we observe that for any $i = 1, \ldots, k$ and $\varepsilon > 0$ ($\varepsilon = 0$ if Γ_i is a maximum for every i) we can find $\mathbf{x}_{i,\varepsilon} \geq \mathbf{0}$ such that

$$\mathbf{x}_{i,\varepsilon}^T \left(\mathbf{B} - (\Gamma_i - \varepsilon + \delta_{\mathbf{x}})\,\mathbf{A}\right) \geq \mathbf{e}_i^T \qquad \Rightarrow \qquad \mathbf{x}_{i,\varepsilon}^T \left(\mathbf{B} - \delta\mathbf{A}\right) \geq \mathbf{e}_i^T + (\Gamma_i - \varepsilon + \delta_{\mathbf{z}})\,\mathbf{x}_{i,\varepsilon}^T \mathbf{A}.$$

Take now $\beta_0 > 0$ and $\beta_1, \ldots \beta_k$ such that $\beta_i \geq 0$. Set

$$\mathbf{x}_{\varepsilon,s} := \beta_0 \sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon} e^{(\Gamma_i - \varepsilon)s}, \qquad s \geq 0.$$

We clearly have that $\mathbf{x}_{\varepsilon,s} \geq \mathbf{0}$, $\mathbf{x}_{\varepsilon,s} \neq \mathbf{0}$ for every $s \geq 0$ and

$$\mathbf{x}_{\varepsilon,s}^T \left(\mathbf{B} - \delta\mathbf{A}\right) \mathbf{e}_j = \beta_0 \sum_{i=1}^k \beta_i e^{(\Gamma_i - \varepsilon)s} \mathbf{x}_{i,\varepsilon}^T \left(\mathbf{B} - \delta\mathbf{A}\right) \mathbf{e}_j \geq \beta_0 \sum_{i=1}^k \beta_i e^{(\Gamma_i - \varepsilon)s} \left[\mathbf{e}_i^T \mathbf{e}_j + (\Gamma_i - \varepsilon + \delta_{\mathbf{z}})\,\mathbf{x}_{i,\varepsilon}^T \mathbf{A} \mathbf{e}_j\right]$$

Consider now the control strategy $\mathbf{x}_t = \mathbf{x}_{\varepsilon,t}$, $\mathbf{c}_t = \beta_0(\beta_1 e^{(\Gamma_1 - \varepsilon)t}, \ldots, \beta_k e^{(\Gamma_k - \varepsilon)t})$ for each $t \geq 0$. Since, for $t \geq 0$, we have

$$\mathbf{x}_{\varepsilon,t}^T \left(\mathbf{B} - \delta\mathbf{A}\right) \mathbf{e}_j - \hat{\mathbf{c}}_t^T \mathbf{e}_j \geq \beta_0 \sum_{i=1}^k \beta_i e^{(\Gamma_i - \varepsilon)t} \left(\Gamma_i - \varepsilon + \delta_{\mathbf{z}}\right) \mathbf{x}_{i,\varepsilon}^T \mathbf{A} \mathbf{e}_j$$

and the associated solution of the state equation (5) is given by:

$$\mathbf{s}_t^T = e^{-\delta_{\mathbf{z}} t} \bar{\mathbf{s}}^T + \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \mathbf{x}_{\varepsilon,s}^T \left[\mathbf{B} - \delta\mathbf{A}\right] ds - \int_0^t e^{-\delta_{\mathbf{z}}(t-s)} \hat{\mathbf{c}}_s ds$$

$$= \quad e^{-\delta_{\mathbf{z}} t} \left[\bar{\mathbf{s}}^T + \int_0^t \left(\mathbf{x}_{\varepsilon,s}^T \left(\mathbf{B} - \delta\mathbf{A}\right) - \hat{\mathbf{c}}_s^T\right) e^{\delta_{\mathbf{z}} s} ds\right],$$

then

$$\mathbf{s}_t^T \mathbf{e}_j \geq e^{-\delta_{\mathbf{z}} t} \left[\bar{\mathbf{s}}^T \mathbf{e}_j + \int_0^t \beta_0 \sum_{i=1}^k \beta_i e^{(\delta_{\mathbf{z}} + \Gamma_i - \varepsilon)s} \left(\Gamma_i - \varepsilon + \delta_{\mathbf{z}}\right) \mathbf{x}_{i,\varepsilon}^T \mathbf{A} \mathbf{e}_j \; ds\right]$$

