A Unified Implementation Theory

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January 2001

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This paper unifies the theories of Nash implementation and Bayesian implementation in a single framework. Environments considered are such that each agent's characteristics include, in addition to a specification of his private information, a commonly known type parameter, while both attributes are unknown to the designer. Each social choice correspondence (SCC) assigns a commonly known type vector to a social choice set, a collection of functions mapping private type vectors to allocations.

Conditions that fully characterize an implementable SCC in economic environments where agents are not satiated generalize and merge respective conditions in the complete information model of Danilov (1992) and the incomplete information model of Jackson (1991). In noneconomic environments there remains to exist a gap between the necessary and sufficient conditions, like in Jackson (1991). In order to narrow down this gap, we employ Danilov's notion of essential elements and develop a stronger necessary condition, termed essential-generalized-Bayesian monotonicity (EGBM).

Keywords: Bayesian implementation, Nash implementation, mechanism, complete information, incomplete information, social choice correspondence.

1. INTRODUCTION

In this paper, we provide a unified framework in which the theories of both Nash implementation and Bayesian (Nash) implementation can be accommodated. We also discuss whether Danilov's (1992) notion of essential elements can be used to fill the gap between the necessary and sufficient conditions of implementation in noneconomic environments.

An environment is called economic if agents cannot be simultaneously satiated, and noneconomic otherwise. The problem of implementing social choice sets in both economic and noneconomic environments involving agents that have incomplete information about the state of the society is examined by Jackson (1991).

He defines social choice functions from states to allocations, and social choice sets as collections of social choice functions. His contributions establish that social choice sets are Bayesian implementable only if they satisfy closure (C), incentive compatibility (IC), and Bayesian monotonicity (BM) conditions. Moreover, these three conditions are sufficient to implement a social choice set in any economic environment involving at least three agents.[1] Unfortunately, the same sufficiency result does not hold in noneconomic environments. Jackson shows that a social choice set in any noneconomic environment is implementable if it satisfies (C), (IC) and monotonicity-no-veto (MNV), a condition combining Bayesian monotonicity and no-veto conditions. Since (MNV) is not necessary, there exists a gap between necessary and sufficient conditions for Bayesian implementation in noneconomic environments. Jackson is quick to realize that Danilov's (1989) single condition, namely essential monotonicity (EM), that characterizes Nash implementable social choice correspondences can be helpful along this line.[2]

Danilov (1989, 1992) shows that any Nash implementable social choice correspondence (SCC) - from preferences to alternatives - is essentially monotone. Conversely, if a SCC is essentially monotone and there are at least three agents in the environment, then the SCC is implementable via Nash equilibria. Essential monotonicity is stronger than monotonicity, a necessary condition for Nash implementation. On the other hand, essential monotonicity is weaker than monotonicity + no-veto power, which are sufficient conditions of Nash implementation when there are at least three agents, a fact proved by Maskin (1977). Analogously, to reduce the gap in Bayesian implementation, we wish carefully translate (EM) to get a condition stronger than (BM) while weaker than (MNV).

The environment that we consider differs from that of Jackson in two aspects. First, an agent's characteristics include, in addition to a specification of his private information, a commonly known type parameter. The two attributes are both unknown to the designer. Second, instead of social choice sets, we deal with social choice correspondences assigning the commonly known types of individuals to social choice sets. Like in Jackson's model, however, each social choice function within a given social choice set maps the private type profiles to allocations. The problem of implementation is then to design a strategic outcome function whose equilibria for any environment coincides with the social choice correspondence.

[1]See, also, Matsushima (1990) for similar results in economic environments.

[2]A full characterization of necessary and sufficient conditions of Nash implementation is due to Moore and Repullo (1990). Danilov's (1989, 1992) single condition reduces Moore and Repullo's three conditions to one.

The distinction in the environment with regard to the previous literature has an important implication. In a single framework, we merge two models of implementation: Nash implementation model and Bayesian implementation model. We show that conditions characterizing implementable social choice correspondences select, up to some required generalizations and modifications, from the respective conditions for Nash implementation and Bayesian implementation.

Any SCC in our framework is implementable only if it satisfies conditions generalizing Danilov's essential monotonicity and Jackson's closure, incentive compatibility and Bayesian monotonicity provided that the domain of preferences is sufficiently rich. In economic environments involving rich preference domains and at least three agents, the same conditions are also sufficient to fully implement an SCC.

However, in noneconomic environments the sufficiency conditions must involve a generalized monotonicity-no-veto (GMNV) condition replacing generalized Bayesian monotonicity. So, with regards to Jackson (1991), the gap between necessary and sufficient conditions in noneconomic environments remains to exist, since (GMNV) is not necessary. To fill this gap, we define a new condition on SCC, which we call essential-generalized-Bayesian monotonicity (EGBM). This condition is a stronger definition of Bayesian monotonicity in the literature. We show that in our unified framework (EGBM) is necessary for Bayesian implementation if the domain of preferences is sufficiently rich. Unfortunately, not all social choice correspondences satisfying (EGBM) together with the other necessary conditions are implementable.

Two particular cases within our unified framework are of a special interest. In one extreme case in which the information set of each agent is a singleton, the model boils down to the Nash implementation model considered by Danilov. In the other extreme case in which the society is known to have a single type, the model coincides with Bayesian implementation model of Jackson.

The paper proceeds as follows: Section 2 introduces the environment that heavily borrows from Jackson (1991), and defines social choice correspondences. Section 3 provides the definitions that generalize and merge the notions in the Bayesian model of Jackson and the complete information model of Danilov. In Section 4 we describe the implementation problem, and in Section 5 we unify the theories of Nash implementation and Bayesian implementation. Finally, in Section 6 we strengthen the necessary conditions for implementation, using Danilov's notion of essential elements.

2. Basic Structures

Environments

There are a finite number, N, of agents. Agent i has two attributes θ^i and s^i. The parameter θ^i is common knowledge while s^i is privately known by agent i. Henceforth, we will use the term *type* for θ^i and *information set* for s^i.

Let Θ^i be the set of possible types of agent i. A type profile is a vector $\theta = (\theta^1, ..., \theta^N)$ and the set of all type profiles is $\Theta = \Theta^1 \times ... \times \Theta^N$. Let S^i describe the finite number of possible information sets of agent i. A *state* is a vector $s = (s^1, ..., s^N)$ and the set of states is $S = S^1 \times ... \times S^N$. Both the type profile and the state of the society are unknown to the designer.

Let A denote the set of feasible allocations. We assume A is fixed across states.

A *social choice function* is a map from states to allocations. The set of all social choice functions is $X = \{x | x : S \to A\}$.

