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On the Neutrality of Redistribution in a General Equilibrium Model with Public Goods*

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Abstract

Models on private provision of public goods typically involve a single private good and linear production technology for the public good. We study a model with several private goods and non-linear (strictly concave) production technology. We revisit the question of “neutrality” of government interventions on equilibrium outcomes and show that relative price effects that are absent with a single private good and linear production technology become a powerful channel of redistribution in this case. Contrary to previous results, redistributing endowments in favor of contributors is shown to be neither necessary nor sufficient for increasing the equilibrium level of public good.

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Running Head: On the Neutrality of Redistribution

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1 Introduction

In this paper, we analyze a general equilibrium model of a completely decentralized pure public good economy. Competitive firms using private goods as inputs produce the public good, which is “financed”, or “privately provided”, or “voluntarily contributed”, by households. Previous studies on private provision of public goods typically use one private good, one public good models in which the public good is produced through a constant returns to scale technology. Two distinguishing features of our model are the presence of *several* private goods and *non-linear*, in fact strictly concave, production technology for the public good. In this more general framework we revisit the question of “neutrality” of government interventions on private provision equilibrium outcomes. We show that relative price effects, which are absent with a single private good and under constant returns to scale technology for public good production, come to play an important role in our more general framework. Relative price effects provide a powerful channel through which government interventions can bring about redistributive wealth effects, which, in turn, will change equilibrium outcomes.

Warr (1983) provides the first statement of the fact that in a private provision model of voluntary public good supply, small income redistributions among contributors to a public good are “neutralized” by changes in amounts contributed in equilibrium. Consumption of the private good and the total supply of the public good remain exactly the same as before redistribution.

Bernheim (1986) and Andreoni (1988) extend Warr’s result to show that distortionary taxes and subsidies may also be neutralized by changes in private contributions. Bergstrom, Blume and Varian (1986) - from now on quoted as BBV - discuss Warr’s results in a simple general equilibrium model with one private and one public good and constant returns to scale in the production of the public good.

BBV show that redistribution is neutral if the amount of income distributed away from any household is less than his private contribution to the public good in the original equilibrium. They also show that changes in the wealth distribution which are small enough to leave unchanged the set of contributing households have the following properties: (i) leaving unchanged the aggregate wealth of current contributors will leave unchanged the equilibrium level of the public good; (ii) increasing the aggregate wealth of current contributors will increase the level of public good.

In our model, we do confirm that the neutrality result of BBV stated above holds true for “small” lump-sum taxes on contributing households. On the other hand, we show that their results on (i) and (ii) above do not hold in a more general setup such as ours.

Observe that in the BBV setup, where there is only one private good and relative price changes are ruled out by assumption, a redistribution (using the only private good which is by default the numeraire good) in favor of a consumer automatically implies that her wealth increases. But with more than one private good and relative price changes allowed, a positive subsidy in terms of the numeraire good does not necessarily imply that the consumer’s overall wealth

will increase. The reason behind this is the role relative price changes play: in the equilibrium after redistribution, changes in overall prices may (more than) offset the increase in the numeraire good due to the subsidy. Therefore, the expressions “subsidize household h (with the numeraire good)” and “increase the wealth of household h (in the equilibrium after redistribution)” are equivalent in BBV’s model, but not in ours. Another consequence of allowing for relative price changes is that a redistribution that involves a subset of households does not only affect the wealth of households in that subset. It also (indirectly) affects the wealth of households outside that subset through changes in relative prices it gives rise to. Allowing for relative price changes is the main reason why our results differ from their counterparts by BBV.

We show that a) the level of public good can be changed without affecting contributors’ total wealth; b) a redistribution in favor of contributors, in either of two meanings described above, is neither a necessary nor a sufficient condition to increase the level of privately provided public good. For example, redistributing the numeraire good among the non-contributors only may increase, or decrease, or leave unchanged the public good level even though the set of contributing agents remains the same in this case.

Our results highlight the crucial role of relative price changes that arise from redistribution of wealth on voluntary contributions to a public good. Relative price changes are simply ruled out in a model with one private good and linear production technology for the public good.

The plan of our paper is as follows. In section 2, we present the set up of the model and the existence and regularity results proved by Villanacci and Zenginobuz (2005a). In Section 3, we state and prove our results on how government intervention in the form of pure redistribution influences the total amount of public good. In Section 4 we discuss our results and remark on the properties of the model we studied.¹

2 Setup of the Model

We consider a general equilibrium model with private provision of a public good.² There are C , $C \geq 1$, private commodities, labelled by $c = 1, 2, \dots, C$. There are H households, $H > 1$, labelled by $h = 1, 2, \dots, H$. Let $\mathcal{H} = \{1, \dots, H\}$ denote the set of households. Let x_h^c denote consumption of private commodity c by household h ; e_h^c embodies similar notation for the endowment in private goods. The following notation is also used: $x \equiv (x_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$, where $x_h \equiv (x_h^c)_{c=1}^C$; $e \equiv (e_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$, where $e_h \equiv (e_h^c)_{c=1}^C$; $p \equiv (p^c)_{c=1}^C$, where p^c is the price of private good c ; $\hat{p} \equiv (p, p^g)$, where p^g is the price of the public good; $g \equiv (g_h)_{h=1}^H$, where $g_h \in \mathbb{R}_+$ is the amount of public good that consumer h provides; $G \equiv \sum_{h=1}^H g_h$ and $G_{\setminus h} \equiv G - g_h$.