$$= e^{-\delta_{\mathbf{z}} t} \left[\bar{\mathbf{s}}^T \mathbf{e}_j + \beta_0 \sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T \mathbf{A} \mathbf{e}_j \int_0^t e^{(\delta_{\mathbf{z}} + \Gamma_i - \varepsilon)s} \left(\Gamma_i - \varepsilon + \delta_{\mathbf{z}}\right) ds\right]$$

$$= e^{-\delta_{\mathbf{z}} t}\left[\bar{\mathbf{s}}^T\mathbf{e}_j + \beta_0\sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T \mathbf{A}\mathbf{e}_j\left(e^{(\delta_{\mathbf{z}}+\Gamma_i-\varepsilon)t}-1\right)\right]$$

$$= e^{-\delta_{\mathbf{z}} t}\left[\bar{\mathbf{s}}^T\mathbf{e}_j - \beta_0\sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T \mathbf{A}\mathbf{e}_j\right] + \beta_0\sum_{i=1}^k \beta_i e^{(\Gamma_i-\varepsilon)t}\mathbf{x}_{i,\varepsilon}^T \mathbf{A}\mathbf{e}_j$$

$$= e^{-\delta_{\mathbf{z}} t}\left[\bar{\mathbf{s}}^T\mathbf{e}_j - \beta_0\sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T \mathbf{A}\mathbf{e}_j\right] + \mathbf{x}_{\varepsilon,t}^T \mathbf{A}\mathbf{e}_j$$

In this case it is clear that the constraints $\mathbf{s}_t^T \geq \mathbf{x}_t^T\mathbf{A}$ are satisfied if, for every $j = 1, \ldots, n$,

$$\bar{\mathbf{s}}^T\mathbf{e}_j - \beta_0\sum_{i=1}^k \beta_i \mathbf{x}_{i,\varepsilon}^T \mathbf{A}\mathbf{e}_j \geq 0.$$

Since $\bar{\mathbf{s}} > 0$, the above is true if we set β_0 sufficiently small. So our control strategy is admissible. Moreover setting $\beta_1 = \ldots = \beta_k = 1$ we have

$$U(\mathbf{c}) = \frac{\beta_0^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{-\rho t}\nu\left(e^{(\Gamma_1-\varepsilon)t}, \ldots, e^{(\Gamma_k-\varepsilon)t}\right)^{1-\sigma} dt$$

$$= \frac{\beta_0^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{[-\rho+(\Gamma_\nu-\varepsilon)(1-\sigma)]t}\left[e^{-\Gamma_\nu t}\nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right)\right]^{1-\sigma} dt$$

Using the definition of Γ_ν and the fact that $a_{\nu,0} < 0$ we get that, for ε sufficiently small, the above integral is $+\infty$ and so the claim.

The case $a_{\nu,0} = 0$ follows simply observing that, in the above equation, since Γ_i is a maximum for $i = 1, \ldots, k$, we can take $\varepsilon = 0$ and, thanks to (44), we can take $\rho = \Gamma_\nu(1-\sigma)$.

Proof of 2. Take now the case when $\sigma = 1$ and $a_{\nu,0} \leq 0$, $\Gamma_\nu > 0$. Since $\Gamma_\nu > 0$ let ε such that $\Gamma_\nu > 2\varepsilon$. Then we take the above control strategy so that

$$U_1(\mathbf{c}) = \int_0^{+\infty} e^{-\rho t}\log\nu\left(e^{(\Gamma_1-\varepsilon)t}, \ldots, e^{(\Gamma_k-\varepsilon)t}\right) ds =$$

$$= \int_0^{+\infty} e^{-\rho t}\left[\log\beta_0 + (\Gamma_\nu - 2\varepsilon)t + \log\left(e^{-(\Gamma_\nu-\varepsilon)t}\nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right)\right)\right] dt.$$

Clearly, for $a_{\nu,0} = \rho \geq 0$ the last integrand is locally bounded, definitely positive, and goes to $+\infty$ for $t \to +\infty$. Then for this strategy we have $U_1(\mathbf{c}) = +\infty$.

Proof of 3. Let $a_{\nu,0} > 0$ and $\sigma \in [1, +\infty)$. We observe that (since admissibility does not depend on the value of σ) the strategy found in point 1 above is admissible. We then have, for $\sigma \in (1, +\infty)$

$$U(\mathbf{c}) = \frac{\beta_0^{1-\sigma}}{1-\sigma}\int_0^{+\infty} e^{[-\rho+(\Gamma_\nu-\varepsilon)(1-\sigma)]t}\left[e^{-\Gamma_\nu t}\nu\left(e^{\Gamma_1 t}, \ldots, e^{\Gamma_k t}\right)\right]^{1-\sigma} dt$$

so, if $\varepsilon > 0$ is such that $-\rho + (\Gamma_\nu - \varepsilon)(1-\sigma) < 0$ we get the claim.

