Factor intensity and order of resource extraction

Freni, Giuseppe

Dipartimento di Studi Economici "Salvatore Vinci", Università "Parthenope" of Napoli

20 November 2009

Factor Intensity and Order of Resource Extraction

Giuseppe Freni

Dipartimento di Studi Economici,

Università "Parthenope" di Napoli,

Via Medina 40, 80133, Napoli

giuseppe.freni@uniparthenope.it

This version: November 20, 2009

Abstract. This paper characterizes the optimal time paths of extraction of several nonrenewable resource deposits with different costs of extraction when the extracted resource can be converted into productive capital and the extraction process, as well as the production of the substitute, requires two primary factors of production. Under a technological assumption granting that the time paths of primary factor prices are monotonic, we show that, for each pair (lower cost/higher cost) of deposits, an intensity condition is necessary in order to have discontinuous extraction of the lower cost deposit. We also show that the same condition is sufficient for discontinuous extraction of the lower cost deposit, provided the stock of the lower cost deposit is sufficiently large and the stocks of all other deposits are sufficiently small.

JEL classification: Q3; Q4

1. Introduction

If an “energy” sector can exploit several deposits of an exhaustible resource, then a problem of order of extraction arises. Partial equilibrium analysis suggests that efficient extraction should occur sequentially from the lowest-cost deposit to the highest-cost one (Herfindahl, 1967) and that, if a high-cost substitute exists, then its production by means of the 'backstop technology' should begin after exhaustion of all deposits. In an attempt to evaluate the generality of the above principles, Kemp and Long (1980) showed that both the above 'folk theorems' are invalid in a “Ricardian” general equilibrium context. Within that framework, the ‘theorems’ fail because the desire to smooth consumption and the fact that the product is non-storable provide an incentive to delaying extraction from low cost deposit. In turn, procrastination of extraction implies the generic existence of time intervals during which (at least) two processes are operated. Building on this argument, Lewis (1982) developed an extended model in which storage is allowed, and proved that the least-cost-first principle is restored, provided the extracted resource can be converted into productive capital, where productive capital means that stored capital grows at a positive rate.

More recently, Amigues, Favard, Gaudet and Moreaux (1998) have modified the Kemp and Long model by introducing a constraint for the capacity of the backstop that is active in the long run equilibrium. Amigues et al. (1998) found that the capacity constraint generates an additional incentive to delay extraction, which may lead to start the backstop well before a lower cost resource is ever put into use. Favard (2002) extended these results to the Lewis (1982) framework. Capacity constraints have been further investigated by Freni (2004) and Holland (2003) in partial equilibrium settings.

Both studies reported a series of negative counter-examples showing, in particular, that deposits may be temporarily abandoned after a period of initial extraction and that high-cost deposits may be either opened or exhausted before low-cost ones. To our knowledge, Freni's (2004) and Holland's (2003) articles are the only studies that reported the possibility of discontinuous extraction of a nonrenewable resource reserve in a single demand setting (for discontinuous extraction with multiple demands see Gaudet, Moreaux and Salant, 2001 and Im, Chackavorty and Roumasset, 2006).

Capacity constraints reflect the existence of specific primary factors of production in fixed supply. In general, therefore, adding a capacity constraint increases by one the number of primary factors of the system and opens the door to 'substitution' effects associated with the transitional dynamics of factor prices. These effects are indeed at the root of the results of Amigues et al. (1998), Favard (2002), Freni (2004) and Holland (2003). For example, in Favard's (2002) model, which involves two factors of production (transferable 'labor' and a backstop-specific factor of production), the specific factor price increases and the price of the transferable factor decreases during the transition to the long run equilibrium. Hence, in Favard's (2002) model, delaying extraction is optimal because a factor that will be *cheaper* in the future is intensively used in exploiting the resource. Analogous mechanisms are at work in the Amigues et al. (1998), Freni (2004), and Holland (2003) models, although in these models consumption smoothing operates as in the Kemp and Long (1980) model, and labor supply is elastic.

The purpose of this paper is to characterize the optimal time paths of extraction and production of an "energy" sector in which two non-specific primary factors of production are required to exploit the reserves and to produce the substitute. The main result is that the incentive to procrastinate extraction in response to monotonic factor price dynamics can lead to discontinuous extraction of lower cost deposits along the optimal path, even if the resource can be converted into productive capital (Solow and Wan, 1976, Lewis, 1982). Given a higher cost and a lower cost deposit, a factor intensity condition turns out to be necessary for a complete cost reversal. The condition is also sufficient, provided the stocks in the higher cost deposit and in all other deposits are sufficiently small. In this case, if the stock in the lower cost deposit is small, then the higher cost deposit is exploited before the lower cost one. On the other hand, if the stock in the lower cost deposit is sufficiently large, it will be optimal to have an initial phase during which the lower cost deposit is exploited.

For the sake of simplicity, we embed the results in a simple endogenous growth model in which, as it is typically assumed in the endogenous growth literature, the instantaneous utility function exhibits a constant inter-temporal elasticity of substitution and the discount rate is smaller than the given maximum rate of growth. Such a set of assumptions ensures an equilibrium in which a constant consumption growth rate is sustained by a constant rate of interest, while only level effects are associated with the existence of non-reproducible factors either in fixed or, as for the exhaustible resources, in decreasing supply.

In Section 2, I present the model and give the optimality conditions for the general case of n, $n \geq 2$, deposits. The structure of the optimal paths for the case of two deposits is discussed in details in Section 3. Section 4 characterizes the optimal extraction path for the case of n deposits. Section 5 presents some concluding remarks.

2. The model

Consider an "energy" sector whose output derives either from the exploitation of n, $n \geq 2$, deposits of an exhaustible resource or from the activation of a backstop technology, or from both. The backstop system is the one sector AK model with a drift, in which the drift is due to the possibility of operating a backstop production process without the use of capital. This process can be run at any scale of operation, but requires the services of two primary resources, called 'labor' and 'land', which are in fixed supply. We normalize the existing amounts of labor and land to 1.

The initial stocks in the n deposits are given by the vector $\bar{\mathbf{y}}$, $\bar{\mathbf{y}} \in R^n_{++}$. As for the backstop, the extraction technology from each deposit exhibits constant returns and requires (the services of) labor and/or land in a given proportion. Storage of the good is possible. The rate of growth of stored capital is a constant $\Gamma, \Gamma \in R$. The capital stock at time zero is denoted by $\bar{s}$, $\bar{s} \in R_+$.

Formally, we have n + 1 production processes for "energy": the backstop and the n extraction processes. At the unitary level, the backstop requires l_0, $l_0 > 0$, units of labor and d_0, $d_0 > 0$, units of land:

$$(l_0, d_0) \rightarrow 1,$$

while the extraction process i, $i \in \{1.2, \ldots, n\}$, requires l_i, $l_i \geq 0$, units of labor, d_i, $d_i \geq 0$, units of land, and depletes deposit i of one unit of the resource:

$$(l_i, d_i, 1) \rightarrow 1.$$

At time t, s(t) denotes the stock of capital, and $\mathbf{y}(t)$ denotes the resource stocks. Moreover, I use $x_0(t)$ to indicate the intensity of the backstop production process and $\mathbf{x}(t)$ to indicate the intensities of the extraction processes. To simplify the notation, whenever the context makes clear which time is referred to, I omit the time argument.

The preference side of the model is standard. There is a representative consumer with an infinite horizon, who derives utility only from consumption of "energy", c(t), c(t) ≥ 0. His utility function is time additive separable, with the instantaneous utility function u(c) taking the form:

$$u(c) = \begin{cases} \dfrac{c^{1-\sigma}}{1-\sigma} & \sigma \neq 1 \\ log(c) & \sigma = 1 \end{cases} \tag{CES}$$

Future utilities are discounted at the constant rate ρ.[1]

The Pareto-optimal allocations of our system are therefore the solutions of the following optimal control problem:

$$V\left(\bar{s},\bar{\mathbf{y}}\right) = \sup \int_0^\infty e^{-\rho t} u(c(t))dt \tag{PO1}$$

$$\text{s. to} \qquad x_0(t)l_0 + \mathbf{x}(t)\mathbf{l} \leq 1 \tag{1}$$

$$x_0(t)d_0 + \mathbf{x}(t)\mathbf{d} \leq 1 \tag{2}$$

$$\dot{s}(t) = x_0(t) + \mathbf{x}(t)\mathbf{e} + \Gamma s(t) - c(t) \tag{3}$$

$$\dot{\mathbf{y}}(t) = -\mathbf{x}(t) \tag{4}$$

$$x_0(t) \geq 0 \quad \mathbf{x}(t) \geq \mathbf{0} \tag{5}$$

$$c(t) \geq 0 \tag{6}$$

$$s(t) \geq 0 \quad \mathbf{y}(t) \geq \mathbf{0} \tag{7}$$

$$s(0)=\bar{s} \quad \mathbf{y}(0)=\bar{\mathbf{y}} \tag{8},$$

where $\mathbf{l}=[l_1,\ l_2,\ \ldots,\ l_n]$, $\mathbf{d}=[d_1,\ d_2,\ \ldots,\ d_n]$, $\mathbf{e}=[1,\ 1,\ \ldots,\ 1]$, and $u(c(t))$ takes the CES form given above.