Each agent i has a probability measure q^i defined on S.[3] It is assumed that if $q^i(s) > 0$ for some i and $s \in S$, then $q^j(s) > 0$ for all $j \neq i$. All agents agree on that T denotes the set of states which occur with positive probability, where $T = \{s \in S | q^i(s) > 0, \ \forall i\}$.

The sets Π^i are partitions of T defined by q^i. For a given information set $s^i \in S^i$, $\pi^i(s^i) = \{t \in S | t^i = s^i \text{ and } q^i(t) > 0\}$ denotes the set of states which agent i believes may be the true state. It is assumed that $\pi^i(s^i) \neq \emptyset$ for all i and $s^i \in S^i$. Let Π denote the finest partition which is coarser than each Π^i. For a given state $s \in S$, let $\pi(s)$ be the element of Π which contains s.

A preference is a linear order on X. The set of all preferences is denoted as $\mathcal{R}$. Each agent has preferences over social choice functions which have a conditional expected utility representation. Given $x, y \in X$, $s^i \in S^i$, and $\theta \in \Theta$, agent i's *weak preference relation* $R^i(s^i, \theta^i) \in \mathcal{R}$ is such that

$$xR^i(s^i, \theta^i)y \Leftrightarrow \sum_{s \in \pi^i(s^i)} q^i(s)U^i[x(s), s, \theta^i] \geq \sum_{s \in \pi^i(s^i)} q^i(s)U^i[y(s), s, \theta^i],$$

where $U^i : A \times S \times \Theta^i \to \mathbb{R}_+$ is a state and type dependent utility function. Preferences are complete and transitive. The strict preference and indifference relations associated with R^i are P^i and I^i, respectively.

[3] For notational simplicity and with no loss of generality in our results, we assume that q^i is type-independent.

An *environment* is a collection $[N, S, \Theta, A, \{q^i\}, \{U^i\}]$, whose structure is assumed to be common knowledge among agents.

Social Choice Correspondences

A *social choice correspondence* (SCC) is a nonempty subset $F \subset \Theta \times X$ (or $F : \Theta \Rightarrow X$). An SCC F assigns to every type profile $\theta \in \Theta$, a social choice set $F(\theta) \subset X$, i.e., a collection of social choice functions.

3. Definitions

Here, we generalize several notions in the Bayesian model of Jackson and the complete information model of Danilov.

Definition 1: Given a vector or vector of functions $v = (v^1, ..., v^N)$, let $(v^{-i}, \tilde{v}^i)$ represent the vector $(v^1, ..., v^{i-1}, \tilde{v}^i, v^{i+1}, ..., v^N)$.

Definition 2: Let $L(x, R^i(s^i, \theta^i))$ be the set of social choice functions to which agent i of type θ^i weakly prefers x at state s^i. This set is defined by $L(x, R^i(s^i, \theta^i)) = \{y \in X | x R^i(s^i, \theta^i) y\}$.

Definition 3: The social choice functions x and y are *equivalent* if $x(s) = y(s)$ for all $s \in T$. The social choice correspondences F and $\hat{F}$ are *equivalent* if for each θ and $x \in F(\theta)$ there exists $\hat{x} \in \hat{F}(\theta)$ which is equivalent to x, and for each θ and $\hat{x} \in \hat{F}(\theta)$ there exists $x \in F(\theta)$ which is equivalent to $\hat{x}$.

Definition 4: Let $x/_C z$ be a splicing of two social choice functions x and z along a set $C \in S$. The social choice function $x/_C z$ is defined by $[x/_C z](s) = x(s)$ $\forall s \in C$, and $[x/_C z](s) = z(s)$ otherwise.

Definition 5: An environment is said to be *economic* if for any $z \in X$, $\theta \in \Theta$ and $s \in S$, there exist i and j $(i \neq j)$, $x \in X$ and $y \in X$ such that x and y are constant, $x/_C z \notin L(z, R^i(s^i, \theta^i))$ and $y/_C z \notin L(z, R^j(s^j, \theta^i))$ for all $C \subset S$ such that $s \in C$. An environment is called *noneconomic* if it is not economic.

Definition 6: Let B and D be any disjoint sets of states such that $B \cup D = T$ and for any $\pi \in \Pi$ either $\pi \subset B$ or $\pi \subset D$. Consider an SCC F, and $\theta \in \Theta$. The social choice set $F(\theta)$ satisfies *closure* (C) if for any $x, y \in F(\theta)$, there exists $z \in F(\theta)$ such that $z(s) = x(s)$ $\forall s \in B$ and $z(s) = y(s)$ $\forall s \in D$. The SCC F satisfies *generalized closure* (GC) if for all θ, $F(\theta)$ satisfies (C).

Definition 7: Given i, $x \in X$, and $t^i \in S^i$, define x_{t^i} by $x_{t^i}(s) = x(s^{-i}, t^i)$, $s \in S$. Consider an SCC F, and $\theta \in \Theta$. The social choice set $F(\theta)$ satisfies *incentive compatibility* (IC) if for all i, $x \in F(\theta)$, and $t^i \in S^i$,

$$x_{t^i} \in L(x, R^i(s^i, \theta^i)) \quad \forall s^i \in S^i.$$

The SCC F satisfies *generalized incentive compatibility* (GIC) if for all θ, $F(\theta)$ satisfies (IC).

Definition 8: A *deception* for i is a mapping $\alpha^i : S^i \to S^i$. Let $\alpha = (\alpha^1, ..., \alpha^N)$ and $\alpha(s) = [\alpha^1(s^1), ..., \alpha^N(s^N)]$. The notation $x \circ \alpha$ represents the social choice function which results in $x[\alpha(s)]$ for each $s \in S$.

Definition 9: Consider an SCC F, $\theta \in \Theta$, $x \in F(\theta)$ and a deception α. The social choice set $F(\theta)$ satisfies *Bayesian monotonicity* (BM) if whenever there is no social choice function in $F(\theta)$ which is equivalent to $x \circ \alpha$, there exists $i, s^i \in S^i$ and $y \in X$ such that

$$y \circ \alpha \notin L(x \circ \alpha, R^i(s^i, \theta^i)) \;\text{ while }\; y_{\alpha^i(s^i)} \in L(x, R^i(t^i, \theta^i)) \quad \forall t^i \in S^i.$$

The SCC F satisfies *generalized Bayesian monotonicity* (GBM) if for all θ, $F(\theta)$ satisfies (BM).

Definition 10: A social choice function $z \in X$ satisfies the *no-veto hypothesis* (NVH) for α, θ and $D \subset T$, if for each $s \in D$ there exists i such that for each $j \neq i$ and $\tilde{z} \in X$ there is a set $C \subset D$ such that $s \in C$ and $\tilde{z} \circ \alpha /_C z \in L(z, R^j(s^j, \theta^j))$.