For $\sigma = 1$ we have

$$U_1(\mathbf{c}) = \int_0^{+\infty} e^{-\rho t} \log \nu \left(e^{(\Gamma_1 - \varepsilon)t}, \dots, e^{(\Gamma_k - \varepsilon)t}\right) ds =$$

$$= \int_0^{+\infty} e^{-\rho t} \left[\log \beta_0 + \Gamma_\nu t + \log \left(e^{-(\Gamma_\nu + \varepsilon)t} \nu \left(e^{\Gamma_1 t}, \dots, e^{\Gamma_k t}\right)\right)\right] dt.$$

Since the last integrand is less than polynomially growing and $\rho = a_{\nu,0} > 0$ then the integral above is finite, so $U_1(\mathbf{c}) > -\infty$. ∎

Remark 4.8 *The above result shows in particular that, when $a_{\nu,0} > 0$ and $\sigma \in [1, +\infty)$, the intertemporal utility functional $U_\sigma(\mathbf{c})$ is not always $-\infty$ so it is bounded from below (recall that from Proposition 4.5 we already know that in these case $U_\sigma(\mathbf{c})$ is bounded from above). Moreover in the cases when*

1. *$\sigma \in (0,1)$ and $a_{\nu,0} < 0$*
2. *$\sigma \in (0,1)$, $a_{\nu,0} = 0$, each Γ_i is a maximum and (44) holds,*
3. *$\sigma = 1$, $a_{\nu,0} \leq 0$ and $\Gamma_\nu > 0$,*

Proposition 4.7 shows that there are no optimal strategies in the sense of Definition 2.1 since the supremum of the utility is $+\infty$. ∎

Summing up the informations taken from Propositions 4.5 and 4.7 we can say the following.

- In the cases when $a_{\nu,0} > 0$ we know that the functional is uniformly bounded (case $\sigma \in (0,1)$) or bounded from from above and not identically $-\infty$ (case $\sigma \geq 1$);
- In the cases when $a_{\nu,0} \leq 0$ we have nonexistence when
  1. $\sigma \in (0,1)$ and $a_{\nu,0} < 0$;
  2. $\sigma \in (0,1)$, $a_{\nu,0} = 0$, each Γ_i is a maximum and (44) holds;
  3. $\sigma = 1$, $a_{\nu,0} \leq 0$, $\Gamma_\nu \neq 0$
  4. $\sigma = 1$, $a_{\nu,0} \leq 0$, $\Gamma_\nu = 0$, each Γ_i is not a maximum and (40) holds;
  5. $\sigma > 1$ and $a_{\nu,0} < 0$
  6. $\sigma > 1$, $a_{\nu,0} = 0$, each Γ_i is not a maximum and (40) holds.

We observe that, to end the treatment of nonexistence result one should deal with a complete treatment of the following limiting cases:

- $\sigma \in (0,1)$ and $a_{\nu,0} = 0$;
- $\sigma = 1$, $a_{\nu,0} \leq 0$, $\Gamma_\nu = 0$;
- $\sigma > 1$ and $a_{\nu,0} = 0$

Proof of Theorem 3.2. It follows directly from Proposition 4.5, Proposition 4.7 and the remarks above. ∎

Now we come to prove existence when $a_{\nu,0} > 0$ using compactness arguments. To do this we need first to prove suitable properties of the set $\mathcal{A}(\bar{\mathbf{s}})$ which are given in the next proposition. First we recall a simple definition: given a measurable function $g_1 : \mathbb{R}^+ \to \mathbb{R}$ we denote by $L^\infty_{g_1}(0,+\infty;\mathbb{R}^m)$ the set of measurable functions $f : \mathbb{R}^+ \to \mathbb{R}^m$ such that the product $f \cdot g_1$ is bounded on $\mathbb{R}^+$. Moreover given a measurable function $g_2 : \mathbb{R}^+ \to \mathbb{R}^k$ we denote by $L^1_{g_2}\left(0,+\infty;\mathbb{R}^k\right)$ the set of measurable functions $f : \mathbb{R}^+ \to \mathbb{R}^k$ such that the product $f_i \cdot g_{2,i}$ is integrable on $\mathbb{R}^+$ for each $i = 1,\ldots,k$. We set $g_1(t) = e^{\lambda t}$ (λ is given by (23) of Lemma 4.2) and $g_2(t) = (e^{(\Gamma_1+\varepsilon)t},\ldots,e^{(\Gamma_k+\varepsilon)t})$ for $\varepsilon > 0$ such that $a_{\nu,\varepsilon} > 0$.