In this section, we study problem (PO1) under the following set of assumptions:

[A1] $(l_i, d_i) \geq \mathbf{0}$, $(l_i, d_i) \neq \mathbf{0}$, $(l_i, d_i) \neq (l_j, d_j)$, $(l_0, d_0) >> (l_i, d_i)$, j, $i \in \{1,2,\ldots,n\}$,

[A2] $\Gamma > 0$,

[A3] $\rho - \Gamma(1 - \sigma) > 0$,

[A4] $\Gamma - \rho > 0$.

Assumption [A1] means that there is not free lunch and that, although deposits differ from one another, each extraction process dominates the backstop production method. The meaning of [A2] is that capital is productive. The inequality under [A3] is a condition ensuring the existence of an optimal solution of problem (PO1) with $-\infty < V(\bar{s}, \bar{\mathbf{y}}) < \infty$ (see Freni, Gozzi and Salvadori, 2006). The condition in [A4] gives the incentive to accumulate. Assumptions [A1] - [A4] imply that the optimal production path is nonincreasing. This conclusion, that is the main result of this section, is presented in Proposition 1 below.

In the two following sections, we specialize problem (PO1) by adding two further assumptions.

[A5] $d_0 < l_0$, $d_i > l_i$ $i \in \{1,2,\ldots,n\}$,

[A6] $(l_1, d_1) >> (l_2, d_2) >> \ldots >> (l_n, d_n)$.

---

[1] As in Rebelo (1991), I do not assume $\rho > 0$.

Assumption [A5] means that the total endowments of the two factor inputs belong to each diversification cone generated by the backstop process and any extraction process. This assumption grants monotonic dynamics of the factor prices and, almost ever, prevents simultaneous extraction from multiple deposits. Finally, Assumption [A6] contains the conditions that allow us to order deposits with costs in a natural way. Whenever this assumption holds, I label the deposits in reverse order of costs.

Define the current value Lagrangian function:

$$L\left(s,\mathbf{y},v,\mathbf{p},x_0,\mathbf{x},c,\mathbf{q},\mu\right)= u(c)+(x_0+\mathbf{xe}+\Gamma s-c)v-\mathbf{xp}+\mathbf{yq}+s\mu$$

where v and **p** are the costate variables corresponding to s and **y**, respectively, and **q** and μ are the multipliers for the non-negativity constraints in (7), and let w(t) and r(t) be the multipliers for the constraints (1) and (2), respectively.[2] Then the following conditions are sufficient for optimality:[3]

$$\dot{s}(t)=x_0(t)+\mathbf{x}(t)\mathbf{e}+\Gamma s(t)-c(t) \qquad (3)$$

$$\dot{\mathbf{y}}(t)=-\mathbf{x}(t) \qquad (4)$$

$$\dot{v}(t)\leq(\rho-\Gamma)v(t) \qquad (9)$$

$$s(t)\geq 0 \qquad (10)$$

[2] The usual interpretation in terms of spot competitive prices applies both to the costates, v(t) and **p**(t), and to the multipliers, w(t) and r(t). Thus, in what follows, I will often refer to v(t) as to the competitive price of energy, to **p**(t) as to the vector of the *in situ* competitive prices for the different grades of the resource, to w(t) as to the wage rate, and to r(t) as to the land rent rate.
[3] See Freni, Gozzi and Pignotti (2008).

$$\dot{v}(t)s(t) = (\rho - \Gamma)v(t)s(t) \tag{11}$$

$$\dot{\mathbf{p}}(t) \le \rho \mathbf{p}(t) \tag{12}$$

$$\mathbf{y}(t) \ge \mathbf{0} \tag{13}$$

$$\dot{\mathbf{p}}(t)\mathbf{y}(t) = \rho \mathbf{p}(t)\mathbf{y}(t) \tag{14}$$

$$\mathbf{p}(t) \ge 0 \tag{15}$$

$$c(t)^{-\sigma} = v(t) > 0 \tag{16}$$

$$\begin{gathered} \min \left[w(t) + r(t) \right] \\ v(t) \le w(t) l_0 + r(t) d_0 \\ \mathbf{e}v(t) - \mathbf{p}(t) \le w(t)\mathbf{l} + r(t)\mathbf{d} \\ w(t) \ge 0, r(t) \ge 0 \end{gathered} \tag{17}$$

$$\begin{gathered} \max \left\{ x_0(t)v(t) + \mathbf{x}(t)\left[\mathbf{e}v(t) - \mathbf{p}(t) \right] \right\} \\ x_0(t) l_0 + \mathbf{x}(t)\mathbf{l} \le 1 \\ x_0(t) d_0 + \mathbf{x}(t)\mathbf{d} \le 1 \\ x_0(t) \ge 0, \mathbf{x}(t) \ge \mathbf{0} \end{gathered} \tag{18}$$

$$\lim_{t \to \infty} e^{-\rho t}\left[s(t)v(t) + \mathbf{y}(t)\mathbf{p}(t) \right] = 0 . \tag{19}$$

Given that exhaustion of a deposit is irreversible, condition (12) can be satisfied as an equality. Moreover, since Assumption [A4] implies that for each optimal path we must have s(t) > 0 for t > 0, we can use Theorem 5.3 in Freni, Gozzi and Pignotti (2008) to claim that the above conditions are also necessary for optimality. This allow us to state condition (9) as an equality without missing any of the optimal solutions. We can therefore immediately derive from condition (16) that the rate of growth of consumption is the constant $g^o = \dfrac{\Gamma - \rho}{\sigma}$ along the whole optimal path.

From the above set of conditions and Assumption [A1], we must have that the scarcity rent vector **p**(t) is positive. Otherwise the demand for a grade of the resource would become infinite. Henceforth, given that conditions (9), (12) and (14) imply that the left side of inequality $\mathbf{e}v(t)-\mathbf{p}(t)\leq w(t)\mathbf{l}+r(t)\mathbf{d}$ becomes negative in finite time, the date of exhaustion of the resources is finite. We denote this date by $T_{\mathrm{y}}, T_{\mathrm{y}}>0$.

Once all deposits are exhausted, the system behavior is given by the solution of the following problem:

$$\sup \int_0^\infty e^{-\rho t} u(c(t))dt \qquad \text{(POB)}$$

$$\dot{s}(t)=\frac{1}{l_0}+\Gamma s(t)-c(t)$$

$$c(t)\geq 0,\ s(t)\geq 0,\ s(0)=\hat{s}\geq 0.$$

Given that the utility function takes the CES form, a straightforward verification procedure (see for example Jones and Manuelli, 1990, or Rebelo, 1991) provides the solutions of (POB):

$$c^B(t)=(\Gamma-g^o)\left(\hat{s}+\frac{1}{\Gamma l_0}\right)e^{gt} \qquad (20)$$

$$s^B(t)=-\frac{1}{\Gamma l_0}+\frac{c^B(t)}{\Gamma-g^o},\ s^B(t)\geq 0. \qquad (21)$$

Figure 1 illustrates the optimal trajectory (21) in the (s(t), c(t)) space. We note that along the optimal path condition (21) must hold from the time T_{y} on, because, once we have $\mathbf{y}(t)=0$, the transversality condition (19) forces the optimal solution of (PO1) on the half-line AB.