¹A more detailed version of the paper, containing even the most elementary proofs, is available upon request from the authors.

²The presence of more than one public good can be incorporated into our model, leaving the basic results unchanged.

The preferences over the private goods and the public good of household h are represented by a utility function $u_h : \mathbb{R}_{++}^C \times \mathbb{R}_{++} \rightarrow \mathbb{R}$. Note that households' preferences are defined over the total amount of the public good, i.e., we have $u_h : (x_h, G) \mapsto u_h(x_h, G)$.

Assumption 1 u_h is a smooth, differentiably strictly increasing (i.e., for every $(x_h, G) \in \mathbb{R}_{++}^{C+1}$, $Du_h(x_h, G) \gg 0$)³, differentiably strictly concave function (i.e., for every $(x_h, G) \in \mathbb{R}_{++}^{C+1}$, $D^2u_h(x_h, G)$ is negative definite), and for each $\underline{u} \in \mathbb{R}$ the closure in the standard topology of \mathbb{R}^{C+1} of the set $\{(x_h, G) \in \mathbb{R}_{++}^{C+1} : u_h(x_h, G) \geq \underline{u}\}$ is contained in \mathbb{R}_{++}^{C+1} .

There is a fixed number F of firms⁴, indexed by subscript f , that use a production technology represented by a transformation function $t_f : \mathbb{R}^{C+1} \rightarrow \mathbb{R}$, where $t_f : (y_f, y_f^g) \mapsto t_f(y_f, y_f^g)$.

Assumption 2 $t_f(y_f, y_f^g)$ is a C^2 , differentiably strictly decreasing (i.e., $Dt_f(y_f, y_f^g) \ll 0$), and differentiably strictly concave function, with $t_f(0) = 0$.

For each f , define $\hat{y}_f \equiv (y_f, y_f^g)$, $\hat{y} \equiv (\hat{y}_f)_{f=1}^F$ and $Y_f \equiv \{\hat{y}_f \in \mathbb{R}^{C+1} : t_f(\hat{y}_f) \geq 0\}$, $t \equiv (t_f)_{f=1}^F$ and $\hat{p} \equiv (p, p^g)$.

The following assumption on the production set $Y \equiv \sum_{f=1}^F Y_f$ is made to ensure existence of equilibria.

Assumption 2' (Bounded reversibility) $Y \cap (-Y)$ is bounded.

Using the convention that input components of the vector \hat{y}_f are negative and output components are positive, the profit maximization problem for firm f is: For given $\hat{p} \in \mathbb{R}_{++}^{C+1}$,

$$\text{Max}_{\hat{y} \in \mathbb{R}^{C+1}} \hat{p}\hat{y} \quad \text{s.t.} \quad t_f(\hat{y}) \geq 0 \quad (1)$$

From Assumption 2, it follows that if problem (1) has a solution, it is unique and it is characterized by Kuhn-Tucker (in fact, Lagrange) conditions.

Let s_{fh} be the share of firm f owned by household h . $s_f \equiv (s_{fh})_{h=1}^H \in \mathbb{R}_+^H$ and $s \equiv (s_f)_{f=1}^F \in \mathbb{R}_+^{FH}$. The set of all shares of each firm f is $S \equiv \{s_f \in [0, 1]^H : \sum_{h=1}^H s_{fh} = 1\}$. $s \equiv (s_f)_{f=1}^F \in S^F$. The set of shares $s_h \equiv (s_{fh})_{f=1}^F$ of household h is $[0, 1]^F$.

Note that $s_{fh} \in [0, 1]$ denotes the proportion of profits of firm f owned by household h . The definition of S simply requires each firm to be completely owned by some households.

³For vectors y, z , $y \geq z$ (resp. $y \gg z$) means every element of y is not smaller (resp. strictly larger) than the corresponding element of z ; $y > z$ means that $y \geq z$ but $y \neq z$.

⁴Thus, we assume there is no possibility of new entry of firms into the market.

Household's maximization problem is then the following: For given $\hat{p} \in \mathbb{R}_{++}^{C+1}$, $s_h \in [0, 1]^F$, $e_h \in \mathbb{R}_{++}^C$, $G_{\setminus h} \in \mathbb{R}_+$, $\hat{y} \in \mathbb{R}^{(C+1)F}$,

$$\begin{aligned} & \text{Max}_{(x_h, g_h) \in \mathbb{R}_{++}^C \times \mathbb{R}} u_h(x_h, g_h + G_{\setminus h}) \\ \text{s.t.} \quad & -p(x_h - e_h) - p^g g_h + \hat{p} \sum_{f=1}^F s_{fh} \hat{y}_f \geq 0 \\ & g_h \geq 0 \end{aligned} \tag{2}$$

From Assumption 1, it follows that problem (2) has a unique solution characterized by Kuhn-Tucker conditions.

The set of all utility functions of household h that satisfy Assumption 1 is denoted by \mathcal{U}_h ; the set of all transformation functions of firm f that satisfy Assumption 2 is denoted by \mathcal{T}_f . Moreover define $\mathcal{U} \equiv \times_{h=1}^H \mathcal{U}_h$ and $\mathcal{T} \equiv \times_{f=1}^F \mathcal{T}_f$.