Proposition 4.9 *Let Assumptions 2.2 and 2.3 be verified. Let also $\sigma \in (0,1) \cup (1,+\infty)$. Given any $\bar{\mathbf{s}} \geq \mathbf{0}$ the set $\mathcal{A}(\bar{\mathbf{s}})$ of admissible control strategies starting at $\bar{\mathbf{s}}$ is a closed, bounded, convex subset of the space $L^\infty_{g_1}(0,+\infty;\mathbb{R}^m) \times L^1_{g_2}\left(0,+\infty;\mathbb{R}^k\right)$. Moreover*

$$(\mathbf{x},\mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}}), \quad \lambda \in [0,1] \Rightarrow (\lambda\mathbf{x},\lambda\mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}}) \tag{45}$$

Finally, if ν is strictly concave the functional U_σ is strictly concave with respect to the argument $\mathbf{c}$. The same holds when $\sigma = 1$ and $\rho > 0$.

Proof. Convexity. Let $i = 1,2$ and let $(\mathbf{x}_i,\mathbf{c}_i) \in \mathcal{A}(\bar{\mathbf{s}})$, and $\lambda \in [0,1]$. Calling

$$(\mathbf{x}_\lambda,\mathbf{c}_\lambda) = \lambda(\mathbf{x}_1,\mathbf{c}_1) + (1-\lambda)(\mathbf{x}_2,\mathbf{c}_2)$$

then due to the linearity of the state equation (5)

$$\mathbf{s}_{t,\bar{\mathbf{s}},(\mathbf{x}_\lambda,\mathbf{c}_\lambda)} = \lambda\mathbf{s}_{t,\bar{\mathbf{s}},(\mathbf{x}_1,\mathbf{c}_1)} + (1-\lambda)\mathbf{s}_{t,\bar{\mathbf{s}},(\mathbf{x}_2,\mathbf{c}_2)}.$$

Since all constraints on $(\mathbf{s},(\mathbf{x},\mathbf{c}))$ (i.e. $\mathbf{x} \geq \mathbf{0}$, $\mathbf{c} \geq 0$, $\mathbf{x}^T\mathbf{A} \leq \mathbf{s}^T$) are linear it follows that, since $(\mathbf{x}_i,\mathbf{c}_i)$ $(i = 1,2)$ satisfy them, then so does $(\mathbf{x}_\lambda,\mathbf{c}_\lambda)$. This yields $(\mathbf{x}_\lambda,\mathbf{c}_\lambda) \in \mathcal{A}(\bar{\mathbf{s}})$ when $\sigma \in (0,1) \cup (1,+\infty)$. If $\sigma = 1$ we also have to prove that $(\mathbf{x}_\lambda,\mathbf{c}_\lambda)$ is semiintegrable. This follows from point (2) of Proposition 4.5. Indeed if $\rho > 0$, thanks to estimate (39) we know all admissible strategies are upper semiintegrable, so also their convex combinations are upper semiintegrable. Boundedness follows from the estimates of Lemma 4.2.

Closedness follows from the fact that all constraints are linear so all of them preserve in the limit in the topology of $L^\infty_{g_1}(0,+\infty;\mathbb{R}^m) \times L^1_{g_2}\left(0,+\infty;\mathbb{R}^k\right)$. For $\sigma = 1$ we need to know that the limit of semiintegrable sequences is again semiintegrable. For $\rho > 0$ this follows from the estimate (39).

Homogeneity (45) follows from convexity and from the fact that the strategy $(\mathbf{0}, \mathbf{0})$ is always admissible.

Strict concavity of the functional U_σ is a standard result (see e.g. [5]) and we omit the proof. ■

Now we move on to the proof of Theorem 3.1.

Proof of Theorem 3.1. The uniqueness property follows from the strict concavity of U_σ proved in Proposition 4.9. The existence result follows applying a suitable modification of Theorems 21 and 22 in [16, p. 406] (see also [15]). We divide it into three cases depending on the value of σ.

Case $\sigma > 1$.