Definition 11: Consider an SCC F, a deception α, and for each $\hat{\theta} \in \Theta$, $x \in F(\hat{\theta})$, and i, a set $B^i_{x,\hat{\theta}} \subset S^i$. Let $B_{x,\hat{\theta}} = B^1_{x,\hat{\theta}} \times ... \times B^N_{x,\hat{\theta}}$. Suppose that there exists z such that for each $\hat{\theta} \in \Theta$, $x \in F(\hat{\theta})$ and $s \in B_{x,\hat{\theta}}$, $z(s) = x \circ \alpha(s)$. Furthermore, suppose that z satisfies (NVH) for α, θ and $T - (\cup_{\hat{\theta} \in \Theta} \cup_{x \in F(\hat{\theta})} B_{x,\hat{\theta}})$. F satisfies *generalized-monotonicity-no-veto* (GMNV) if whenever there is no social choice function in $F(\theta)$ which is equivalent to z, there exist i, $\hat{\theta} \in \Theta$, $x \in F(\hat{\theta})$, $s \in \cup_{\bar{\theta} \in \bar{\Theta}_x} B_{x,\bar{\theta}}$ where $\bar{\Theta}_x = \{\theta : x \in F(\theta)\}$, and $y, \tilde{z}$, and $\bar{z} \in X$, such that $\bar{z}(t) = y \circ \alpha(t)$ when $t \in \cup_{\bar{\theta} \in \bar{\Theta}_x} B_{x,\bar{\theta}}$; $\bar{z}(t) = z(t)$ when $t^{-i} \in \cup_{\bar{\theta} \in \bar{\Theta}_{\bar{x}}} B^{-i}_{\bar{x},\bar{\theta}}$ for some $\bar{x}$ such that $\bar{x} \neq x$; and $\bar{z}(t) = \tilde{z} \circ \alpha(t)$ otherwise; and

$$\bar{z} \notin L(z, R^i(s^i, \theta^i)), \quad \text{while} \quad y_{\alpha^i(s^i)} \in L(x, R^i(t^i, \theta^i)) \quad \forall t^i \in S^i.$$

If Θ is a singleton, every SCC is a social choice set; hence (GC), (GIC), (GBM) and (GMNV), respectively, reduce to the conditions (C), (IC), (BM) and (MNV) defined by Jackson (1991) for social choice sets.

Definition 12: Let i be an agent and $Y \subset X$. A social choice function $y \in Y$ is *essential* for i in set Y if $y \in F(\theta)$ for some $\theta \in \Theta$ and $L(y, R^i(s^i, \theta^i)) \subset Y$ for all $s^i \in S^i$.

Given the social choice correspondence F, the set of all essential elements for i in $Y \subset X$ is denoted by $Ess(F; i, Y)$ or simply $Ess(i, Y)$. Obviously $Ess(F; i, Y) \subset Y$, and if $Z \subset Y \subset X$ then $Ess(i, Z) \subset Ess(i, Y)$. Moreover, $Ess(F; i, X) = \cup_{\theta \in \Theta} F(\theta)$.

Definition 13: The SCC F satisfies *generalized-essential monotonicity* (GEM) if for $\theta, \hat{\theta} \in \Theta$, and $x \in F(\theta)$ the relations

$$Ess(F; i, L(x, R^i(t^i, \theta^i))) \subset L(x, R^i(t^i, \hat{\theta}^i)) \quad \forall t^i \in S^i \text{ and } \forall i$$

imply $x \in F(\hat{\theta})$.

Generalized-essential-monotonicity means that the social choice function x survives not only at an improvement of position of x at all states but also when its position gets nonessentially worse at some states of the society. In the case in which S is a singleton, (GEM) boils down to Danilov's essential monotonicity condition (EM).

Finally, we will define environments that are sufficiently rich in the preferences of its agents.

Definition 14: Consider any linear order $\tilde{R} \in \mathcal{R}$. The environment satisfies *rich domain hypothesis* (RDH) if for each i there exists $\theta^i \in \Theta^i$ such that $R^i(s^i, \theta^i) = \tilde{R}$ for all $s^i \in S^i$.

4. Implementation

A *mechanism* is an action space $M = M^1 \times ... \times M^N$ and a map $\mu : M \to A$.

A *strategy* for agent i is a map $\sigma^i : S^i \to M^i$. Denote by Σ^i the set of all strategies for agent i, and define $\Sigma = \Sigma^1 \times ... \times \Sigma^N$.

For any $\sigma \in \Sigma$, $\mu(\sigma)$ represents the social choice function which results when σ is played.

Let θ be a type profile. A vector of strategies $\sigma \in \Sigma$ is a *Bayesian (Nash) equilibrium* in the game $G(M, \mu, \theta)$ if $\mu(\sigma^{-i}, \tilde{\sigma}^i) \in L(\mu(\sigma), R^i(s^i, \theta^i))$ for all i, s^i and $\tilde{\sigma}^i \in \Sigma^i$. In other words, $\mu(\sigma^{-i}, \Sigma^i) \subset L(\mu(\sigma), R^i(s^i, \theta^i))$ for all i and s^i.

Let $BE(\mu, \theta)$ be the set of all Bayesian equilibria in the game $G(M, \mu, \theta)$. Then the set of all *equilibrium outcomes* in this game is defined by $E(\mu, \theta) = \mu(BE(\mu, \theta))$.

A mechanism (M, μ) implements a social choice correspondence F if:

(i) for any $\theta \in \Theta$ and $x \in F(\theta)$ there exists an equilibrium $\sigma \in BE(\mu, \theta)$ with $\mu[\sigma(s)] = x(s)$ for all $s \in T$, and

(ii) for any $\theta \in \Theta$ and any equilibrium $\sigma \in BE(\mu, \theta)$ there exists $x \in F(\theta)$ with $\mu[\sigma(s)] = x(s)$ for all $s \in T$.

In other words, the mechanism (M, μ) implements F if $E(\mu)$ is equivalent to F. A social choice set F is *implementable* if there exists a mechanism (M, μ) which implements F.

5. Unifying Theories of Nash Implementation and Bayesian Implementation

This section begins with the description of essential elements for the equilibrium outcomes correspondence, $E(\mu)$. Next, we establish the result that $E(\mu)$ satisfies the condition (GEM) if the domain of preferences is sufficiently rich. First two lemmas stating the above results actually extend similar results obtained by Danilov (1992) for the complete information case to our Bayesian framework.

LEMMA 1: *Assume the environment satisfies (RDH). Consider a mechanism* (M, μ)*, a set* $Y \subset X$*, a social choice function* $y \in Y$ *and agent* i*. Then,* $y \in Ess(E(\mu); i, Y)$ *if and only if* $y = \mu(\sigma^{-i}, \sigma^i)$ *where* $\mu(\sigma^{-i}, \Sigma^i) \subset Y$.