Let us define $\hat{\mathbf{p}}(t)=\frac{1}{v(t)}\mathbf{p}(t)$, $\hat{w}(t)=\frac{w(t)}{v(t)}$ and $\hat{r}(t)=\frac{r(t)}{v(t)}$. Substituting these in the above optimality conditions, we first rearrange the linear problems (17) and (18) as follows:

$$\begin{gathered}\min\left[\hat{w}(t)+\hat{r}(t)\right]\\ 1\le\hat{w}(t)l_0+\hat{r}(t)d_0\\ \mathbf{e}-\hat{\mathbf{p}}(t)\le\hat{w}(t)\mathbf{l}+\hat{r}(t)\mathbf{d}\\ \hat{w}(t)\ge 0,\hat{r}(t)\ge 0\end{gathered}\tag{22}$$

$$\begin{gathered}\max\left\{x_0(t)+\mathbf{x}(t)\left[\mathbf{e}-\hat{\mathbf{p}}(t)\right]\right\}\\ x_0(t)l_0+\mathbf{x}(t)\mathbf{l}\le 1\\ x_0(t)d_0+\mathbf{x}(t)\mathbf{d}\le 1\\ x_0(t)\ge 0,\mathbf{x}(t)\ge\mathbf{0}\end{gathered},\tag{23}$$

and then, taking conditions (9) and (12) as equalities, we get:

$$\dot{\hat{\mathbf{p}}}(t)=\Gamma\hat{\mathbf{p}}(t),\tag{24}$$

which can be interpreted as the Hotelling rule,[4]

Using equation (24) in conjunction with the dual linear problems (22) and (23), we can now derive the following proposition:

Proposition 1: Let Assumptions [A1]-[A4] hold. Then the optimal production paths, $\bar{x}_0(t)+\bar{\mathbf{x}}(t)\mathbf{e}$, is a decreasing step function with $\bar{\mathbf{x}}(t)\neq\mathbf{0}$ and $\bar{x}_0(t)+\bar{\mathbf{x}}(t)\mathbf{e}$ > $\max\left(\frac{1}{l_0},\ \frac{1}{d_0}\right)$ for $t<T_{\mathrm{y}}$, and $\bar{\mathbf{x}}(t)=\mathbf{0}$ and $\bar{x}_0(t)=\max\left(\frac{1}{l_0},\ \frac{1}{d_0}\right)$ for $t>T_{\mathrm{y}}$.

Proof. Note that equation (24) implies constancy in the scarcity rents ratios. Thus, fixing $\hat{\mathbf{p}}(0)$, the family of vectors $\left[\frac{\hat{p}_1(0)}{\hat{p}_n(0)},\ \frac{\hat{p}_2(0)}{\hat{p}_n(0)},\ \ldots,\ \frac{\hat{p}_{n-1}(0)}{\hat{p}_n(0)},\ 1\right]\hat{p}_n(t)$ can be used in (22) to generate a family of linear programs. Linear parametric programming theory then implies that the minimum value, $m(\hat{p}_n(t))=\min\left(\hat{w}(t)+\hat{r}(t)\right)$, is a continuous, convex and piecewise linear function of $\hat{p}_n(t)$. Furthermore, given that the

[4] Since $\dot{v}(t)=(\rho-\Gamma)v(t)$, the maximum rate of uniform growth and the rate of interest in the dual price system are equal as in von Neumann (1945) (see also Rebelo, 1991).

admissible region of problem (22) is non-decreasing with $\hat{p}_n(t)$, the function is non-increasing and becomes the constant $\max\left(\frac{1}{l_0}, \ \frac{1}{d_0}\right)$ for $\hat{p}_n(t) \geq \hat{p}_n(T_y)$. An example of the minimum function is depicted in Figure 2. Then, from the Duality Theorem of Linear Programming we get:

$$m(\hat{p}_n(t)) = \bar{x}_0(t) + \bar{\mathbf{x}}(t)\mathbf{e} - \bar{\mathbf{x}}(t)\left[\frac{\hat{p}_1(0)}{\hat{p}_n(0)}, \ \frac{\hat{p}_2(0)}{\hat{p}_n(0)}, \ \ldots, \ \frac{\hat{p}_{n-1}(0)}{\hat{p}_n(0)}, \ 1\right]^T \hat{p}_n(t),$$

where $\bar{x}_0(t) + \bar{\mathbf{x}}(t)\mathbf{e}$ is the output associated with an optimal basic solution.

We therefore conclude that the optimal production paths $\bar{x}_0(t) + \bar{\mathbf{x}}(t)\mathbf{e}$, is a decreasing step function with the stated properties. ■

An implication of Proposition 1 is that the optimal stock trajectory solves the following piecewise linear differential equation:

$$\dot{s}(t) = \bar{x}_0(t) + \bar{\mathbf{x}}(t)\mathbf{e} + \Gamma s(t) - c(0)e^{g^o t}, \qquad (25)$$

in which consumption at time zero is jointly determined by the initial condition $s(0) = \bar{s}$, and by the "final" condition

$$s(T_y) = -\frac{1}{\Gamma l_0} + \frac{c(T_y)}{\Gamma - g^o}, \ s(T_y) \geq 0 \ . \qquad (26)$$

What is left out is the analysis of the production path and, hence, the determination of date T_y given in equations (25) and (26). For the case where Assumptions [A5] and [A6] hold, this is the task we accomplish in the following two sections. To fix ideas, in the next section we take n = 2 and give a complete characterization of both the extraction and the substitute production optimal paths.

3. Order of extraction with two deposits

Let us now consider the case where n = 2 and Assumptions [A5] and [A6] hold. The factor intensity conditions in Assumption [A5] have two main implications. First, given that each extraction process is relatively more land-intensive and the backstop is relatively more labor-intensive than is the overall system, then we have $\begin{vmatrix} l_0 & d_0 \\ l_i & d_i \end{vmatrix} > 0$

and $\begin{bmatrix} 1, & 1 \end{bmatrix} \begin{bmatrix} l_0 & d_0 \\ l_i & d_i \end{bmatrix}^{-1} >> \begin{bmatrix} 0 & 0 \end{bmatrix}, i \in \{1,2\}$. On the other hand, either

$$\begin{vmatrix} l_1 & d_1 \\ l_2 & d_2 \end{vmatrix} = 0,$$

and $\begin{bmatrix} x_1, & x_2 \end{bmatrix} \begin{bmatrix} l_1 & d_1 \\ l_2 & d_2 \end{bmatrix} = \begin{bmatrix} 1, & 1 \end{bmatrix}$ does not have a solution by Assumption [A1],

or

$$\begin{bmatrix} 1, & 1 \end{bmatrix} \begin{bmatrix} l_1 & d_1 \\ l_2 & d_2 \end{bmatrix}^{-1} \ngeq \begin{bmatrix} 0, & 0 \end{bmatrix}.$$

Therefore six basic feasible solutions of the linear problem (23) exist and we can compute them to be:

$$\begin{bmatrix} x_0^1, \mathbf{x}^1 \end{bmatrix} = \begin{bmatrix} 0, & \left(0, \quad \frac{1}{d_2}\right) \end{bmatrix}, \begin{bmatrix} x_0^2, \mathbf{x}^2 \end{bmatrix} = \begin{bmatrix} 0, & \left(\frac{1}{d_1}, \quad 0\right) \end{bmatrix},$$

$$\left[x_0^3, \mathbf{x}^3\right] = \left[\frac{d_1 - l_1}{l_0 d_1 - l_1 d_0}, \quad \left(\frac{l_0 - d_0}{l_0 d_1 - l_1 d_0}, \quad 0\right)\right], \left[x_0^4, \mathbf{x}^4\right] = \left[\frac{d_2 - l_2}{l_0 d_2 - l_2 d_0}, \quad \left(0, \quad \frac{l_0 - d_0}{l_0 d_2 - l_2 d_0}\right)\right]$$

$$\left[x_0^5, \mathbf{x}^5\right] = \left[\frac{1}{l_0}, \quad (0, \quad 0)\right], \left[x_0^6, \mathbf{x}^6\right] = \left[0, \quad (0, \quad 0)\right].$$

We note that the first five of these solutions can be optimal and that the fifth is indeed the long-run optimal solution.

Now we can use Proposition 1 to get

$$m(\hat{p}_2(t)) = \max\left\{\frac{1}{l_0}, \quad G_1(\hat{p}_2(t)), \quad G_2(\hat{p}_2(t))\right\}, \tag{27}$$

where

$$G_1(\hat{p}_2(t)) = \max\left\{\mathbf{x}^2\mathbf{e}(1 - \frac{\hat{p}_1(0)}{\hat{p}_2(0)}\hat{p}_2(t)), \quad \mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^3\mathbf{e}\frac{\hat{p}_1(0)}{\hat{p}_2(0)}\hat{p}_2(t)\right\} \tag{28}$$

and

$$G_2(\hat{p}_2(t)) = \max\left\{\mathbf{x}^1\mathbf{e}(1 - \hat{p}_2(t)), \quad \mathbf{x}^4\mathbf{e} + x_0^4 - \mathbf{x}^4\mathbf{e}\hat{p}_2(t)\right\}. \tag{29}$$

Both function $G_1(\hat{p}_2(t))$ and $G_2(\hat{p}_2(t))$ are continuous, piecewise linear, convex, and decreasing, and both the graphs have a kink, the first at $(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}(1 - \frac{d_1}{d_0}), \frac{1}{d_0})$ and the second one at $((1 - \frac{d_2}{d_0}), \frac{1}{d_0})$. The graph of $m(\hat{p}_2(t))$ for a given value of $\frac{\hat{p}_2(0)}{\hat{p}_1(0)}$ is depicted in Figure 3.

Thus, we can conclude that deposits extraction will always end with a phase during which the substitute is produced and that an initial phase during which the backstop is inactive will exist only if the stocks are sufficiently large.