Here we can apply directly Theorem 22 and note 26 of [16, p. 406] plus [16, note 20, p.137]. In fact this theorem asks the following:

1. the set U where the controls take values is closed (in our case U is $\mathbb{R}_+^m \times \mathbb{R}_+^k$ i.e. the positive orthant of $\mathbb{R}^{m+k}$);

2. the functions defining the running utility $((\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \to e^{-\rho t} u_\sigma(\nu(\mathbf{c})))$, the dynamics of the state equation $((\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \to -\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T(\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}})$ and the constraints $((\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \to \mathbf{s}^T - \mathbf{x}^T \mathbf{A})$ are defined on the set
$$S = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \in \mathbb{R}_+^n \times \left[\mathbb{R}_+^m \times \mathbb{R}_+^k \right] \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\},$$
are linear (or sum of linear and nondecreasing) in the variable $\mathbf{s}$ and continuous on the set
$$S' = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \in \mathbb{R}_+^n \times \left[\mathbb{R}_+^m \times \mathbb{R}_+^k \right] \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\};$$

3. for each $t \geq 0$ the set
$$S'(t) = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c})) \in \mathbb{R}_+^n \times \left[\mathbb{R}_+^m \times \mathbb{R}_+^k \right] : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\}$$
is contained in the closure $\overline{S^0(t)}$ of the set
$$S^0(t) = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c})) \in \mathbb{R}_+^n \times U : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} > \mathbf{0} \right\};$$

4. for each $n \in \mathbb{N}$ and $t \geq 0$ the set
$$\Gamma_t^n = \left\{ (\mathbf{s}, (\mathbf{x},\mathbf{c})) : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x},\mathbf{c}) \in U, \right.$$
$$\left. \left|\left| \left(e^{-\rho t} u_\sigma(\nu(\mathbf{c})), -\delta_{\mathbf{z}} \mathbf{s}^T + \mathbf{x}^T(\mathbf{B} - \delta \mathbf{A}) - \hat{\mathbf{c}} \right) \right|\right| \leq n, \right\}$$
is closed and is contained in $S'(t)$. The same for the set
$$\Gamma^n = \left\{ \mathbf{s} : (\mathbf{s}, (\mathbf{x},\mathbf{c})) \in \Gamma_t^n \right\};$$

5. there exists an admissible strategy with finite value;
6. the set
$$N(\mathbf{s},U,t) = \left\{ \left(e^{-\rho t} u_\sigma (\nu(\mathbf{c})) + \gamma, -\delta_\mathbf{z}\mathbf{s} + \mathbf{x}^T(\mathbf{B}-\delta\mathbf{A}) - \hat{\mathbf{c}} + \gamma\right) : \\ (\gamma,\gamma) \le 0, \mathbf{s} - \mathbf{x}^T\mathbf{A} \ge \mathbf{0}, (\mathbf{x},\mathbf{c}) \in U \right\}$$
is convex for all $(\mathbf{s},t) \in \mathbb{R}^n \times [0,+\infty)$;
7. the set $N(\mathbf{s},U,t)$ has closed graph for each t as a function of $\mathbf{s} \in \Gamma^n$. Closed graph means that
$$\mathbf{s}_n \in \Gamma^n, \mathbf{v}_n \in N(\mathbf{s},U,t), \mathbf{s}_n \to \mathbf{s}, \mathbf{v}_n \to \mathbf{v} \Rightarrow \mathbf{s} \in \Gamma^n.$$
8. there exists $\mathbf{q}' \in \mathbb{R}^{n+1}$, $\mathbf{q}' \ge 0$ such that for every $\mathbf{q} \ge \mathbf{q}'$ $(\mathbf{q} = (q_0, q_1, ..., q_n) = (q_0, \mathbf{q}_1))$ there exists locally integrable functions $\phi_\mathbf{q}$ and $\psi_\mathbf{q}$ defined for $t \in [0,+\infty)$ such that
$$e^{-\rho t} u_\sigma(\nu(\mathbf{c})) q_0 + \left(-\delta_\mathbf{z}\mathbf{s}^T + \mathbf{x}^T(\mathbf{B}-\delta\mathbf{A}) - \hat{\mathbf{c}}\right)\mathbf{q}_1 \le \phi_\mathbf{q}(t) + \psi_\mathbf{q}(t) \cdot \max\left[0, \mathbf{s}^T\mathbf{e}_1, ..., \mathbf{s}^T\mathbf{e}_n\right]$$
for every $(\mathbf{s}, (\mathbf{x},\mathbf{c}), t) \in S$.
9. for every $i = 1, ..., n$ and every admissible state trajectory $\mathbf{s}_t$, we have $\mathbf{s}_t^T\mathbf{e}_i \ge 0$ for every $t \ge 0$. Moreover for every $q_0 \in \mathbb{R}$, $q_0 \ge 0$, there exists an integrable function ν_{q_0} defined for $t \in [0,+\infty)$ such that, ,
$$e^{-\rho t} u_\sigma(\nu(\mathbf{c}_t))(1+q_0) \le \nu_{q_0}(t),$$
for every admissible strategy $(\mathbf{x},\mathbf{c}) \in \mathcal{A}(\bar{\mathbf{s}})$.