PROOF: To show the "only if" part, let $y \in Ess(E(\mu); i, Y)$. Then $y \in E(\mu, \theta)$ for some θ such that $y = \mu(\hat{\sigma}^{-i}, \hat{\sigma}^i)$ where $\mu(\hat{\sigma}^{-i}, \Sigma^i) \subset L(y, R^i(s^i, \theta^i)) \subset Y$ for all $s^i \in S^i$. Conversely, let $y = \mu(\sigma^{-i}, \sigma^i)$ and $\mu(\sigma^{-i}, \Sigma^i) \subset Y$. Since the environment satisfies (RDH), let θ be such that $L(y, R^i(s^i, \theta^i)) = Y$ for all s^i and $y = \max_j R^j(s^j, \theta^j)$ for all $j \neq i$ and s^j. It is obvious that (σ^{-i}, σ^i) is Bayesian equilibrium in the game $G(M, \mu, \theta)$, and therefore $y \in Ess(E(\mu); i, Y)$. *Q.E.D.*

LEMMA 2: *If the environment satisfies (RDH), then, for any mechanism* (M, μ)*, the correspondence* $E(\mu)$ *satisfies (GEM).*

PROOF: Let σ be a Bayesian equilibrium in the game $G(M, \mu, \theta)$, and $x = \mu(\sigma)$. Then, $\mu(\sigma^{-i}, \Sigma^i) \subset L(x, R^i(t^i, \theta^i))$ for all i and t^i. By Lemma 1, $\mu(\sigma^{-i}, \Sigma^i) \subset Ess(E(\mu); i, L(x, R^i(t^i, \theta^i)))$ for all i and t^i. Let $\hat{\theta}$ be a type profile satisfying

$$Ess(F; i, L(x, R^i(t^i, \theta^i))) \subset L(x, R^i(t^i, \hat{\theta}^i))$$

for all i and t^i. It follows that $\mu(\Sigma^i, \sigma^{-i}) \subset L(x, R^i(t^i, \hat{\theta}^i))$, for all i and t^i. Therefore, σ is a Bayesian equilibrium in the game $G(M, \mu, \hat{\theta})$, and $x \in E(\mu, \hat{\theta})$. *Q.E.D.*

THEOREM 1: *Assume the environment is economic, satisfies (RDH) and* $N \geq 3$*. An SCC* F *is implementable if and only if there exists an SCC* $\hat{F}$ *which is equivalent to* F *and satisfies (GC), (GIC), (GEM), and (GBM).*

The proof of Theorem 1 appears in the Appendix. The assumptions that the environment is economic and $N \geq 3$ are only needed for the sufficiency part of the Theorem. If we drop the assumption that the environment is economic, we have the following sufficiency theorem.

THEOREM 2: *Assume the environment satisfies (RDH) and* $N \geq 3$*. An SCC* F *is implementable if there exists an SCC* $\hat{F}$ *which is equivalent to* F *and satisfies (GC), (GIC), (GEM), and (GMNV).*

The proof of Theorem 2 appears in the Appendix. Note here that when the type space Θ is finite, the environment $[N, S, \Theta, A, \{q^i\}, \{U^i\}]$ can be shown to have the same information structure and preferences as the environment $[N, \hat{S}, A, \{\hat{q}^i\}, \{\hat{U}^i\}]$ (a standart setting in Bayesian models) with $\hat{S}, \{\hat{q}^i\}$ and $\{\hat{U}^i\}$ appropriately defined. Thus, Theorems 1 and 2 could be proven as corollaries to the corresponding theorems in Jackson (1991) in situations where Θ is finite. This means that (GEM) is redundant when Θ is finite.

In analyzing our first two results, two particular cases are of a special interest. First, consider an environment with a single state of the society. Then, the collection of SCC's which are equivalent to an SCC F consists simply of F itself, and every SCC satisfies (GC), (GIC), (EGBM) and (GMNV) regardless (RDH) holds. So, in both economic and noneconomic environments (GEM) becomes the unique sufficiency condition if S is a singleton. Moreover, (GEM) reduces to (EM) in such a case. Thus, we obtain the following result by Danilov (1992) as a straightforward corollary to our previous two theorems.

Corollary 1: Assume the environment satisfies (RDH), $\#S = 1$ and $N \geq 3$. A social choice correspondence F is implementable if and only if F satisfies (EM).

Consider now the other extreme case in which the type space contains a single element. In this case, (GEM) has no bite, whereas (GC), (GIC), (GBM) and (GMNV) reduce to (C), (IC), (BM) and (MNV), respectively. In addition, any SCC is a social choice set now. Thus, we obtain Theorem 1 and Theorem 2 in Jackson (1991) as separate corollaries to our first and second Theorems, respectively.

Corollary 2: *Assume the environment is economic, $\#\Theta = 1$ and $N \geq 3$. An SCC F is implementable if and only if there exists an SCC $\hat{F}$ which is equivalent to F and satisfies (C), (IC) and (BM).*

Corollary 3: *Assume $\#\Theta = 1$ and $N \geq 3$. An SCC F is implementable if there exists an SCC $\hat{F}$ which is equivalent to F and satisfies (C), (IC), and (MNV).*

Remark 1: Assume the environment is noneconomic and satisfies (RDH). The conditions (GC), (GIC), (GEM) and (GBM) are not sufficient for implementation.

In what follows, we extend Example 1 in Jackson (1991) in order to prove the claim in Remark 1.

Example 1: Consider the environment in which $N = 4$, $A = \{a, b\}$, $\Theta^i = \{\theta^i_1, \theta^i_2, \theta^i_3, \theta^i_4\}$, $S^i = \{s^i, t^i\}$, and $T = \{s_1 = (s^1, s^2, s^3, s^4); s_2 = (s^1, s^2, t^3, t^4); s_3 = (t^1, t^2, t^3, t^4)\}$, the partitions pictured below represent the information structure implied by T:

	States
Agents 1 and 2	$[s_1 \quad s_2] \quad [s_3]$
Agents 3 and 4	$[s_1] \quad [s_2 \quad s_3]$

The functional form of the utility functions of agents 1 and 2 is the same as is that of agents 3 and 4. The utilities representing the preferences are given below.