A second implication of Assumption [A5] is that, along the optimal path, the price of the factor that is used intensively in the extraction processes cannot increase and the price of factor that is used intensively in the production of the substitute cannot decrease. To prove this, consider the family of linear problems (22). We note that when the backstop is inactive the solution is $\hat{w}=0, \hat{r}=\max\left(G_1, \quad G_2\right)$, while in the long run we have $\hat{w}=\frac{1}{l_0}, \hat{r}=0$. Furthermore, when the substitute is produced in the phase preceding exhaustion of the resource, the equilibrium factor prices solve the system of equations

$$\begin{bmatrix}\hat{w}\\ \hat{r}\end{bmatrix}\begin{bmatrix}l_0 & d_0\\ l_{i*} & d_{i*}\end{bmatrix}=\begin{bmatrix}1\\ 1-p_{i*}\end{bmatrix},$$

where i* is the cost-minimizing extraction process. The desired result follows from the fact that the entries of $\begin{bmatrix}l_0 & d_0\\ l_{i*} & d_{i*}\end{bmatrix}^{-1}$ are positive at the diagonal and negative off the diagonal.[5]

We are now ready to determine the structure of the optimal extraction path. In constructing Figure 3, we showed that the extraction path ends with a phase during which $\bar{x}_0(t)>0$. The maximum length of this phase, L, is the solution of the following equation:

$$\hat{p}_2(T_y)=\max\left\{1-\frac{d_2}{d_0},(1-\frac{d_1}{d_0})\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right\}e^{\Gamma t},$$

where $\hat{p}_2(T_y)=\max\left\{1-\frac{l_2}{l_0},(1-\frac{l_1}{l_0})\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right\}$. Thus,

$$L=\frac{1}{\Gamma}\log\frac{\max\left\{1-\frac{l_2}{l_0},(1-\frac{l_1}{l_0})\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right\}}{\max\left\{1-\frac{d_2}{d_0},(1-\frac{d_1}{d_0})\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right\}}. \qquad (30)$$

[5] Note that this result is a version of the Samuelson-Stolper Theorem.

Since $\frac{\hat{p}_2(0)}{\hat{p}_1(0)} < \min\left\{\frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}, \frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}\right\}$ implies $m(\hat{p}_2(t)) > G_1(\hat{p}_2(t))$ for each $\hat{p}_2(t)$, the resource price ratio that supports an optimal path must satisfy the inequality $\frac{\hat{p}_2(0)}{\hat{p}_1(0)} \geq \min\left\{\frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}, \frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}\right\}$. If $\frac{\hat{p}_2(0)}{\hat{p}_1(0)} \geq \max\left\{\frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}, \frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}\right\} > \min\left\{\frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}, \frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}\right\}$, then, as shown in Figure 3, only the higher cost deposit is exploited during the phase in which the substitute is produced. On the other hand, if $\min\left\{\frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}, \frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}\right\} < \frac{\hat{p}_2(0)}{\hat{p}_1(0)} < \max\left\{\frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}, \frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}\right\}$, then a part of the time L is spent in exploiting the higher cost deposit and the rest in exploiting the lower cost deposit. In this case, the precise sequence of extraction will depend on the value of $\max\left\{\frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}, \frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}\right\}$. If $\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} > \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$, then the lower cost deposit is extracted before the higher cost deposit. On the contrary, if $\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} < \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$, then a cost reversal occurs and the higher cost deposit is used first. Finally, if $\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} = \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}} = \frac{\hat{p}_2(0)}{\hat{p}_1(0)}$, then there is a continuum of optimal extraction paths and the sequence of extraction is therefore indeterminate. Figures 4(a), 4(b) and 4(c) depict the graphs of $m(\hat{p}_2(t))$ for the three case discussed above. Figure 4(a) portrays a "normal" case where $\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} > \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$ and Figure 4(c) depicts the pathological situation where

$\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} = \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}} = \frac{\hat{p}_2(0)}{\hat{p}_1(0)}$. On the other hand, Figure 4(b) illustrates that, when a cost reversal occurs in the phase just preceding the transition to the backstop, then it can be optimal to exploit the lower cost deposit over two disjoint intervals and, from the above analysis, we expect that for any given initial stock $\bar{y}_1$ below a critical value, there is a threshold level on $\bar{y}_2$ which will determine whether or not discontinuous extraction will occur.

In order to pursue all these cases more deeply, we now study the minimum value of the family of linear programs (22) as a function of the two *in situ* prices, $(\hat{p}_1, \hat{p}_2)$. The graph of this function consists of flat faces, each of which is associated with a specific production of the substitute/extraction profile, so changes in the production of the substitute/extraction strategy occur when a $(\hat{p}_1(t), \hat{p}_2(t))$ ray from the origin crosses the projection of the edges of the graph in the $(\hat{p}_1, \hat{p}_2)$ plane (i. e., where the minimum function is not differentiable). Substituting $\hat{p}_1(t) = \frac{\hat{p}_1(0)}{\hat{p}_2(0)}\hat{p}_2(t)$ in (27) and (28) and rearranging the terms in (27), we first calculate:

$$m(\hat{p}_1(t), \hat{p}_2(t)) = \max\left\{\mathbf{x}^2\mathbf{e}(1-\hat{p}_1(t)), \quad \mathbf{x}^1\mathbf{e}(1-\hat{p}_2(t))\right\}$$

$$\text{for } \min\left\{\frac{1-\hat{p}_1(t)}{d_1}, \quad \frac{1-\hat{p}_2(t)}{d_2}\right\} \le \frac{1}{d_0}, \tag{31}$$

$$m(\hat{p}_1(t),\hat{p}_2(t)) = \max\left\{\mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^3\mathbf{e}\hat{p}_1(t), \quad \mathbf{x}^4\mathbf{e} + x_0^4 - \mathbf{x}^4\mathbf{e}\hat{p}_2(t)\right\} \quad \text{for}$$

$$\min\left\{\frac{1-\hat{p}_1(t)}{d_1}, \quad \frac{1-\hat{p}_2(t)}{d_2}\right\} \geq \frac{1}{d_0}, \min\left\{\frac{1-\hat{p}_1(t)}{l_1}, \quad \frac{1-\hat{p}_2(t)}{l_2}\right\} \leq \frac{1}{l_0}, \text{(32)}$$

and

$$m(\hat{p}_1(t),\hat{p}_2(t)) = \frac{1}{l_0} \quad \text{for } \min\left\{\frac{1-\hat{p}_1(t)}{l_1}, \quad \frac{1-\hat{p}_2(t)}{l_2}\right\} \geq \frac{1}{l_0}, \quad (33)$$

and then, using expression (31), (32) and (33), we identify the five regions in the $(\hat{p}_1,\hat{p}_2)$ plane where the basic solutions $\left[x_0^1,\mathbf{x}^1\right], \left[x_0^2,\mathbf{x}^2\right], \left[x_0^3,\mathbf{x}^3\right], \left[x_0^4,\mathbf{x}^4\right]$, and $\left[x_0^5,\mathbf{x}^5\right]$ are optimal.

In Figure 5(a), we depict these regions for the case $\dfrac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} > \dfrac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$. Extraction occurs from the lower (higher) cost deposit when $(\hat{p}_1(t),\hat{p}_2(t))$ belongs to the union of sets 1 and 4 (2 and 3). The long run is reached when a ray from the origin crosses set 5. Making time runs backwards, we note that the interior of the union of set 1 and 4 absorbs the trajectories generated by equation (24). We therefore conclude that extraction occurs in order of costs. We also note that $\dfrac{l_2}{d_2} \geq \dfrac{l_1}{d_1} \Rightarrow \dfrac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} > \dfrac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$.