All points 1-4 and 6-7 are easily checked in our case thanks to the linearity of the state equation and of the constraints. We omit the verification of them for brevity. Point 5 is known from previous results (Proposition 4.5 and Proposition 4.7). Point 9 comes simply recalling that for $\sigma > 1$ the utility is negative and so one can choose $\nu_{q_0}(t) = 0$ for every $t \ge 0$. Point 8 is more delicate. Setting
$$g(\mathbf{c}) = e^{-\rho t} u_\sigma(\nu(\mathbf{c})) q_0 - \hat{\mathbf{c}}^T\mathbf{q}_1$$
we have
$$g(\mathbf{c}) \le 0$$
Moreover
$$-\delta_\mathbf{z}\mathbf{s}^T + \mathbf{x}^T(\mathbf{B}-\delta\mathbf{A}) = -\delta_\mathbf{z}\left(\mathbf{s}^T - \mathbf{x}^T\mathbf{A}\right) + \mathbf{x}^T(\mathbf{B}-\delta_\mathbf{x}\mathbf{A}) \le \mathbf{x}^T\mathbf{B}.$$
Now, recalling the proof of (23) we have that
$$\mathbf{x}^T\mathbf{B} \le \mathbf{s}^T\mathbf{D} \le M \max\left[0, \mathbf{s}^T\mathbf{e}_1, ..., \mathbf{s}^T\mathbf{e}_n\right]$$
where M depends only on the coefficient of $\mathbf{D}$. Setting, for $t \ge 0$, $\phi_\mathbf{q}(t) = 0$ and $\psi_\mathbf{q}(t) = M$ we see that $\phi_\mathbf{q}$ and $\psi_\mathbf{q}$ are locally integrable functions and satisfy point 8.

Case $\sigma = 1$ and $\sigma \in (0,1)$.

Also this case goes applying Theorem 22 and note 26 of [16, p. 406] (see also [15] or [16, Exercise 6.8.3, p.410]). In fact this theorem asks the following:

1. the set U where the controls take values is closed (in our case U is $\mathbb{R}_+^m \times \mathbb{R}_+^k$ i.e. the positive orthant of $\mathbb{R}^{m+k}$);

2. the functions defining the running utility $((\mathbf{s}, (\mathbf{x}{,}\mathbf{c}), t) \rightarrow e^{-\rho t} u_\sigma(c))$, the dynamics of the state equation $((\mathbf{s}, (\mathbf{x}{,}\mathbf{c}), t) \rightarrow -\delta_\mathbf{z} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B}{-}\delta \mathbf{A}) - \hat{\mathbf{c}})$ and the constraints $((\mathbf{s}, (\mathbf{x}{,}\mathbf{c}), t) \rightarrow \mathbf{s}^T - \mathbf{x}^T \mathbf{A})$ are defined on the set
$$S = \left\{ (\mathbf{s}, (\mathbf{x}{,}\mathbf{c}), t) \in \mathbb{R}_+^n \times U \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\}$$
are linear (or sum of linear and nondecreasing) in the variable $\mathbf{s}$ and continuous on the set
$$S' = \left\{ (\mathbf{s}, (\mathbf{x}{,}\mathbf{c}), t) \in \mathbb{R}_+^n \times U \times \mathbb{R}_+ : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\};$$

3. for each $t \geq 0$ the set
$$S'(t) = \left\{ (\mathbf{s}, (\mathbf{x}{,}\mathbf{c})) \in \mathbb{R}_+^n \times U : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0} \right\}$$
is contained in the closure $\overline{S^0(t)}$ of the set
$$S^0(t) = \left\{ (\mathbf{s}, (\mathbf{x}{,}\mathbf{c})) \in \mathbb{R}_+^n \times U : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} > \mathbf{0} \right\};$$

4. for each $n \in \mathbb{N}$ and $t \geq 0$ the set
$$\Gamma_t^n = \left\{ (\mathbf{s}, (\mathbf{x}{,}\mathbf{c})) : \mathbf{s}^T - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x}{,}\mathbf{c}) \in U, \right.$$
$$\left. \left|\left| \left(e^{-\rho t} u_\sigma(\nu(\mathbf{c})), -\delta_\mathbf{z} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B}{-}\delta \mathbf{A}) - \hat{\mathbf{c}} \right) \right|\right| \leq n, \right\}$$
is closed and is contained in $S'(t)$. The same for the set
$$\Gamma^n = \{ \mathbf{s} : (\mathbf{s}, (\mathbf{x}{,}\mathbf{c})) \in \Gamma_t^n \};$$

5. there exists an admissible strategy with finite value;