Assumption 3 \mathcal{U} and \mathcal{T} are endowed with the subspace topology of the C^3 uniform convergence topology on compact sets^{5, 6}.

Definition 1 An economy is a vector $\pi \equiv (e, s, u, t) \in \Pi \equiv \mathbb{R}_{++}^{CH} \times S^F \times \mathcal{U} \times \mathcal{T}$.

Observe that the market clearing condition for one good, say good C , is redundant. Moreover, the price of that good can be normalized without affecting the budget constraints of any household. With little abuse of notation, we denote the normalized private and public good prices with $p \equiv (p^{\setminus}, 1)$ and p^g , respectively.

We are now able to give the following definition:

Definition 2 A vector $(x, g, p^{\setminus}, p^g, \hat{y})$ is an equilibrium for an economy $\pi \in \Pi$ if:

1. firms maximize, i.e., for each f , \hat{y}_f solves problem (1) at $\hat{p} \in \mathbb{R}_{++}^{C+1}$;
2. households maximize, i.e., for each h , (x_h, g_h) solves problem (2) at $p^{\setminus} \in \mathbb{R}_{++}^{C-1}$, $p^g \in \mathbb{R}_{++}$, $e_h \in \mathbb{R}_{++}^C$, $G_{\setminus h} \in \mathbb{R}_+$, $s_h \in [0, 1]^F$, $\hat{y} \in \mathbb{R}^{(C+1)F}$; and
3. markets clear, i.e., (x, g, \hat{y}) solves

$$\begin{aligned} -\sum_{h=1}^H x_h^{\setminus} + \sum_{h=1}^H e_h^{\setminus} + \sum_{f=1}^F y_f^{\setminus} &= 0 \\ -\sum_{h=1}^H g_h + \sum_{f=1}^F y_f^g &= 0 \end{aligned} \tag{3}$$

where for each h and f , $x_h^{\setminus} \equiv (x_h^c)_{c \neq C}$, $e_h^{\setminus} \equiv (e_h^c)_{c \neq C} \in \mathbb{R}_{++}^{C-1}$ and $y_f^{\setminus} \equiv (y_f^c)_{c \neq C} \in \mathbb{R}^{C-1}$.

⁵A sequence of functions f^n whose domain is an open set \mathcal{O} of \mathbb{R}^m converges to f if and only if f^n , Df^n , $D^2 f^n$ and $D^3 f^n$ uniformly converge to f , Df , $D^2 f$ and $D^3 f$ respectively, on any compact subset of \mathcal{O} .

⁶In the proof of existence of equilibria the fact that utility and transformation functions are C^2 suffices. We need the stronger form presented in Assumption 3 to apply the Transversality Theorem to a function whose components contain the Hessian of utility and transformation functions.

By definition of u_h , observe that we must have $\sum_h g_h > 0$ and, therefore (i) since $g_h \geq 0$ for all h , there exists h' such that $g_{h'} > 0$; and (ii) $\sum_{f=1}^F y_f^g > 0$. That is, there will exist at least one contributor to the public good and, hence, there will be a strictly positive level of public good production. Note household h is called a contributor if $g_h > 0$, and a non-contributor if $g_h = 0$.

Given our assumptions, we can now characterize equilibria in terms of the system of Kuhn-Tucker conditions to problems (1) and (2), and market clearing conditions (3).

Define

$$\xi \equiv \left(\hat{y}, \alpha, x, g, \mu, \lambda, p \setminus, p^g \right) \in \Xi \equiv \mathbb{R}^{(C+1)F} \times (-\mathbb{R}_{++}^F) \times \mathbb{R}_{++}^{CH} \times \mathbb{R}^H \times \mathbb{R}^H \times \mathbb{R}_{++}^H \times \mathbb{R}_{++}^{C-1} \times \mathbb{R}_{++}$$

and

$$\mathcal{F} : \Xi \times \Pi \rightarrow \mathbb{R}^{\dim \Xi}, \quad \mathcal{F} : (\xi, \pi) \mapsto \text{left hand side of (4) below}$$

$$\begin{aligned} & \dots \\ & \hat{p} + \alpha_f D t_f(\hat{y}_f) & = 0 \\ & t_f(\hat{y}_f) & = 0 \\ & \dots \\ & D_{x_h} u_h(x_h, g_h + G \setminus_h) - \lambda_h p & = 0 \\ & D_{g_h} u_h(x_h, g_h + G \setminus_h) - \lambda_h p^g + \mu_h & = 0 \\ & \min\{g_h, \mu_h\} & = 0 \\ & -p(x_h - e_h) - p^g g_h + \hat{p} \sum_{f=1}^F s_{fh} \hat{y}_f & = 0 \\ & \dots \\ & -\sum_{h=1}^H x_h \setminus + \sum_{h=1}^H e_h \setminus + \sum_{f=1}^F y_f \setminus & = 0 \\ & -\sum_{h=1}^H g_h + \sum_{f=1}^F y_f^g & = 0 \end{aligned} \tag{4}$$

where α_f and λ_h, μ_h are the Kuhn-Tucker multipliers associated to the firm and the household's maximization problems.

Observe that $(\hat{y}, x, g, p \setminus, p^g)$ is an equilibrium associated with an economy π if and only if there exists (α, μ, λ) such that $\mathcal{F}(\hat{y}, \alpha, x, g, \mu, \lambda, p \setminus, p^g, \pi) = 0$. With innocuous abuse of terminology, we will call $\xi \equiv (\hat{y}, \alpha, x, g, \mu, \lambda, p \setminus, p^g)$ an equilibrium.