	Agents 1 and 2		Agents 3 and 4	
	a	b	a	b
$U^i(., s_1, \theta^i_1)$	2	1	1	2
$U^i(., s_2, \theta^i_1)$	2	1	1	2
$U^i(., s_3, \theta^i_1)$	2	1	1	2
$U^i(., s_1, \theta^i_2)$	1	1	1	1
$U^i(., s_2, \theta^i_2)$	1	1	1	1
$U^i(., s_3, \theta^i_2)$	1	1	1	1
$U^i(., s_1, \theta^i_3)$	1	2	2	1
$U^i(., s_2, \theta^i_3)$	1	2	2	1
$U^i(., s_3, \theta^i_3)$	1	2	2	1
$U^i(., s_1, \theta^i_4)$	2	1	1	2
$U^i(., s_2, \theta^i_4)$	1	1	1	1
$U^i(., s_3, \theta^i_4)$	2	1	1	2

Preferences satisfy rich domain hypothesis since the set $\times_{i=1}^4 \{\theta^i_1, \theta^i_2, \theta^i_3\}$, alone, constitutes a rich domain.

Consider the social choice set $F(\theta) = \{x, \bar{x}\}$ for all $\theta \in \Theta$, where $x(s) = a$ for all $s \in S$ and $\bar{x}(s) = b$ for all $s \in S$.

F satisfies (GEM) since F is constant on Θ. F satisfies (GC) since the common knowledge concatenation satisfies $\Pi = \{T\}$. Condition (GIC) is satisfied since x and $\bar{x}$ are constant. Since $x \circ \alpha = x$ and $\bar{x} \circ \alpha = \bar{x}$ for every deception α, it follows that for every $\theta \in \Theta$, $x\circ\alpha \in F(\theta)$ and $\bar{x}\circ\alpha \in F(\theta)$ for every deception α, and so (GBM) is satisfied.

Although F satisfies (GC), (GIC), (GEM) and (GBM), it is not implementable. To see this, suppose that a mechanism (M, μ) implements F. Let $\theta_4 = (\theta_4^1, \theta_4^2, \theta_4^3, \theta_4^4)$. Then there exist equilibrium sets of strategies $\sigma_x, \sigma_{\bar{x}} \in BE(\mu, \theta_4)$ resulting in x and $\bar{x}$ on T, respectively. Consider the set of strategies $\tilde{\sigma}$ defined by $\tilde{\sigma}^i(s^i) = \sigma_x(s^i)$ and $\tilde{\sigma}^i(t^i) = \sigma_{\bar{x}}(t^i)$. Since each agent i is completely indifferent at (s_2, θ_4^i), $\tilde{\sigma}$ is an equilibrium. Notice that $\mu[\tilde{\sigma}(s_1)] = a$ and $\mu[\tilde{\sigma}(s_3)] = b$. However, there is no social choice function in $F(\theta_4)$ which coincides with $\mu[\tilde{\sigma}]$ on T, which is a contradiction. Therefore, F is not implementable.

6. Strengthening the Necessary Conditions

We use Danilov's notion of essential elements to strengthen the condition (GBM), and show that the stronger condition, termed essential-generalized-Bayesian monotonicity (EGBM), is necessary for implementation.

Definition 15: Consider an SCC F, $\theta \in \Theta$, $x \in F(\theta)$ and deception α. F satisfies *essential-generalized-Bayesian monotonicity* (EGBM) if whenever there is no social choice function in $F(\theta)$ which is equivalent to $x\circ\alpha$, there exists $i, s^i \in S^i$ and $y \in X$ such that

$$y\circ\alpha \notin L(x\circ\alpha, R^i(s^i, \theta^i)), \quad \text{while} \quad y_{\alpha^i(s^i)} \in Ess(F; i, L(x, R^i(t^i, \theta^i)))\ \forall t^i \in S^i.$$

Lemma 3: *Any social choice correspondence satisfies (GBM) if it satisfies (EGBM).*

Proof: Obvious from the inclusions $Ess(F; i, L(x, R^i(t^i, \theta^i))) \subset L(x, R^i(t^i, \theta^i))$ for all i, x, θ^i and t^i.

The preceding result is valid for both economic and noneconomic environments. However, Examples 2 and 3 below show that the converse of Lemma 3 is not true in either type of environments.

Example 2: Consider the environment in which $N = 6$, $A = \{a, b, c, d\}$, $\Theta = \{\theta = (\theta^1, ..., \theta^6)\}$, $S^1 = \{v^1, w^1\}$, $S^i = \{v^i\}$ for all $i \neq 1$, and $T = S =$

$\{s_1 = (v^1, ..., v^6); s_2 = (w^1, v^2, ..., v^6)\}$. The partitions pictured below represent the information structure implied by T:

	States	
Agent 1	$[s_1]$	$[s_2]$
Agents 2-6	$[s_1$	$s_2]$

Agents 1 and 2 have identical and state-independent utility functions at all states as do agents 3, 4 and 5, 6:

	Agents 1 and 2				Agents 3 and 4				Agents 5 and 6			
	a	b	c	d	a	b	c	d	a	b	c	d
$U^i(., s, \theta^i)$	10	10	12	4	10	10	4	12	10	10	4	4

for all $s \in S$.

The priors $\{q^i\}$ are given as follows: $q^i(s) = 0.5$ for all i and s. It follows that $q^1(s_1|v^1) = 1$, $q^1(s_2|w^1) = 1$, and $q^i(s_j|v^i) = 0.5$, for all $i \neq 1$ and $j = 1, 2$.

Consider the following social choice functions in X that we will refer to.

	s_1	s_2		s_1	s_2
$x_1(.)$	a	a	$x_5(.)$	d	d
$x_2(.)$	a	b	$x_6(.)$	d	c
$x_3(.)$	b	a	$x_7(.)$	c	d
$x_4(.)$	b	b	$x_8(.)$	c	c

Let SCC be given by the formula $F(\theta) = \{x_1, x_2, x_3\}$.

To check that F satisfies (GBM), consider $x \in F(\theta)$. Let α be any deception such that $x \circ \alpha \notin F(\theta)$. It is clear that $x \neq x_1$ since $x_1 \circ \alpha = x_1 \in F(\theta)$. Thus, $x \in \{x_2, x_3\}$ and $x \circ \alpha = x_4$. Without loss of generality assume $x = x_2$, i.e. $x(s_1) = a$ and $x(s_2) = b$. Then, α must be such that $\alpha(s_1) = \alpha(s_2) = s_2$, i.e. $\alpha^1(v^1) = \alpha^1(w^1) = w^1$ and $\alpha^i(v^i) = v^i$ for all $i \neq 1$.