Using Figure 5(a) we can construct the optimal policy of extraction as follows. Assume $\dfrac{\hat{p}_2(0)}{\hat{p}_1(0)} \geq \dfrac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$, let $\hat{p}_2(t) = \dfrac{\hat{p}_2(0)}{\hat{p}_1(0)}\hat{p}_1(t)$ be a ray from the origin generated by solving equation (24) on the time interval $(-\infty, \quad \infty)$, and let $L_1^3\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$ be the interval of time that the trajectory spends in set 3. Once $L_1^3\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$ is known, the time spent in set 4, $L_2^4\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$, can be found using equation (30) as follows:

$$L_2^4\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right) = \frac{1}{\Gamma}\log\frac{(1-\frac{l_1}{l_0})\frac{\hat{p}_2(0)}{\hat{p}_1(0)}}{\max\left\{1-\frac{d_2}{d_0}, (1-\frac{d_1}{d_0})\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right\}} - L_1^3\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right), \qquad (34)$$

If $\frac{\hat{p}_2(0)}{\hat{p}_1(0)} \geq \frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}$, then $L_1^3\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right) = \frac{1}{\Gamma}\log\frac{1-\frac{l_1}{l_0}}{1-\frac{d_1}{d_0}}$ and $L_2^4\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right) = 0$. On the other side, if $\frac{\hat{p}_2(0)}{\hat{p}_1(0)} = \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$, then $L_1^3\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right) = 0$ and $L_2^4\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right) = \frac{1}{\Gamma}\log\frac{1-\frac{l_2}{l_0}}{1-\frac{d_2}{d_0}}$. For intermediate values of the resources price ratio, using (32) to find the coordinates of the point where a trajectory enters region 3 and using equation (24) to evaluate $L_1^3\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$, we get

$$e^{\Gamma L_1^3\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)} = \frac{(1-\frac{l_1}{l_0})(\mathbf{x}^3\mathbf{e} - \mathbf{x}^4\mathbf{e}\frac{\hat{p}_2(0)}{\hat{p}_1(0)})}{\mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^4\mathbf{e} - x_0^4}, \qquad (35).$$

Then, substituting from (35) for $\frac{\hat{p}_2(0)}{\hat{p}_1(0)}$ in (34), we obtain

$$L_2^4 = \frac{1}{\Gamma}\log\frac{(1-\frac{l_1}{l_0})\frac{\mathbf{x}^3\mathbf{e}}{\mathbf{x}^4\mathbf{e}} - \frac{\mathbf{x}^3\mathbf{e}+x_0^3-\mathbf{x}^4\mathbf{e}-x_0^4}{\mathbf{x}^4\mathbf{e}}e^{\Gamma L_1^3}}{1-\frac{d_2}{d_0}} - L_1^3 \qquad (36).$$

Finally, since the amounts extracted in regions 3 and 4, y_1^* and y_2^*, are given by $\mathbf{x}^3\mathbf{e}L_1^3$ and $\mathbf{x}^4\mathbf{e}L_2^4$, respectively, substituting these values in (36) we get

$$y_2^* = \mathbf{x}^4\mathbf{e}\frac{1}{\Gamma}\log\frac{(1-\frac{l_1}{l_0})\frac{\mathbf{x}^3\mathbf{e}}{\mathbf{x}^4\mathbf{e}} - \frac{\mathbf{x}^3\mathbf{e}+x_0^3-\mathbf{x}^4\mathbf{e}-x_0^4}{\mathbf{x}^4\mathbf{e}}e^{\Gamma\frac{y_1^*}{\mathbf{x}^3\mathbf{e}}}}{1-\frac{d_2}{d_0}} - \frac{\mathbf{x}^4\mathbf{e}}{\mathbf{x}^3\mathbf{e}}y_1^* \qquad (37)$$

Figure 5(b) portraits the projection of the optimal paths in the $(y_1(t), y_2(t))$-space. In the figure, the graph of function (37) is the decreasing dashed curve that identifies the boundary of the region where $\bar{x}_0(t) > 0$

In a similar way, we can use Figure 6(a) to fully characterize the optimal extraction paths when $\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} < \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$ and Figure 7(a) for the singular case $\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} = \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$. In the first case, $\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}} < \frac{\hat{p}_2(0)}{\hat{p}_1(0)} < \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}$ implies that a ray will re-enter the union of set 1 and 4 after leaving it. Therefore, instead of the single function in (37), we need two different functions to define the threshold levels where the lower cost deposit is temporary abandoned and where $\bar{x}_0(t) > 0$, respectively. We graph the two curves in Figure 6(b), where we also portrait the projection of the optimal paths in the $(y_1(t), y_2(t))$-space. The algebra is relegated in the Appendix.

In the second case, the extraction policy is determined, and no cost reversal occurs, only if the stock in the higher cost deposit is sufficiently large. Otherwise, the optimal path is indeterminate. Figure 7(b) depicts the region of indeterminacy. In the Appendix we provide a formal derivation of the results.

The above findings are summarized in Proposition 2.

Proposition 2: Let Assumptions [A1]-[A6] hold and let n = 2. Then:

(i) for each ray $(\lambda y_1(t), \lambda y_2(t)) \in R_+^2$, $\lambda \geq 0$ there is a number M > 0 such that $\lambda < M \Rightarrow \bar{x}_0(t) > 0$ and $\lambda > M \Rightarrow \bar{x}_0(t) = 0$.

(ii) the real rental rate of factor used intensively in the extraction processes is not increasing and the real rental rate of factor that is used intensively in the production of the substitute is not decreasing along any dual optimal path,

(iii) if $\dfrac{1-\dfrac{d_2}{d_0}}{1-\dfrac{d_1}{d_0}} > \dfrac{1-\dfrac{l_2}{l_0}}{1-\dfrac{l_1}{l_0}}$, then the optimal order of extraction is the order of costs. If $\dfrac{1-\dfrac{d_2}{d_0}}{1-\dfrac{d_1}{d_0}} \leq \dfrac{1-\dfrac{l_2}{l_0}}{1-\dfrac{l_1}{l_0}}$, then there is a number N > 0 such that: (a) deposits are optimally extracted in order of costs if $\bar{y}_1 \geq N$, (b) a cost reversal occurs along the optimal path if $\bar{y}_1 < N$ and $\dfrac{1-\dfrac{d_2}{d_0}}{1-\dfrac{d_1}{d_0}} < \dfrac{1-\dfrac{l_2}{l_0}}{1-\dfrac{l_1}{l_0}}$. In this case, for each $\bar{y}_1 < N$ there is a threshold value on $\bar{y}_2$, $P(\bar{y}_1)$, such that the lower cost resource is optimally extracted on two disjoint intervals if and only if $\bar{y}_2 > P(\bar{y}_1)$, (c) If $\bar{y}_1 < N$ and $\dfrac{1-\dfrac{d_2}{d_0}}{1-\dfrac{d_1}{d_0}} = \dfrac{1-\dfrac{l_2}{l_0}}{1-\dfrac{l_1}{l_0}}$, then a portion of the optimal extraction path is indeterminate.

4. Order of extraction with any number of deposits

Consider the general case of an arbitrary number of deposits n and let Assumptions [A5] and [A6] hold. By extending the argument used in section 3, we see that $\begin{bmatrix}1, & 1\end{bmatrix}\begin{bmatrix}l_0 & d_0 \\ l_i & d_i\end{bmatrix}^{-1} >> \begin{bmatrix}0 & 0\end{bmatrix}, i \in \{1,2,\cdots,n\}$ and no system $\begin{bmatrix}x_i, & x_j\end{bmatrix}\begin{bmatrix}l_i & d_i \\ l_j & d_j\end{bmatrix} = \begin{bmatrix}1, & 1\end{bmatrix}$ has a non-negative solution for $i, j \in \{1,2,\cdots,n\}, i \neq j$. Therefore, only 2n – 1 semi-positive basic feasible solutions of the linear problem (23) exist and, hence, by using Proposition 1, we can get

$$m(\hat{p}_n(t)) = \max\left\{\frac{1}{l_0}, G_1(\hat{p}_n(t)), G_2(\hat{p}_n(t)), \ldots, G_n(\hat{p}_n(t))\right\}, \tag{38}$$

where

$$G_i(\hat{p}_n(t)) = \max\left\{\frac{1-\dfrac{\hat{p}_i(0)}{\hat{p}_n(0)}\hat{p}_n(t)}{d_i}, \quad \frac{d_i - l_i + l_0 - d_0}{l_0 d_i - l_i d_0} - \frac{l_0 - d_0}{l_0 d_i - l_i d_0}\frac{\hat{p}_i(0)}{\hat{p}_n(0)}\hat{p}_n(t)\right\}$$

$$i \in \{1,2,\cdots,n\}. \tag{39}$$

Since the graph of each $G_i(\hat{p}_n(t))$ is kinked at $(\frac{\hat{p}_n(0)}{\hat{p}_i(0)}(1-\frac{d_i}{d_0}), \frac{1}{d_0})$, as for the two-deposits case, extraction will always ends with a phase during which $\bar{x}_0(t) > 0$, and an extraction phase during which the backstop is inactive will exists only if the stocks are sufficiently large. Thus, along the optimal dual path the "wage rate" is still not decreasing, while the "land rent rate" is still not increasing and, furthermore, the maximum number of disjoint intervals during which a single deposit can be used is two. We have, therefore, constrained the optimal extraction path, and established that point (ii) and the general analog of point (i) of Proposition 2 hold with an arbitrary number of deposits.

We can further characterize the optimal extraction path as in Proposition 3.