6. the set
$$N(\mathbf{s}{,}U, t) = \left\{ \left(e^{-\rho t} u_\sigma(\nu(\mathbf{c})) + \gamma, -\delta_\mathbf{z} \mathbf{s} + \mathbf{x}^T (\mathbf{B}{-}\delta \mathbf{A}) - \hat{\mathbf{c}}^T + \gamma \right) : \right.$$
$$\left. (\gamma, \gamma) \leq 0, \mathbf{s} - \mathbf{x}^T \mathbf{A} \geq \mathbf{0}, (\mathbf{x}{,}\mathbf{c}) \in U \right\}$$
is convex for all $(\mathbf{s}{,}t) \in \mathbb{R}^n \times [0, +\infty)$;

7. the set $N(\mathbf{s}{,}U, t)$ has closed graph for each t as a function of $\mathbf{s} \in \Gamma^n$. Closed graph means that
$$\mathbf{s}_n \in \Gamma^n, \mathbf{v}_n \in N(\mathbf{s}{,}U, t), \mathbf{s}_n \rightarrow \mathbf{s}, \mathbf{v}_n \rightarrow \mathbf{v} \Rightarrow \mathbf{s} \in \Gamma^n.$$

8. there exists $\mathbf{q}' \in \mathbb{R}^{n+1}$, $\mathbf{q}' \geq 0$ such that for every $\mathbf{q} \geq \mathbf{q}'$ $(\mathbf{q} = (q_0, q_1, ..., q_n) = (q_0, \mathbf{q}_1))$ there exists locally integrable functions $\phi_\mathbf{q}$ and $\psi_\mathbf{q}$ defined for $t \in [0, +\infty)$ such that
$$e^{-\rho t} u_\sigma(\nu(\mathbf{c})) q_0 + \left(-\delta_\mathbf{z} \mathbf{s}^T + \mathbf{x}^T (\mathbf{B}{-}\delta \mathbf{A}) - \hat{\mathbf{c}}^T \right) \mathbf{q}_1 \leq \phi_\mathbf{q}(t) + \psi_\mathbf{q}(t) \cdot \max\left[0, \mathbf{s}^T \mathbf{e}_1, ..., \mathbf{s}^T \mathbf{e}_n \right]$$
for every $(\mathbf{s}, (\mathbf{x}{,}\mathbf{c}), t) \in S$.

9. for every $i=1,...,n$ and every admissible state trajectory $\mathbf{s}_t$, we have $\mathbf{s}_t^T\mathbf{e}_i \geq 0$ for every $t \geq 0$. Moreover for every $q_0 \in \mathbb{R}$, $q_0 \geq 0$, there exists an integrable function ν_{q_0}, continuous functions $\chi^i_{q_0}$ and $\theta^i_{q_0}$ $(i=1,...,n)$ defined for $t \in [0,+\infty)$ such that, for every admissible strategy $(\mathbf{x},\mathbf{c}) \in \mathcal{A}(\overline{\mathbf{s}})$,

$$e^{-\rho t}u_\sigma(\nu(\mathbf{c}_t)\,(1+q_0)+\sum_{i=1}^{n}\chi^i_{q_0}(t)\left(-\delta_{\mathbf{z}}\mathbf{s}_t^T+\mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A})-\hat{\mathbf{c}}_t^T\right)\mathbf{e}_i \leq \nu_{q_0}(t)\,,$$

and

$$-\int_s^{+\infty}\chi^i_{q_0}(t)\left(-\delta_{\mathbf{z}}\mathbf{s}_t^T+\mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A})-\hat{\mathbf{c}}_t^T\right)\mathbf{e}_i dt \leq \theta^i_{q_0}(s)$$

where $\lim_{s\to+\infty}\theta^i_{q_0}(s)=0$.

All points 1-4 and 6-7 are easily checked in our case thanks to the linearity of the state equation and of the constraints. We omit the verification of them for brevity. Point 5 is known from previous results (Proposition 4.5 and Proposition 4.7). Point 8 follows arguing exactly as in the case $\sigma>1$ except for the estimate of the function $g(\mathbf{c})$ which is done as follows. First observe that $g(\mathbf{c})$ goes to $-\infty$ as any $c_i \to +\infty$, then observe that g is positive on a compact set depending on $\mathbf{q}$ and t which is bounded uniformly when $\mathbf{q}$ and t belong to a bounded set. This is enough to guarantee that g has a maximum point and that the value of the maximum is uniformly bounded for $\mathbf{q}$ and t on bounded sets.