Now, consider x_6 and agent 2. Notice that $x_6 \circ \alpha = x_8$, and $x_8 \notin L(x_2, R^2(v^2, \theta^2))$ since

$$\sum_{s \in \pi^2(v^2)} q^2(s|v^2) U^2[x_8(s), s, \theta^2] = (0.5)(12) + (0.5)(12) = 12 >$$
$$\sum_{s \in \pi^2(v^2)} q^2(s|v^2) U^2[x_4(s), s, \theta^2] = (0.5)(10) + (0.5)(10) = 10.$$

On the other hand, $x_6(.,\alpha^2(v^2)) \in L(x_2, R^2(v^2,\theta^2))$ since $x_6(.,\alpha^2(v^2)) = x_6$ and

$$\sum_{s\in\pi^2(v^2)} q^2(s|v^2)U^2[x_6(s),s,\theta^2] = (0.5)(4) + (0.5)(12) = 8 <$$
$$\sum_{s\in\pi^2(v^2)} q^2(s|v^2)U^2[x_2(s),s,\theta^2] = (0.5)(10) + (0.5)(10) = 10.$$

(Note that for $x = x_3$, the social choice function x_6 can be changed with x_7 in the above lines, with the relevant inequalities still holding.) Therefore, F satisfies (GBM).

Despite this fact, F does not satisfy (EGBM). This is seen as follows. Suppose there exists a social choice function $y \in X$ such that for some i and s^i, $y{\circ}\alpha \notin L(x{\circ}\alpha, R^i(s^i,\theta^i))$ while $y_{\alpha^i(s^i)} \in Ess(F; i, L(x, R^i(t^i,\theta^i)))$ for all t^i. Then $i \in \{1,2,3,4\}$ since it can be easily verified that $x{\circ}\alpha = x_4$ is among the most preferred social choice functions for agents 5 and 6. It must be true that $y(t) = a$ for some t by the supposition that $y_{\alpha^i(s^i)} \in F(\theta)$. Moreover, there exists t such that $y(t) = c$ if $i \in \{1,2\}$ and $y(t) = d$ if $i \in \{3,4\}$, by the supposition that $y{\circ}\alpha \notin L(x{\circ}\alpha, R^i(s^i,\theta^i))$ and the construction of U^i's. So, given $x = x_2$ and $x \circ \alpha = x_4$, we must have $y(s_1) = a$, $y(s_2) = c$ if $i \in \{1,2\}$ and $y(s_2) = d$ if $i \in \{3,4\}$. Consider agent 1, first. Since $\alpha^1(v^1) = \alpha^1(w^1) = w^1$, $y_{\alpha^1(s^1)} = x_8$ for all $s^1 \in S^1$. But, $x_8 \notin F(\theta)$. Next, consider agents 2,3 and 4. We have $\alpha^i(v^i) = v^i$, and hence $y_{\alpha^i(v^i)} = y$ for all $i \in \{1,2,3\}$. But, $y \notin F(\theta)$. Therefore, we have established that $y^i_\alpha(s^i) \notin F(\theta)$ for all $i \in \{1,2,3,4\}$ and s^i, which is a contradiction. Therefore, F does not satisfy (EGBM).

Finally, we will show that the environment is economic. Let z be a social choice function in X. Consider the following cases:

Case 1: $z(s) \in \{a,b,d\}$ for all s. Notice that x_8 is constant, and for all $C \subset S$ such that $s \in C$, $x_8/_C z \notin L(z, R^i(s^i,\theta^i))$ for $i = 1,2$.

Case 2: $z(s) \in \{a,b,c\}$ for all s. Notice that x_5 is constant, and for all $C \subset S$ such that $s \in C$, $x_5/_C z \notin L(z, R^i(s^i,\theta^i))$ for $i = 3,4$.

Case 3: $z(s) \in \{c,d\}$ for all s. Notice that x_1 is constant, and for all $C \subset S$ such that $s \in C$, $x_1/_C z \notin L(z, R^i(s^i,\theta^i))$ for $i = 5,6$.

Thus, for any given social choice function and state, there are at least two agents who prefer to alter the social choice function at that state, and therefore the environment is economic.

Example 3: Drop agents 3,4,5 and 6 in the environment described in Example 2. Obviously, this environment, like the one in Example 2, satisfies (GBM) but not (EGBM). However, in this case the environment is noneconomic. To see this, consider x_8. For all $x \in X$ such that x is constant, $x/_C x_8 \in L(x_8, R^i(s^i, \theta^i))$ for all $i \in \{1, 2\}$ and for all $C \subset S$ such that $s \in C$. That is to say, agents 1 and 2 are simultaneously satiated at x_8.

LEMMA 4: *If the environment satisfies (RDH), then, for any mechanism μ, the correspondence $E(\mu)$ satisfies (EGBM).*

PROOF: Let σ be a Bayesian Nash equilibrium in the game $G(M, \mu, \theta)$, and $x = \mu(\sigma)$, i.e. $x(s) = \mu[\sigma(s)]$ for all $s \in S$. Consider that for some deception α, there exists no $z \in F$ such that $z(s) = x \circ \alpha(s)$ for all $s \in T$. It must be that $\sigma \circ \alpha$ is not an equilibrium at some $s \in T$. Therefore there exist i and $\hat{m} \in M^i$ such that $\mu[(\sigma^{-i}, \hat{\sigma}^i) \circ \alpha] \notin L(\mu[\sigma \circ \alpha], R^i(s^i, \theta^i))$, where $\hat{\sigma}^i(t^i) = \hat{m}^i$ for all $t^i \in S$. Let $y = \mu(\sigma^{-i}, \hat{\sigma}^i)$. From above $y \circ \alpha \notin L(x \circ \alpha, R^i(s^i, \theta^i))$. Since $\hat{\sigma}^i$ is constant, $y_{\alpha^i(s^i)} = y = \mu(\sigma^{-i}, \hat{\sigma}^i)$. We know that $\mu(\sigma^{-i}, \Sigma^i) \subset L(\mu(\sigma), R^i(t^i, \theta^i))$ for all $t^i \in S^i$, since σ is Bayesian equilibrium. From Lemma 1, $\mu(\sigma^{-i}, \Sigma^i) \subset Ess(E(\mu); i, L(\mu(\sigma), R^i(t^i, \theta)))$ for all $t^i \in S^i$. Thus, $y_{\alpha^i(s^i)} \in Ess(E(\mu); i, L(x, R^i(t^i, \theta)))$ for all $t^i \in S^i$. *Q.E.D.*

In the light of Lemma 4, we can now state a stronger necessity result.

THEOREM 3: *Assume the environment satisfies (RDH). A social choice correspondence F is implementable only if there exists a social choice correspondence $\hat{F}$ which is equivalent to F and satisfies (GC), (GIC), (GEM), and (EGBM).*

The proof of Theorem 3 is omitted as it is similar to that of the necessity portion of Theorem 1. It is important to note here that replacing the necessary condition (GBM) with a stronger condition (EGBM) is possible if the domain of preferences is sufficiently rich. Such a strengthening in necessary conditions may not be possible if we drop the hypothesis (RDH). In order to illustrate this claim, we will show that (EGBM) is not necessary to implement an SCC in Jackson's environments, which do not satisfy (RDH), unless, of course, the set of social choice functions X is a singleton.