Proposition 3: Let Assumptions [A1]-[A6] hold. Then:

(i) if $\frac{1-\frac{d_j}{d_0}}{1-\frac{d_i}{d_0}} > \frac{1-\frac{l_j}{l_0}}{1-\frac{l_i}{l_0}}$ $\forall\, i, j \in \{1,2,\cdots,n\}, i < j$, then the optimal order of extraction is the order of costs,

(ii) if $\exists\, i, j \in \{1,2,\cdots,n\}, i < j$ such that $\frac{1-\frac{d_j}{d_0}}{1-\frac{d_i}{d_0}} < \frac{1-\frac{l_j}{l_0}}{1-\frac{l_i}{l_0}}$, then there exist a non zero measure subset of R^n_{++}, U, such that $\bar{\mathbf{y}} \in U$ implies that deposit j is extracted on two disjoint intervals.

Proof. First, note that if $\frac{1-\frac{d_j}{d_0}}{1-\frac{d_i}{d_0}} > \frac{1-\frac{l_j}{l_0}}{1-\frac{l_i}{l_0}}$, $i, j \in \{1,2,\cdots,n\}, i < j$, then $G_i(\hat{p}_n^*) = G_j(\hat{p}_n^*) \Rightarrow G_i(\hat{p}_n) > G_j(\hat{p}_n)$ for each $\hat{p}_n > \hat{p}_n^*$. This proves point (i). Assume now that $\exists\, i, j \in \{1,2,\cdots,n\}, i < j$ such that $\frac{1-\frac{d_j}{d_0}}{1-\frac{d_i}{d_0}} < \frac{1-\frac{l_j}{l_0}}{1-\frac{l_i}{l_0}}$. By choosing sufficiently high resource price ratios $\frac{\hat{p}_z(0)}{\hat{p}_i(0)}$ and $\frac{\hat{p}_z(0)}{\hat{p}_j(0)}, z \neq i, j$, we know from point (iii) of

Proposition 2 that we can choose $\frac{\hat{p}_j(0)}{\hat{p}_i(0)}$ in such a way that deposit j is cost-minimizing on two disjoint intervals of the range of $\hat{p}_n$. Now, we can progressively diminish prices $\hat{p}_z(0)$ until each deposit will appear on the $m(\hat{p}_n(t))$ frontier on (at least) an interval of the range of $\hat{p}_n$, leaving at the same time deposit j on the frontier on two disjoint intervals. This proves point (ii). ■

5. Concluding remarks

We have examined the optimal order of extraction of several nonrenewable resource deposits with different costs of extraction when the extracted resource can be converted into productive capital and the extraction process, as well as the production of the substitute, requires two primary factors of production. As we have shown, even if the time paths of primary factor prices are monotonic, when high cost resources are not abundant, then complete cost reversals can occur depending on whether or not an intensity condition is satisfied for each pair of deposits. In turn, these cost reversals will determine discontinuous extraction from low cost reserves if the initial endowment of these low cost deposits is sufficiently large.

Our analysis extends to a single demand setting in which resources are differentiated by cost and the extracted resource can be converted into productive capital a phenomenon that is known can arise with multiple demands (Gaudet, Moreaux and Salant, 2001, Im, Chackavorty and Roumasset, 2006), with resources that are differentiated by their polluting characteristics (Chackavorty, Moreaux and Tidball, 2008), and with capacity constraints on the extraction rate of a non storable resource (Freni, 2004, Holland, 2003).

Two important assumptions in the model are that there is in incentive to accumulate (Assumption [A4]), and that the factor intensity of the overall system is intermediate between the factor intensity of the backstop and that of each extraction process (Assumption [A5]), implying that transitional dynamics of factor price is monotonic. It may be of some interest to know what kind of new phenomena can arise without these assumptions.

Appendix

Consider in Figure 6(a) a ray from the origin whose slope $\frac{\hat{p}_2(0)}{\hat{p}_1(0)}$ lies in the interval $\left(\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}, \frac{1-\frac{l_2}{l_0}}{1-\frac{l_1}{l_0}}\right)$. Let $\left(p_1^*\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right), \frac{\hat{p}_2(0)}{\hat{p}_1(0)} p_1^*\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)\right)$ be the coordinates of the point where the ray intersects the set $(3 \cap 4)$ and let $\left(p_1^{**}\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right), \frac{\hat{p}_2(0)}{\hat{p}_1(0)} p_1^{**}\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)\right)$ be the coordinates of the point where the ray intersect the set $(1 \cap 2)$. Using first equation (24) to calculate the time the trajectory stays in the different regions, and then (31) and (32) to evaluate $p_1^*\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$ and $p_1^{**}\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$ we get

$$e^{\Gamma L_1^3\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)} = \frac{p_1^*\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)}{1-\frac{d_1}{d_0}} \tag{A1}$$

$$e^{\Gamma L_2^4\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)} = \frac{1-\frac{l_2}{l_0}}{\frac{\hat{p}_2(0)}{\hat{p}_1(0)} p_1^*\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)} \tag{A2}$$

$$e^{\Gamma L_1^2\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)} = \frac{1-\frac{d_1}{d_0}}{p_1^{**}\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)} \tag{A3}$$

$$p_1^*\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right) = \frac{\mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^4\mathbf{e} - x_0^4}{\mathbf{x}^3\mathbf{e} - \mathbf{x}^4\mathbf{e}\frac{\hat{p}_2(0)}{\hat{p}_1(0)}} \tag{A4}$$

and

$$p_1^{**}\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right) = \frac{\mathbf{x}^2\mathbf{e} - \mathbf{x}^1\mathbf{e}}{\mathbf{x}^2\mathbf{e} - \frac{\hat{p}_2(0)}{\hat{p}_1(0)}\mathbf{x}^1\mathbf{e}}, \tag{A5}$$

where $L_1^2\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$ is the time spent in set 2. Then, substituting from (A4) for $p_1^*\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$ in (A1) and (A2) and from (A4) for $p_1^{**}\left(\frac{\hat{p}_2(0)}{\hat{p}_1(0)}\right)$ in (A3), we have

$$L_1^3 = \frac{1}{\Gamma}\log\frac{\mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^4\mathbf{e} - x_0^4}{(1-\frac{d_1}{d_0})(\mathbf{x}^3\mathbf{e} - \mathbf{x}^4\mathbf{e}\frac{\hat{p}_2(0)}{\hat{p}_1(0)})} \tag{A6}$$

$$e^{\Gamma L_2^4} = \frac{(1-\frac{l_2}{l_0})(\mathbf{x}^3\mathbf{e} - \mathbf{x}^4\mathbf{e}\frac{\hat{p}_2(0)}{\hat{p}_1(0)})}{(\mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^4\mathbf{e} - x_0^4)\frac{\hat{p}_2(0)}{\hat{p}_1(0)}} \tag{A7}$$

$$L_1^2 = \frac{1}{\Gamma}\log\frac{(1-\frac{d_1}{d_0})(\mathbf{x}^2\mathbf{e} - \frac{\hat{p}_2(0)}{\hat{p}_1(0)}\mathbf{x}^1\mathbf{e})}{\mathbf{x}^2\mathbf{e} - \mathbf{x}^1\mathbf{e}} \tag{A8}.$$

Finally, since the amount of the resource extracted in set 2, y_1^{**}, is given by $\mathbf{x}^2\mathbf{e}L_1^2$, substituting from (A7) for $\frac{\hat{p}_2(0)}{\hat{p}_1(0)}$ into (A6) and (A8) and remembering that $y_1^* = \mathbf{x}^3\mathbf{e}L_1^3$ and $y_2^* = \mathbf{x}^4\mathbf{e}L_2^4$, we get

$$y_1^* = \frac{\mathbf{x}^3\mathbf{e}}{\Gamma}\log\frac{\mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^4\mathbf{e} - x_0^4}{(1-\frac{d_1}{d_0})(\mathbf{x}^3\mathbf{e} - \mathbf{x}^4\mathbf{e}\frac{(1-\frac{l_2}{l_0})\mathbf{x}^3\mathbf{e}}{(\mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^4\mathbf{e} - x_0^4)e^{\frac{\Gamma y_2^*}{\mathbf{x}^4\mathbf{e}}} + (1-\frac{l_2}{l_0})\mathbf{x}^4\mathbf{e}})} \tag{A9}$$

$$y_1^{**} = \frac{\mathbf{x}^2\mathbf{e}}{\Gamma}\log\frac{(1-\frac{d_1}{d_0})(\mathbf{x}^2\mathbf{e} - \frac{(1-\frac{l_2}{l_0})\mathbf{x}^3\mathbf{e}}{(\mathbf{x}^3\mathbf{e} + x_0^3 - \mathbf{x}^4\mathbf{e} - x_0^4)e^{\frac{\Gamma y_2^*}{\mathbf{x}^4\mathbf{e}}} + (1-\frac{l_2}{l_0})\mathbf{x}^4\mathbf{e}}\mathbf{x}^1\mathbf{e})}{\mathbf{x}^2\mathbf{e} - \mathbf{x}^1\mathbf{e}} \tag{A10}.$$

The graph of function (A9) and that of the sum of the functions (A9) and (A10) give the two threshold curves in Figure 6(b).