Using a homotopy argument applied to the above function, Villanacci and Zenginobuz (2005a) prove existence of equilibria.⁷

Theorem 3 *For every economy $(e, s, u, t) \in \mathbb{R}_{++}^{CH} \times \mathcal{S}^F \times \mathcal{U} \times \mathcal{T}$, an equilibrium exists.*

Villanacci and Zenginobuz (2005a) also show that there is a large set of the endowments (the so-called regular economies) for which associated equilibria are

⁷In fact, weaker assumptions on the utility and transformation functions than the ones made in this paper suffice to prove existence and generic regularity of equilibria. However, to prove the comparative statics results of this paper we need the stronger assumptions presented in Assumption 1 and Assumption 2. For an account of these and other technical remarks we make in this paper about existence and generic regularity of equilibria, see Villanacci and Zenginobuz (2005a).

finite in number, and that equilibria change smoothly with respect to endowments. Theorem 4 below summarizes their generic regularity results. To this end, the set of utility functions need to be restricted to a “large and reasonable” subset $\tilde{\mathcal{U}}$ of \mathcal{U} :

Assumption 4 For all h , $x_h \in \mathbb{R}_{++}^C$ and $G \in \mathbb{R}_{++}$, it is the case that

$$\det \begin{bmatrix} D_{x_h x_h} u_h(x_h, G) & [D_{x_h} u_h(x_h, G)]^T \\ D_{G x_h} u_h(x_h, G) & D_G u_h(x_h, G) \end{bmatrix} \neq 0. \quad (5)$$

Let $\tilde{\mathcal{U}}_h$ be the subset of \mathcal{U}_h satisfying Assumption 4 and define $\tilde{\mathcal{U}} \equiv \times_{h=1}^H \tilde{\mathcal{U}}_h$ and $\tilde{\Pi} \equiv \mathbb{R}_{++}^{CH} \times S^F \times \tilde{\mathcal{U}} \times \mathcal{T}$.

Assumption 4 has an easy and appealing economic interpretation. It is easy to see that it is equivalent to the public good being a normal good, as long as the household is a contributor.

Define

$$pr_{(s,u,t)} : (\mathcal{F}_{(s,u,t)})^{-1}(0) \rightarrow \mathbb{R}_{++}^{CH}, \quad pr_{(s,u,t)} : (\xi, e) \mapsto e,$$

that is, $pr_{(s,u,t)}$ is the projection of the equilibrium manifold onto the endowment space. We then have the following result:

Theorem 4 For each $(s, u, t) \in S^F \times \tilde{\mathcal{U}} \times \mathcal{T}$, there exists an open and full measure subset \mathcal{R}^* of \mathbb{R}_{++}^{CH} such that $\forall e \in \mathcal{R}^*$

1. there exists $r \in \mathbb{N}$ such that $\mathcal{F}_{(s,u,t,e)}^{-1}(0) = \{\xi^i \equiv (\hat{y}^i, \alpha^i, x^i, g^i, \mu^i, \lambda^i, p^i, p^{g^i})\}_{i=1}^r$; moreover, there exist an open neighborhood Y of e in \mathbb{R}_{++}^{CH} , and for each i an open neighborhood U_i of (ξ^i, e) in $(\mathcal{F}_{(s,u,t)})^{-1}(0)$, such that $U_j \cap U_k = \emptyset$ if $j \neq k$, $(pr_{(s,u,t)})^{-1}(Y) = \cup_{i=1}^r U_i$ and $pr_{(s,u,t)|U_i} : U_i \rightarrow Y$ is a diffeomorphism.
2. $\forall i$ and $\forall h$, either $g_h^i > 0$ or $\mu_h^i > 0$.⁸

The Theorem says that *typically* - i.e., for almost all the economies - (i) the number of equilibria are finite and, locally, equilibrium variables depends smoothly from the endowments; (ii) no household h is at the “border line case” $g_h = 0$ and $\mu_h = 0$, and therefore, by continuity, small enough changes in endowments do not change the set of contributors.

As indicated by the existence and regularity results we just cited, as well as the setup of our model, we use differential techniques in our analysis. In terms of demonstrating the main results of this paper, this amounts to computing the derivative of the *equilibrium* values of the “goal function” - e.g., the total amount of provided public good or household welfare levels - with respect to some policy tools - e.g., redistributive lump-sum taxes or transfers.⁹ We describe

⁸The proof of the Theorem is contained in Villanacci and Zenginobuz (2005a).

⁹Note that all our arguments will therefore be “local” in their nature.

in some detail the general strategy we use to prove our main results in the proof of Theorem 8¹⁰.

3 Redistribution of Wealth and Quantity of Public Good

In this Section, we show the following results.

1. For all economies, “local redistributions”¹¹ of endowments of a private good *among contributors* do not change the set of equilibria.

2. For a generic, i.e., open and dense, subset of the economies for which there exists at least one non-contributor, there exists a redistribution of endowments of a private good *between contributors and non-contributors* which increases the level of public good provided;

3. For a generic subset of economies for which there exist at least two non-contributors, there exists a redistribution of the endowments of a private good among contributors and non-contributors such that regardless of whether the contributors have more, or less, or the same amount of the private good after redistribution (or, alternatively, regardless of whether their wealth is increased, or decreased, or remain unchanged) the level of public good provided may be increased, or decreased, or left unchanged in a manner completely unrelated to the change in the endowments (or wealth) of contributors.