We consider the economic environment in Example 2, which conforms with Jackson's Bayesian model. There, we have $\#\Theta = 1$, $S = T$ and $N = 6$. We established in Example 2 that the SCC satisfies (GBM). Clearly, (GEM) has no bite when $\#\Theta = 1$. The SCC satisfies (GC) since $\Pi = \{T\}$. The SCC also satisfies (GIC), since for all $x \in F(\theta)$ and $s \in S$, $x(s) \in \{a, b\}$, and $U^i(a, s, \theta^i) = U^i(b, s, \theta^i)$

for all i and $s \in S$. Given $\#\Theta = 1$, the SCC is actually a social choice set; so it satisfies (BM), (C) and (IC). Therefore, the SCC in Example 2 is implementable, since a result (Corollary 1) in Jackson (1991) states that in economic environments where $S = T$ and $N \geq 3$, a social choice set is implementable if it satisfies (C), (IC) and (BM). However, we showed in Example 2 that the SCC does not satisfy (EGBM).

The condition (EGBM) certainly narrows down the gap between necessary and sufficient conditions in noneconomic environments that satisfy (RDH). An important question is to see if (EGBM) closes the gap.

Remark 2: Assume the environment is noneconomic and satisfies (RDH). The conditions (GC), (GIC), (GEM) and (EGBM) are not sufficient for implementation.

The above claim is true, as it can be easily verified that the nonimplementable SCC F in Example 1 satisfies (EGBM) since for all θ the social choice set $F(\theta)$ consists of constant social choice functions.

APPENDIX

The proofs of Theorem 1 and Theorem 2 closely follow the respective proofs in Jackson (1991) established for social choice sets.

Proof of Theorem 2: The following mechanism, which slightly extends the mechanism proposed by Jackson for social choice sets, implements the SCC F if the conditions of Theorem 2 are met. Let $\bar{S} = \max_i \#S^i$ and $n = N + N\bar{S}$. Let $V = \{0, 1, ..., \bar{S}^2\}^n$. Thus $v \in V$ is an $(N + N\bar{S})$-dimensional vector such that each entry is an integer between 0 and $\bar{S}^2$. Let $M^i = \{m^i \in \Theta \times S^i \times \cup_\theta F(\theta) \times \{\emptyset \cup V\} \times X \times \{\emptyset \cup X\} \,|\, m^i_3 \in F(m^i_1)\}$ and $M = M^1 \times ... \times M^N$. Partition M into sets:

$$\begin{aligned}
d_0 &= \{m \in M | \exists x \in F(\theta) \text{ s.t. } m^j = (\theta, \cdot, x, \emptyset, \cdot, \emptyset)\ \forall j\},\\
d_1^i &= \{m \in M | m \notin d_0,\ \exists x \in F(\theta) \text{ s.t. } m^j = (\theta, \cdot, x, \emptyset, \cdot, \emptyset)\ \forall j \neq i\\
&\qquad \text{and } m^i = (\cdot, \cdot, x, \cdot, \cdot, \emptyset) \text{ or } (\cdot, \cdot, \bar{x}, \cdot, \cdot, \cdot)\},\\
d_2^i &= \{m \in M | \exists x \in F(\theta) \text{ s.t. } m^j = (\theta, \cdot, x, \emptyset, \cdot, \emptyset)\ \ \forall j \neq i\\
&\qquad \text{and } m^i = (\cdot, \cdot, x, \cdot, \cdot, y)\},\\
d_3 &= \{m \in M | m \notin d_1 \cup d_2\}.
\end{aligned}$$

Let $d_2 = \cup_i d_2^i$ and $d_1 = \cup_i d_1^i$.

Define the payoff function $\mu : M \to X$ by

$$
\begin{array}{lll}
\mu(m) = x(m_2), & m \in d_0 \cup d_1, & \\
\mu(m) = y(m_2), & m \in d_2^i \quad \text{and} & y_{m_2^i} \in L(x, R^i(t^i, \theta^i)) \text{ for all } t^i \in S^i, \\
\mu(m) = x(m_2), & m \in d_2^i \quad \text{and} & y_{m_2^i} \notin L(x, R^i(t^i, \theta^i)) \text{ for some } t^i \in S^i, \\
\mu(m) = m_5^{i^*}(m_2), & m \in d_3, &
\end{array}
$$

where i^* is determined as follows: Let $I^* = \{i | m_4^i \neq \emptyset\}$ and for $i \in I^*$ denote m_4^i by v^i. Let $J(i)$ be the number of $j \in I^*$ such that $v_l^i = v_i^j$ for an integer l where $N + (j-1)\bar{S} < l \ \square\ N + j\bar{S}$. If there exists $i \in I^*$ such that $J(i) > J(k)$ for all $k \in I^*$, then $i^* = i$, otherwise $i^* = 1$.

Remark 3: For any i and σ there exists $v^i \in V$ such that such that $\tilde{\sigma}^i$, where $\tilde{\sigma}_4^i(s^i) = v^i$ for all s^i and $\tilde{\sigma} = \sigma$ otherwise, is such that $i^* = i$ whenever $[\sigma^{-i}, \tilde{\sigma}^i](s) \in d_3$.

The following lemmas establish Theorem 2.

Lemma 5: *If F satisfies (GIC), then for each θ and $x \in F(\theta)$ there is a set of strategies σ which form an equilibrium to the game $G(M, \mu, \theta)$ such that $\mu(\sigma) = x$.*

Proof: Given an arbitrary $\theta \in \Theta$, $x \in F(\theta)$, we consider σ defined by $\sigma^i(s^i) = (\theta, s^i, x, \emptyset, \cdot, \emptyset)$. Notice that $\mu[\sigma(s)] = x(s)$ for all $s \in S$. We verify that σ is an equilibrium by showing that there are no improving deviations. Consider a deviation $\tilde{m}^i$ by i at $s^i \in S^i$.

If $\tilde{m}^i = (\tilde{\theta}, \tilde{s}^i, x, \cdot, \cdot, \emptyset)$ or $\tilde{m}^i = (\tilde{\theta}, \tilde{s}^i, \bar{x}, \cdot, \cdot, \cdot)$ then $[\sigma^{-i}(s^{-i}), \tilde{m}^i] \in d_0 \cup d_1$ (where it is possible that $\tilde{\theta} = \theta$ and $\tilde{s}^i = s^i$). The resulting allocation is $x_{\tilde{s}^i}$ (on $\pi^i(s^i)$). From (GIC) we know that this is not improving.