Consider now the case depicted in Figure 7(a). Note that $\mathbf{x}^3\mathbf{e}\frac{1}{\mathbf{\Gamma}}\left[\log(1-\frac{l_1}{l_0})-\log(1-\frac{d_1}{d_0})\right]$ is the maximum amount of the resource that can be extracted from the higher cost deposit when $\frac{\hat{p}_2(0)}{\hat{p}_1(0)}=\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}$ and that $\frac{\hat{p}_2(0)}{\hat{p}_1(0)}>\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}$ implies that the minimum amount extracted from the higher cost deposit exceeds $\mathbf{x}^3\mathbf{e}\frac{1}{\mathbf{\Gamma}}\left[\log(1-\frac{l_1}{l_0})-\log(1-\frac{d_1}{d_0})\right]$. We have, therefore

$$\bar{y}_1>\mathbf{x}^3\mathbf{e}\frac{1}{\mathbf{\Gamma}}\left[\log(1-\frac{l_1}{l_0})-\log(1-\frac{d_1}{d_0})\right]\Rightarrow\frac{\hat{p}_2(0)}{\hat{p}_1(0)}>\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}$$

$$\bar{y}_1\le\mathbf{x}^3\mathbf{e}\frac{1}{\mathbf{\Gamma}}\left[\log(1-\frac{l_1}{l_0})-\log(1-\frac{d_1}{d_0})\right]\Rightarrow\frac{\hat{p}_2(0)}{\hat{p}_1(0)}=\frac{1-\frac{d_2}{d_0}}{1-\frac{d_1}{d_0}}.$$

Hence, by inspecting Figure 7(a) we can conclude that the optimal extraction path is determined if $\bar{y}_1>\mathbf{x}^3\mathbf{e}\frac{1}{\mathbf{\Gamma}}\left[\log(1-\frac{l_1}{l_0})-\log(1-\frac{d_1}{d_0})\right]$.

Assume now $\frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}}\le\frac{1}{\mathbf{\Gamma}}\left[\log(1-\frac{l_1}{l_0})-\log(1-\frac{d_1}{d_0})\right]$. Note that the maximum amount that can be extracted from the low cost deposit in set $(3\cap 4)$ is given by $\mathbf{x}^4\mathbf{e}\left\{\frac{1}{\mathbf{\Gamma}}\left[\log(1-\frac{l_1}{l_0})-\log(1-\frac{d_1}{d_0})\right]-\frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}}\right\}$. Thus, the optimal extraction strategy will have a support in $(3 \cap 4)$ if and only if $\frac{\bar{y}_2}{\mathbf{x}^4\mathbf{e}} \le \left\{ \frac{1}{\Gamma}\left[\log(1-\frac{l_1}{l_0}) - \log(1-\frac{d_1}{d_0}) \right] - \frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}} \right\}$. In this case, any (measurable and locally integrable) function $\theta(t)$, such that $\theta(t) \ge 0$, $\int_0^{\frac{\bar{y}_2}{\mathbf{x}^4\mathbf{e}}+\frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}}} \theta(t)dt = \frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}}$ and $\int_0^{\frac{\bar{y}_2}{\mathbf{x}^4\mathbf{e}}+\frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}}} (1-\theta(t))dt = \frac{\bar{y}_2}{\mathbf{x}^4\mathbf{e}}$ will generate the optimal extraction strategy $\left[\bar{x}_1(t) \quad \bar{x}_2(t)\right] = \left[\theta(t)\mathbf{x}^3\mathbf{e}, \quad (1-\theta(t))\mathbf{x}^4\mathbf{e}\right]$. If the stock in the low cost deposit exceeds the given threshold, $\frac{\bar{y}_2}{\mathbf{x}^4\mathbf{e}} > \left\{ \frac{1}{\Gamma}\left[\log(1-\frac{l_1}{l_0}) - \log(1-\frac{d_1}{d_0}) \right] - \frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}} \right\}$, then the initial price support lies in region 1, so the optimal extraction path has an initial segment during which $\left[\bar{x}_1(t) \quad \bar{x}_2(t)\right] = \left[0, \quad \mathbf{x}^1\mathbf{e}\right]$. This phase will end when $\frac{\bar{y}_2 - \mathbf{x}^1\mathbf{e}t}{\mathbf{x}^4\mathbf{e}} = \left\{ \frac{1}{\Gamma}\left[\log(1-\frac{l_1}{l_0}) - \log(1-\frac{d_1}{d_0}) \right] - \frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}} \right\}$. Then the system will enter the indeterminacy region if $\left\{ \frac{1}{\Gamma}\left[\log(1-\frac{l_1}{l_0}) - \log(1-\frac{d_1}{d_0}) \right] - \frac{\bar{y}_1}{\mathbf{x}^3\mathbf{e}} \right\} > 0$. We have, therefore, the result stated in point (iii)-(c) of Proposition 2.

References

Amigues, J.P., Favard, P., Gaudet, G., Moreaux, M. (1998), “On the Optimal Order of Natural Resource Use When the Capacity of the Inexhaustible Substitute Is Limited”, *Journal of Economic Theory* 80, 153-170.

Chakravorty, U., Moreaux, M., Tidball, M. (2008), “Ordering the Extraction of Polluting Nonrenewable Resources”, *American Economic Review*, 98 (3), 1128-44.

Favard, P. (2002), “Does productive capital affect the order of resource exploitation?”, *Journal of Economic Dynamics and Control*, 26 (6), 911–918.

Freni, G. (2004), “'Partial Equilibrium Analysis' and Order of Resource Extraction”, under revision.

Freni, G., Gozzi, F., Salvadori, N. (2006) “Existence of Optimal Strategies in Linear Multisector Models”, *Economic Theory*, 29(1), 25-48.

Freni, G., Gozzi, F., Pignotti, C. (2008), “Optimal strategies in linear multisector models: Value function and optimality conditions”, *Journal of Mathematical Economics*, 44 (1) , 55-86.

Gaudet, G., Moreaux, M., Salant, S. (2001), “Intertemporal Depletion of Resource Sites by Spatially Distributed Users,” *American Economic Review*, 91, 1149-59.

Herfindahl, O.C. (1967) “Depletion and Economic Theory”, in Gaffney, M. (Ed.), *Extractive Resources and Taxation*, Univ. of Wisconsin Press, Madison, pp. 63-90.

Holland, S. P. (2003), “Extraction Capacity and the Optimal Order Of Extraction”, *Journal of Environmental Economics and Management,* 45(3), pp. 569-588.

Kemp, M., Long N. V. (1980), “On Two Folk Theorems Concerning the Extraction of Exhaustible Resources”, *Econometrica*, 48, 663-673.

Im, E. I., Chakravorty, U., Roumasset, J. (2006), "Discontinuous extraction of a nonrenewable resource”, *Economic Letters*, 90(1), 6-11.

Jones, L. E., Manuelli R. (1990), “A Convex Model of Equilibrium Growth: Theory and Policy Implications”, *Journal of Political Economy*, 98, 1008-38.

Lewis, T.R. (1982), “Sufficient Conditions for Extracting Least Cost Resource First”, *Econometria*, 50, 1081-1083.

von Neumann, J. (1945), “A Model of General Economic Equilibrium”, *Review of Economic Studies*, 13, 1-9.

Rebelo, S. (1991), “Long-run Policy Analysis and Long-run Growth”, *Journal of Political Economy*, 99, pp. 500-21.

Solow, R.M., Wan, F.Y. (1976), “Extraction Costs in the Theory of Exhaustible Resources”, *Bell Journal of Economics* 7, 359-370.