Point 9 is more delicate. We show it in the case $\sigma=1$ as the other case is analogous. Set $\chi^i_{q_0}(t)=e^{-dt}$ for suitable d to choose later and consider the term containing $\mathbf{c}_t$ first. They are

$$e^{-\rho t}\ln\nu(\mathbf{c}_t)\,(1+q_0)-e^{-dt}\sum_{i=1}^{k}c_{i,t}.$$

Then, setting

$$g(\mathbf{c})=e^{-\rho t}\ln\nu(\mathbf{c})\,(1+q_0)-e^{-dt}\sum_{i=1}^{k}c_i$$

we have, arguing as for the g above, that for every $\mathbf{c}\geq 0$, $g(\mathbf{c})$ is estimated from above by an integrable function depending only on ρ, d, q_0

$$g(c) \leq e^{-\rho t}(1+q_0)\cdot\left[\ln\left(\frac{e^{-\rho t}(1+q_0)}{e^{-dt}}\right)-1\right]=e^{-\rho t}(1+q_0)\cdot[(-\rho+d)\,t+\ln(1+q_0)-1]\,.$$

The right hand side is integrable for $\rho>0$. Moreover

$$-\delta_{\mathbf{z}}\mathbf{s}_t^T+\mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A})=-\delta_{\mathbf{z}}\left(\mathbf{s}_t^T-\mathbf{x}_t^T\mathbf{A}\right)+\mathbf{x}_t^T(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A})\leq\mathbf{x}_t^T(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A})\,.$$

Then, for every $\varepsilon>0$,

$$\begin{aligned}\left[-\delta_{\mathbf{z}}\mathbf{s}_t^T+\mathbf{x}_t^T(\mathbf{B}-\delta\mathbf{A})\right]\mathbf{v}_F e^{-dt} &\leq \mathbf{x}_t^T(\mathbf{B}-\delta_{\mathbf{x}}\mathbf{A})\,\mathbf{v}_F e^{-dt}\\ &\leq \mathbf{x}_t^T\mathbf{A}\mathbf{v}_F(\Gamma+\varepsilon)\,e^{-dt}\leq e^{(\Gamma+\varepsilon)t}\overline{\mathbf{s}}^T\mathbf{v}_F(\Gamma+\varepsilon)\,e^{-dt}.\end{aligned}$$

So if $d > \Gamma$ the first part of point 9 is true. Now observe that

$$\begin{aligned} & -\int_s^{+\infty} \chi_{q_0}^i(t)\left(-\delta_{\mathbf{z}} \mathbf{s}_t^T + \mathbf{x}_t^T(\mathbf{B}-\delta \mathbf{A}) - c_t \mathbf{e}_1^T\right) \mathbf{e}_i dt \\ = & -\int_s^{+\infty} e^{-dt} \mathbf{e}_i^T \mathbf{v}_F \cdot \left(-\delta_{\mathbf{z}} \mathbf{s}_t^T + \mathbf{x}_t^T(\mathbf{B}-\delta \mathbf{A}) - c_t \mathbf{e}_1^T\right) \mathbf{e}_i dt \\ \leq & \int_s^{+\infty} e^{-dt} \mathbf{e}_i^T \mathbf{v}_F \cdot \left(\delta_{\mathbf{z}} \mathbf{s}_t^T + \delta_{\mathbf{x}} \mathbf{x}_t^T \mathbf{A} + c_t \mathbf{e}_1^T\right) \mathbf{e}_i dt. \end{aligned}$$

Now from the estimates 19.21

$$\mathbf{e}_i^T \mathbf{v}_F \cdot \left(\delta_{\mathbf{z}} \mathbf{s}_t^T + \delta_{\mathbf{x}} \mathbf{x}_t^T \mathbf{A}\right) \mathbf{e}_i \leq M e^{(\Gamma+\varepsilon) t}$$

so that

$$\int_s^{+\infty} e^{-dt} \mathbf{e}_i^T \mathbf{v}_F \cdot \left(\delta_{\mathbf{z}} \mathbf{s}_t^T + \delta_{\mathbf{x}} \mathbf{x}_t^T \mathbf{A}\right) \mathbf{e}_i dt \leq M_1 e^{-(d-\Gamma-\varepsilon) s}.$$

Moreover, thanks to stima 22-19 we get, for $d > \Gamma + \varepsilon$

$$\int_s^{\tau} e^{-dr} c_r dr \leq e^{-(d-\Gamma-\varepsilon) s} \overline{\mathbf{s}}^T \mathbf{v}_F$$

so that, sending $\tau \to +\infty$,

$$\int_s^{+\infty} e^{-dt} c_t dt \leq M_2 e^{-(d-\Gamma-\varepsilon) s}.$$

and this completes the proof. ∎

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