If $\tilde{m}^i = (\tilde{\theta}, \tilde{s}^i, x, \cdot, \cdot, y)$, then $[\sigma^{-i}(s^{-i}), \tilde{m}^i] \in d_2$ (where it is possible that $\tilde{\theta} = \theta$ and $\tilde{s}^i = s^i$). If $y_{\tilde{s}^i} \in L(x, R^i(t^i, \theta^i))$ for all $t^i \in S^i$, then the allocation is $y_{\tilde{s}^i}$ (on $\pi^i(s^i)$), which is not improving. Otherwise the allocation is $x_{\tilde{s}^i}$ (on $\pi^i(s^i)$), which is not improving by (GIC).

Lemma 6: *If F satisfies (GC), (GEM) and (GMNV), then for each set of strategies σ which form an equilibrium to the game $G(M, \mu, \theta)$ there exists $z \in F(\theta)$ which is equivalent to $\mu(\sigma)$.*

Proof: Let σ be an equilibrium to $G(M, \mu, \theta)$ and let α describe the announcement of s (m_2 as a function of s) under σ. For each i, $\hat{\theta} \in \Theta$ and $x \in F(\hat{\theta})$, let $B^i_{x,\hat{\theta}} = \{s^i : \sigma^i(s^i) = (\hat{\theta}, \alpha^i(s^i), x, \emptyset, ., \emptyset)\}$.

Since σ is an equilibrium, $\mu(\sigma)$ satisfies (NVH) for α, θ and $T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$. This is seen as follows. Suppose that $\mu(\sigma)$ does not satisfy (NVH) for α, θ and $T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$. Then there exist $s \in T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$, j, and z^j such that $z^j \circ \alpha \,/_C \mu(\sigma) \notin L(\mu(\sigma), R^j(s^j, \theta^j))$ for all $C \subset T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$ such that $s \in C$. Since the failure of (NVH) guarantees the existence of two such agents, and since $s \notin (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$, j can be chosen such that $\sigma(s) \notin d^j_1 \cup d^j_2$. Let $\tilde{m}^j$ be the same as $\sigma^j(s^j)$ except that $\tilde{m}^j_4 = v^j$ as defined in Remark 3, and $m^j_5 = z^j$. Let C be the set of $t \in \pi^j(s^j)$ such that $[\sigma^{-j}(t^{-j}), \tilde{m}^j] \in d_3$. The outcome on C is thus $z^j \circ \alpha$. Furthermore, $s \in C$, since $[\sigma^{-j}(s^{-j}), \tilde{m}^j] \in d_3$, and $C \subset T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$. From the design of $\tilde{m}^j$ it follows that if $t \in \pi^j(s^j)$ and $t \notin C$, then $[\sigma^{-j}(t^{-j}), \tilde{m}^j]$ leads to the same outcome as σ. Hence, the outcome of the deviation is $z^j \circ \alpha$ on $C \cap \pi^j(s^j)$ and $\mu(\sigma)$ otherwise. This is improving for j, which contradicts the fact that σ is an equilibrium.

It has been established that $\mu(\sigma)$ satisfies (NVH) for α, θ, and $T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$. Next, (MNV) is applied to find a social choice function in $F(\theta)$ which is equivalent to $\mu(\sigma)$.

Suppose that there does not exist a social choice function in $F(\theta)$ which is equivalent to $\mu(\sigma)$. By $(GMNV)$ there exist i, $\hat{\theta} \in \Theta$, $x \in F(\hat{\theta})$, $y, \tilde{z}, \bar{z}$ and $s^i \in \cup_{\bar{\theta}\in\bar{\Theta}_x} B^i_{x,\bar{\theta}}$, where $\bar{\Theta}_x = \{\theta : x \in F(\theta)\}$, such that $\bar{z}(s) = y \circ \alpha$ when $s \in \cup_{\bar{\theta}\in\bar{\Theta}_x} B_{x,\bar{\theta}}$; $\bar{z}(s) = \mu[\sigma(s)]$ when $s^{-i} \in \cup_{\bar{\theta}\in\bar{\Theta}_{\bar{x}}} B^{-i}_{\bar{x},\bar{\theta}}$ for some $\bar{x}$ such that $\bar{x} \neq x$; and $\bar{z}(s) = \tilde{z} \circ \alpha$ otherwise; and such that $\bar{z} \notin L(\mu(\sigma), R^i(s^i, \theta^i))$, while $y_{\alpha^i(s^i)} \in L(x, R^i(t^i, \theta^i))$ $\forall t^i \in S^i$. Therefore i is better off submitting $[\hat{\theta}, \alpha^i(s^i), x, v^i, \tilde{z}, y]$ (where v^i is defined in Remark 3) whenever s^i is observed, since the resulting outcome is $\bar{z}$ on $\pi^i(s^i)$. This is shown as follows: The deviation puts the action in d^i_2 for all $s \in \cup_{\bar{\theta}\in\bar{\Theta}_x} B_{x,\bar{\theta}}$, and the outcome is $y \circ \alpha$. The action is in d^i_1 for all $s \in \pi^i(s^i)$ such that $s^{-i} \in \cup_{\bar{\theta}\in\bar{\Theta}_{\bar{x}}} B^{-i}_{\bar{x},\bar{\theta}}$ for some $\bar{x}$ such that $\bar{x} \neq x$, and the outcome remains $\mu[\sigma(s)]$. For any other $s \in \pi^i(s^i)$ the deviation puts the action in d_3 with $i^* = i$ and the outcome $\tilde{z} \circ \alpha(s)$. Thus the outcome is $\bar{z}$ on $\pi^i(s^i)$ which is strictly preferred by i to $\mu(\sigma)$ on $\pi^i(s^i)$. This contradicts the fact that σ is an equilibrium, and so the supposition was wrong. Q.E.D.

Proof of Theorem 1: The sufficiency part follows from Theorem 2. In an environment which satisfies (E), (NVH) can never be satisfied. Therefore given (GC), (GMNV) and (GBM) are equivalent. The necessity part of the theorem is now checked.

Let μ implement F and define $\hat{F}$ such that

$$\hat{F}(\theta) = \{x | x = \mu(\sigma) \text{ for some equilibrium } \sigma \text{ in the game } G(M, \mu, \theta)\}.$$

From the definition of implementation $\hat{F}$ is equivalent to F. It is obvious that $\hat{F}$ satisfies (GC). Consider any $\theta \in \Theta$. $\hat{F}(\theta)$ satisfies (IC) and (BM), by the proof of Theorem 1 in Jackson (1991). So, $\hat{F}$ satisfies (GIC) and (GBM). Since the environment satisfies (RDH), $E(\mu)$ satisfies (GEM) by Lemma 2. Thus, $\hat{F}$ satisfies (GEM) since $\hat{F} = E(\mu)$. Q.E.D.

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