Note that the last result covers the case where redistribution of endowments involves only the non-contributors, with no change in the endowments or wealth of contributors.

3.1 Redistributions among Contributors

The following theorem is a restatement of a theorem by BBV for the case of many private goods, and its proof is a straight forward adaptation of their proof.

Theorem 5 *Consider an equilibrium associated with an arbitrary economy and a redistribution of the private numeraire good among contributing households such that no household loses more wealth than her original contribution. All the equilibria after the redistribution are such that the consumption of private goods and the total amount of consumed public good are the same as before the redistribution.*

The basic intuition behind Theorem 5 is that if, in an equilibrium *after* the redistribution, each household h expects that in the new equilibrium (i) all prices stay the same; (ii) level of production of private and public goods stay the same; and (iii) each other household $h' \neq h$ changes her contribution by the

¹⁰For more detailed descriptions of the techniques and strategies of proof we use, also see the papers by Cass and Citanna (1998), and Citanna, Kajii and Villanacci (1998).

¹¹By “local redistribution” we mean redistribution in an arbitrary small neighborhood of a given endowment.

exact amount of the change in her wealth, then it will be optimal for household h to change her contribution by the exact amount of the change in her own wealth, leading to overall level of public good remaining the same as before the redistribution.

As a simple Corollary to Theorem 5, we get the following:

Proposition 6 *The set of equilibria after a local redistribution from an arbitrary set of non-contributors to one contributor is equal to the set of equilibria after a local redistribution from that same arbitrary set of non-contributors to an arbitrary set of contributors.*

This result follows from the fact that each equilibrium with only 1 contributor being subsidized can be obtained from each equilibrium with more than one contributor being subsidized using appropriate redistributions among contributors. Making use of this result, we consider lump-sum taxes or transfers on only one contributor in all of the different types of planner interventions we study below.

As we have already stated, the basic difference between BBV's model and ours is the importance of relative prices, which arise due to the presence of more than one private good. To further clarify the role of relative prices, using BBV's transformation of the consumers' problem we can rewrite the demand function for the public good for a contributing household in our case as

$$g_h : \mathbb{R}_{++}^C \times \mathbb{R}_{++}^2 \rightarrow \mathbb{R}, \quad g_h : (p, p^g, w_h + G_{\setminus h}) \mapsto g_h(p, p^g, w_h + G_{\setminus h}).$$

BBV's Fact 2 (see page 34 of BBV) says that *in equilibrium*, there exists a function $B(G, \mathcal{H}^+)$, where \mathcal{H}^+ is the set of contributors, which is differentiable and increasing in G and such that $B(G, \mathcal{H}^+) = \sum_{h=1}^H w_h$. In fact, it turns out that

$$B(G, \mathcal{H}^+) = \sum_{h \in \mathcal{C}} \Psi_h(G) + (1 - \#\mathcal{H}^+) G$$

where $\Psi_h(\cdot)$ is an increasing function of G . Since the above equation holds in equilibrium, BBV's statements in Fact 2 are true as long as a change in G does not change \mathcal{H}^+ . In fact, what BBV does amounts to computing the *total derivative* of $B(\cdot)$ with respect to G *in equilibrium*. In other words, they analyze how a small change in G changes the function $B(\cdot)$ when G moves in the equilibrium set. Clearly, if G changes, some other endogenous variable will in general change in equilibrium. In their case the only other relevant endogenous variable is \mathcal{H}^+ . But in our case, following the same procedure as theirs, the function $B(\cdot)$ is

$$B(G, \mathcal{H}^+, p, p^g) = \sum_{h \in \mathcal{C}} \Psi_h(G, p, p^g) + (1 - \#\mathcal{H}^+) G$$

and, in general, changing G in the equilibrium set will change the values of prices, too. Therefore, the total derivative of $B(\cdot)$ with respect to G will have to contain partial derivatives with respect to prices. In fact, since we deal with

”small” redistributions of income the contributing set \mathcal{H}^+ (generically) does not change in our case, and the only channel through which changes in the overall public good level occur is the relative prices.¹²

3.2 Redistributions between Contributors and Non-contributors

We now look at the case in which the planner redistributes endowments of one private good between a (strictly) contributing household, say $h = 1$, and one or two (strictly) non-contributing household, say $h = 2, 4$.

First, the reader can check that the set of economies for which there exists at least one or two non-contributors is open (and non-empty).¹³ The following theorem covers the case where redistribution involves at least one contributor and one non-contributor to the public good.

Theorem 7 *For an open and dense subset S_1^* of the set Π' of the economies for which there exists at least one non-contributor, there exists a redistribution of the endowments of private good C between one contributor and one non-contributor which increases (or decreases) the equilibrium level of provided public good.*

We omit the proof of Theorem 7 since it is similar to that of Theorem 8, which is provided in detail below.

Theorem 7 generalizes to our framework the similar non-neutrality result by BBV - see their Theorem 4, part (ii)) - where they show that a redistribution of income from non-contributors to contributors will affect the level of public good.

The main result of our paper which is presented below looks at the case where there is redistribution from contributors to non-contributors as well as the case where redistribution takes place only among non-contributors.