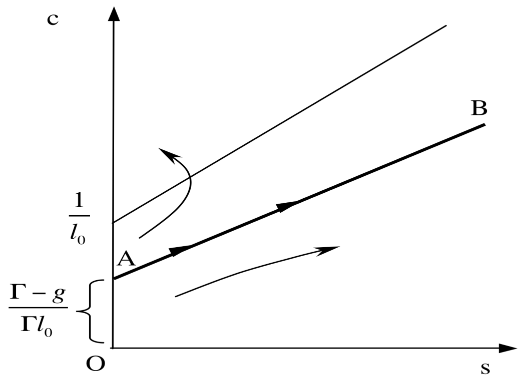


Figure 1

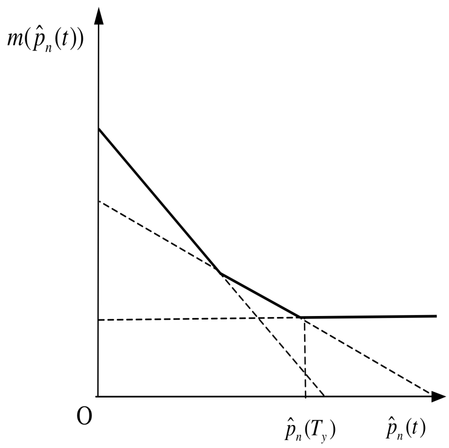


Figure 2

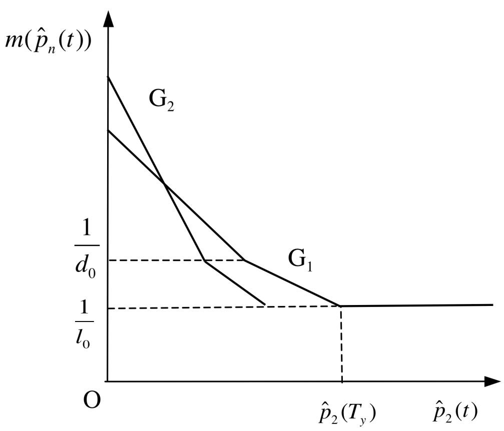
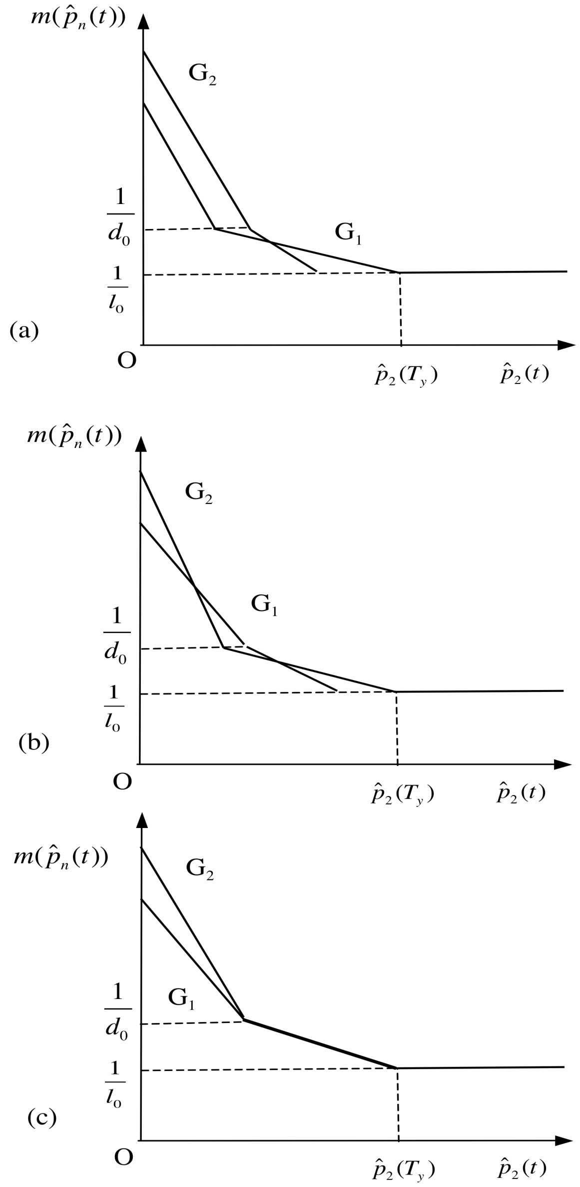


Figure 3



$m(\hat{p}_n(t))$
G_2
G_1
$\frac{1}{d_0}$
$\frac{1}{l_0}$
(a)
O
$\hat{p}_2(T_y)$
$\hat{p}_2(t)$
$m(\hat{p}_n(t))$
G_2
G_1
$\frac{1}{d_0}$
$\frac{1}{l_0}$
(b)
O
$\hat{p}_2(T_y)$
$\hat{p}_2(t)$
$m(\hat{p}_n(t))$
G_2
G_1
$\frac{1}{d_0}$
$\frac{1}{l_0}$
(c)
O
$\hat{p}_2(T_y)$
$\hat{p}_2(t)$

Figure 4

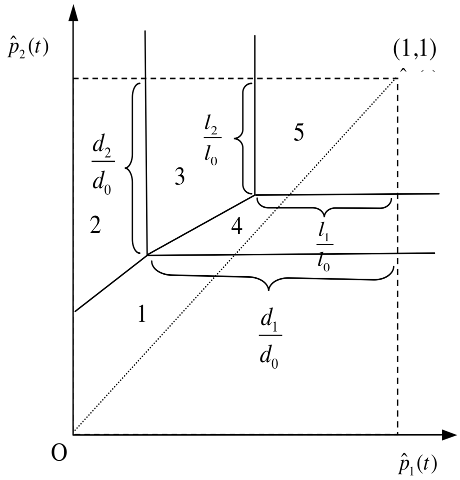


Figure 5(a)

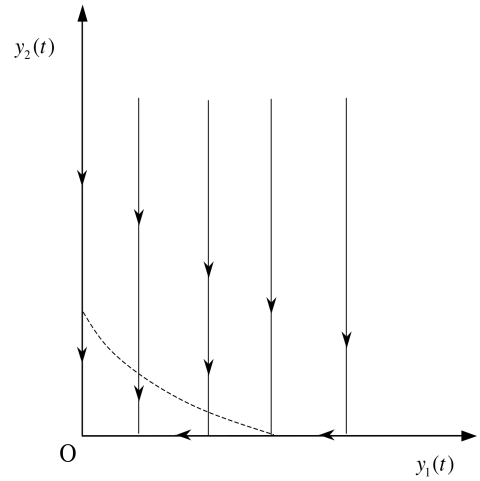


Figure 5(b)

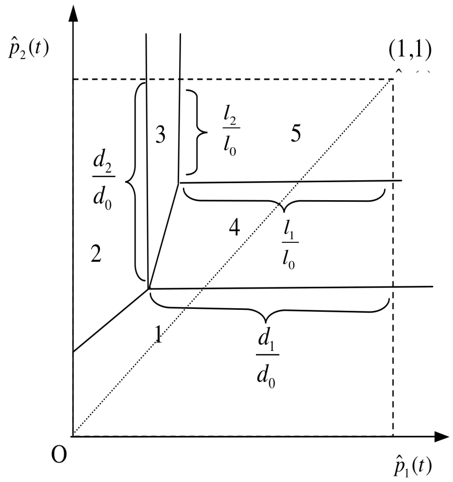


Figure 6(a)

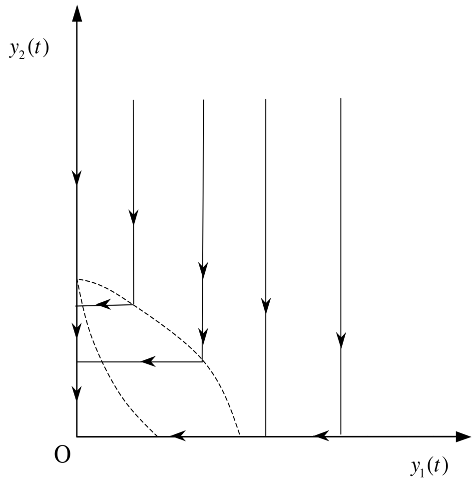


Figure 6(b)

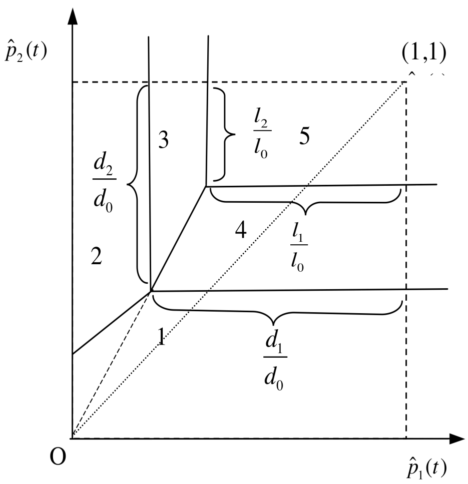


Figure 7(a)

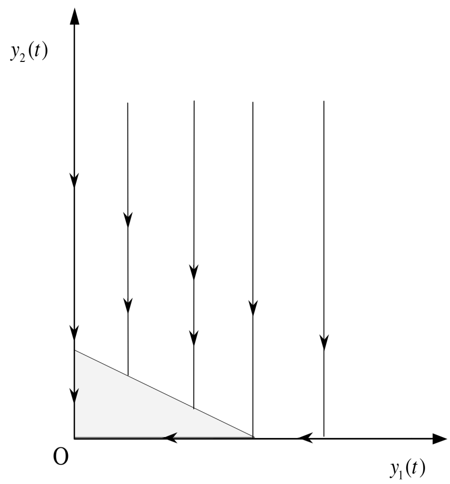


Figure 7(b)