Theorem 8 *Assume that $C \geq 2$, and let Π'' be the set of the economies for which there exist at least two non-contributors. There exist open and dense subsets S_2^* and S_3^* of Π'' with the following properties:*

a. For any economy in S_2^ and any equilibrium associated with it, there exists a redistribution of the endowments of a numeraire good among one contributor and two non-contributors such that, regardless of whether the contributor is given more, or less, or the same amount of the numeraire good, the level of public good provided may be increased, or decreased, or left unchanged in a manner completely unrelated to the change in the numeraire good endowment of the contributor;*

¹² Another point about BBV’s above result is that it amounts to showing that set of equilibria before a local redistribution among contributors is a subset of the set of equilibria after a local redistribution among contributors. In fact, the set inclusion in the opposite direction also holds. This is so because an economy after a local redistribution is an economy sufficiently close to the starting one, and, with regularity, the number of equilibria is locally constant.

¹³ The proof for openness is contained in the extended version of the paper, which is available from authors upon request. (Non-emptiness follows from a Cobb-Douglas utility function exercise.)

b. For any economy in S_3^* and any equilibrium associated with it, there exists a redistribution of the endowments of a numeraire good among one contributor and two non-contributors such that regardless of whether the total wealth (in the post-redistribution equilibrium) of the contributor is increased, or decreased, or left constant, the level of public good provided may be increased, or decreased, or left constant in a manner completely unrelated to the change in the wealth of contributor.

A better understanding of Theorem 8 can be gained reading the proof we give below.

Proof of Theorem 8. First of all, we define a new equilibrium function $F_1(\xi, \rho, \pi)$, taking into account the impact of planner's intervention on agents' behaviors via the policy tools $\rho \equiv (\rho_h)_{h=1,2,3} \in \mathbb{R}^3$, where ρ_h denotes the lump-sum tax, in terms of the numeraire good, imposed on household h ; household 1 is assumed to be a contributor, and households 2 and 3 are non-contributors.

$$\mathcal{F}_1 : \Xi \times \mathbb{R}^3 \times \Pi \rightarrow \mathbb{R}^{\dim \Xi}, \quad \mathcal{F}_1 : (\xi, \rho, \pi) \mapsto (\text{Left Hand Side of (6) below})$$

$$\begin{aligned} \widehat{p} + \alpha_f D t_f(\widehat{y}_f) &= 0 \\ t_f(\widehat{y}_f) &= 0 \\ \dots & \\ D_{x_h} u_h(x_h, g_h + G_{\setminus h}) - \lambda_h p &= 0 \\ D_{g_h} u_h(x_h, g_h + G_{\setminus h}) - \lambda_h p^g + \mu_h &= 0 \\ \min\{g_h, \mu_h\} &= 0 \\ -p(x_h - e_h) + \rho_h - p^g g_h + \sum_{f=1}^F s_{fh} \widehat{p} \widehat{y}_f &= 0 \\ \dots & \\ -\sum_{h=1}^H x_h \setminus + \sum_{h=1}^H e_h \setminus + \sum_{f=1}^F y_f \setminus &= 0 \\ -\sum_{h=1}^H g_h + \sum_{f=1}^F y_f^g &= 0 \end{aligned} \tag{6}$$

Observe that the only difference between the equilibrium system without planner intervention, i.e. system(4), and the above system is the presence of the lump-sum tax ρ_h in household h 's budget constraint.

We then introduce a function $\mathcal{F}_2(\xi, \rho, \pi)$, describing the constraints on the planner intervention, i.e., the fact that the planner can only *redistribute* resources among households:

$$\mathcal{F}_2 : \Xi \times \mathbb{R}^3 \times \Pi \rightarrow \mathbb{R}^2, \quad \mathcal{F}_2 : (\xi, \rho, \pi) \mapsto \rho_1 + \rho_2 + \rho_3$$

Zeros of the function $\widetilde{\mathcal{F}} \equiv (\mathcal{F}_1, \mathcal{F}_2)$ are "equilibria with planner's intervention".

We then partition the vector ρ of tools into two subvectors $(\rho_1, \rho_2) \in \mathbb{R}^2$ and $\rho_3 \in \mathbb{R}$, where (ρ_1, ρ_2) can be interpreted as the vector of independent tools and ρ_3 as the value of the dependent tool: once the values of the first two tools is chosen, the value of the third one is uniquely determined. Observe that there exists $(\bar{\rho}_1, \bar{\rho}_2)$ (and associated $\bar{\rho}_3$) at which equilibria with and without planner's intervention coincide: this value is simply *zero*.

We then define a goal function $G(\xi, \rho, \pi)$. To fix ideas we can take the goals as described in part a. of the Theorem: the level of lump-sum tax on the

contributor and the total provision of the public good. Therefore, we define

$$G : \Xi \times \mathbb{R}^3 \times \Pi \rightarrow \mathbb{R}, \quad G : (\xi, \rho, \pi) \mapsto \left(\rho_1, \sum_{h=1}^H g_h \right)$$

We want to analyze the local effect of a change in (ρ_1, ρ_2) around $(\bar{\rho}_1, \bar{\rho}_2)$ on G when its arguments assume their equilibrium (with planner intervention) values.¹⁴ We proceed through the following more technical steps.

We show¹⁵ that for every (s, u, t) , there exists an open and full measure subset Ω of R_{++}^{CH} such that for every $e \in \Omega$ and for every $\xi \in \Xi$ such that $\tilde{\mathcal{F}}(\xi, (\bar{\rho}_1, \bar{\rho}_2), \bar{\rho}_3, e, s, u, t) = 0$, there exist an open neighborhood N of $(\bar{\rho}_1, \bar{\rho}_2)$ and a unique C^1 function

$$h(\bar{\rho}_1, \bar{\rho}_2) \equiv (\xi(\bar{\rho}_1, \bar{\rho}_2), \rho_3(\bar{\rho}_1, \bar{\rho}_2))$$

defined on N , such that

$$\tilde{\mathcal{F}}(\xi(\rho_1, \rho_2), \rho_3(\rho_1, \rho_2), (\rho_1, \rho_2), \pi) = 0$$

The function h describes how equilibrium variables ξ and the dependent tool ρ_3 adjust to (small) changes in planner's independent tools (ρ_1, ρ_2) . In particular the component $g_h(\rho_1, \rho_2)$ of $\xi(\rho_1, \rho_2)$ says how the equilibrium-with-planner-intervention value of g_h reacts to the planner policy. Therefore the function

$$g : N \rightarrow \mathbb{R}^2, \quad (\rho_1, \rho_2) \mapsto \left(\rho_1, \sum_{h=1}^H g_h(\rho_1, \rho_2) \right)$$

describes how the goal values change when the planner uses her policy tools and variables move in the equilibrium set defined by \tilde{F} .

The purpose of the analysis is to show that there exists an open and dense subset $S^* \subseteq \Pi$ such that for each $\pi \in S^*$, the planner can “move” the equilibrium values of the goal function in any directions locally around $g(\bar{\rho}_1, \bar{\rho}_2)$, the value of the goal function in the case of no intervention.¹⁶ More explicitly, we want to show that the function g has the following property.

Take an arbitrary pair of signs in the changes Δ of the equilibrium-with-planner-intervention values of ρ_1 and $\sum_h g_h$, say $\Delta\rho_1 = \rho_1 > 0$, i.e. a positive lump-sum tax on the contributor, and $\Delta(\sum_h g_h) > 0$, i.e., an increase in the level of provided public good.

¹⁴If the goal of planner intervention is to potentially change the wealth of all contributors, as in part *b.* of the Theorem, the first argument of the function G becomes

$$\sum_{h \in \mathcal{H}^+} \left(pe_h + \sum_{f=1}^F s_{fh} \tilde{p} \tilde{y}_f \right) + \rho_1$$

¹⁵That result is a consequence of the transversality theorem - see, for example, Hirsch (1976), Theorem 2.7, page 79 - and of the fact that zero is a regular value for $\tilde{\mathcal{F}}$.

¹⁶In other words, we want to show that for any direction of movement away from $g(\bar{\rho}_1, \bar{\rho}_2)$ and for any neighborhood $N_1 \subseteq N$ of $(\bar{\rho}_1, \bar{\rho}_2)$, there exists a point $(\rho_1^*, \rho_2^*) \in N_1$ such that $g(\rho_1^*, \rho_2^*)$ belong to that direction.

Then there exists a well chosen pair of values of ρ_1 and of ρ_2 that “induces changes” in $\sum_h g_h$, besides ρ_1 itself, of the desired signs.

The above describe property is what is sometimes called essential surjectivity, which is defined as follows: g is essentially surjective at $(\bar{\rho}_1, \bar{\rho}_2)$ if the image of each open neighborhood of $(\bar{\rho}_1, \bar{\rho}_2)$ contains an open neighborhood of $g((\bar{\rho}_1, \bar{\rho}_2))$ in \mathbb{R}^2 .

A sufficient condition for g to be essentially surjective is that¹⁷

$$\text{rank} [D_{(\rho_1, \rho_2)}g((\bar{\rho}_1, \bar{\rho}_2))] = 2 \quad (7)$$

Verification of the above condition is straightforward but quite involved technically, and, therefore, it is omitted.¹⁸

In their theorem 4, BBV say that if we restrict the analysis to changes in the wealth distribution that leave unchanged the set of contributors the following result is true: Any change in the wealth distribution that leaves unchanged (increase) the aggregate wealth of current contributors will leave unchanged (increase) the equilibrium supply of the public good. As a consequence of Theorem 8, BBV’s statements do not hold true in our model. We noted in the Introduction that, unlike BBV’s one private good model, in our model with more than one private good the expressions “redistributing the numerarie good in favor of household h ” and “increase the wealth of household h (in the equilibrium after redistribution)” are not equivalent to each other. Parts a. and b. in Theorem 8, respectively, cover redistributions in the first and the latter senses. Transfers of the numeraire good that involves only the non-contributors, without any transfer to or from any of the contributors (or, without any change in their wealth), can move G in any direction. Even transferring numeraire good from a contributor or decreasing her wealth can move G in any direction. ■

The requirement $C \geq 2$ in the theorem brings out the importance of having more than one private good in obtaining non-neutrality results in our analysis.¹⁹ To see why having more than one private good is essential to affect relative price changes, consider the case of one public and one private good. Redistributing the private good among non-contributors will not change the demand of the public good because contributors are not affected by this intervention and non-contributors do not become contributors (because, generically, we are not on the border line cases and taxes are small). Therefore, there will also be no change in the overall demand for the single private good. With no other private good available, the overall effect is just a reallocation of the demand for the private good from a non-contributor to another.

¹⁷See for example, Chapter 1 in Golubitsky and Guillemin (1973).

¹⁸In fact, the differentiability strict concavity of both the utility and the transformation functions, which implies that the associated Hessian matrices have full rank, is used to show that the above mentioned rank condition holds.

¹⁹More technically, the condition on the number of goods is required in the proof of condition (7).

4 Discussion and Concluding Remarks

The interest in a general equilibrium model with private provision of public goods lies in the fact that it serves as a benchmark extension of an analysis of completely decentralized private good economies to public good economies. Moreover, there are some relevant situations in which public goods are in fact privately provided: e.g., private donations to charity at a national and international level, campaign funds for political parties or special interests groups, and certain economic activities inside a family.

With only one private good, assuming constant returns to scale, and therefore linearity of the production function, implies that profits of firms are equal to zero, and the presence of firms basically plays no role in the model.

With more than one private good and non-constant returns to scale, modeling of how the public good is produced becomes an issue. If a profit-maximizing (private) firm is assumed to produce the public good, then how the (non-zero) profits of the firm are apportioned among its shareholders will have an impact on equilibrium outcomes. Alternatively, one can consider the production of the public good as being carried out by a non-profit (public) firm subject to a balanced budget constraint. In that case the contributions in monetary amounts collected from households would finance the cost of producing the public good. The amount of public good to be produced by the non-profit firm can be taken as the maximum amount that can be produced with the amount collected.²⁰

In the present paper, we studied the alternative that we believe is the one most consistent with a decentralized framework, namely that profit-maximizing firms produce the public good in a competitive market. Thus, the government was not involved in the production of the public good and only had the role of enforcing lump-sum taxes and transfers on households and firms. It was also assumed that government made purchases from the firms at market prices. We took this set of assumptions as describing a completely free market oriented policy benchmark applicable in principle to provision of any type of public good. The model can also be seen as a descriptive one covering cases in which a public institution purchases from private producers goods that will be consumed by households involved as public goods. Examples include fluoride purchased by a public agency to fluoridate a public water supply, pesticides purchased by government, packages of medicine bought by an international charitable organization for use in an underdeveloped country to control an epidemic disease, and so on.²¹

²⁰Villanacci and Zenginobuz (2005b) analyze such a model.

²¹An interesting feature of the model analyzed in the paper is that it allows an answer to a more general problem. Does including one more “market imperfection” in the presence of an initial one make government intervention more or less effective? It is well known that, under certain assumptions, a well chosen local redistribution among all households in a model with incomplete markets leads to a Pareto superior equilibrium (see the papers by Geanakoplos and Polemarchakis (1986) and Citanna et al. (1998)). On the other hand, neutrality results to the effect that when all households are contributors no local redistribution affects equilibria apply to general equilibrium models with incomplete markets *and* public goods. These observations suggest that a government intervention that would be effective against a single imperfection

One other issue that is widely considered in the private provision of public good models is the *crowding-out* effects. There is said to be a crowding-out effect if, when the government taxes household and uses collected taxes to increase the amount of public good, households reduce their provision, partially or completely, thereby offsetting the intervention by the government to increase the overall level of the public good. The results we have obtained so far can be used to cover certain cases regarding crowding-out effects. For the same reasons presented in Section 3.1, if the planner taxes only the contributors (in lump-sum amounts), crowding-out effect will be total. If, in an equilibrium after the redistribution, each contributing household h expects that in the new equilibrium (i) all prices stay the same; (ii) level of production of private and public goods stay the same; (iii) the government uses taxes to purchase public good at market prices; and (iv) each other household $h' \neq h$ changes her contribution by the exact amount of the change in her wealth, then it will be optimal for household h to change her contribution by the exact amount of the change in her own wealth, leading to overall level of public good remaining the same as before the taxation. On the other hand, in a similar manner to what was stated and shown in Theorem 7 and Theorem 8, it can be proved that if taxes that the government will use to purchase public goods are imposed jointly on non-contributors and contributors, the total supply of public good may increase or decrease.

A final point to mention is about the welfare analysis of government interventions that aim to alter equilibrium outcomes under private provision of public goods. Most of the related literature analyzes the impact of government intervention on the total level G of privately provided public good, which is known to be (almost always) underprovided relative to the “efficient” level.²² The underlying assumption behind that approach is that “welfare” will increase through bringing G closer to the “efficient” level. There are two problems with such an indirect approach to welfare analysis in private public good provision models. Firstly, the efficient level of public good is generically not independent of distribution of endowments, and hence an intervention that involves redistribution of endowments will in general alter the “efficient” level that the intervention is aimed at. Secondly, if intervention involves taxing non-contributors (in lump-sum amounts), then welfare analysis becomes more complicated as taxing non-contributors to increase the public good level may decrease their welfare. Therefore, a direct approach to welfare analysis will be needed to study interventions that has the goal of Pareto improving upon the market outcome. Our techniques allow for such an exploration of types of government policies that will achieve that aim.²³

may turn out to be ineffective when another imperfection is present.

²²An exception is a paper by Cornes and Sandler (2000), where they investigate, in a one private good and one public good setting, the possibility for a government to increase all households’ welfare via an increase in the total supply of the public good.

²³Villanacci and Zenginobuz (2005c) provides an analysis of such policies.

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