College Admissions under Early Decision

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In this paper, we model college admissions under early decision in a many-to-one matching framework with two periods. We show that there exists no stable matching system, involving an early decision matching rule and a regular decision matching rule, which is nonmanipulable via early decision quotas by colleges or via early decision preferences by colleges or students. We then analyze the Nash equilibria of the game, in which the preferences of colleges and students in each period are common knowledge and every college determines a quota for the early decision period given its total capacity for the two periods. Under college-optimal and student-optimal matching systems, we show that a pure strategy equilibrium may not exist. However, when colleges or students have common preferences over the other set of agents, 'terminating early decision program' becomes a weakly dominant strategy for each college if every student, choosing to act early, always applies early to his or her top choice college.

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1 Introduction

The competition among colleges to attract high quality students has led to the adoption of early admissions programs over the last five decades, and turned the college admissions process into a complicated 'admissions game' in the United States.[3] Today the most prominent colleges offer a choice over a variety of admissions programs: 'early action', 'single-choice early action', 'early decision', and 'regular decision'. According to the *2005 Early College Application Directory* of the National Association for College Admission Counseling (NACAC), of all four-year postsecondary institutions about seven percent offer early decision plan while about eight percent offer early action plan.[4] These figures are indeed much higher for top private institutions. As documented by Avery et.al. (2003), nearly 70 percent of the 281 private institutions ranked by the *U.S. News College Guide* as the "Best National Universities" and the "Best Liberal Arts Colleges" in the United States offer an early admissions program.

While early action allows a student a chance to gain an admission decison in the fall without a commitment to attend, early decision requires the student to apply early to only one college and matriculate if admitted. Single-choice early action programs restrict the student to apply to the early action programs of only one college. Early action and decision programs usually require highschool seniors to apply near November with a decision by late December. Regular decision offers a later application deadline (January 1) and time to decide whether to matriculate until the National Common Reply Date (May 1).[5]

[3]In 1954-55 Harvard, Princeton and Yale introduced the A-B-C program that give students of feeder highschools, from where these colleges mostly recruited, a preliminary indication of the likely outcome of their applications. In 1959, Barnard, Bryn Mawr, Mount Holyoke, Radcliff, Smith, Vassar, and Wellesley introduced the Early Decision programs. See Avery et.al. (2003) for an extensive account of the history of early admissions in United States.

[4]*NACAC 2005 Early College Application Directory* also indiates that the institutions "... most likely to offer early decision include highly selective colleges (50 percent), colleges with lower yield rates (24 percent of colleges that enroll less than 30 percent of admitted applicants and 27 percent of colleges that enroll 30 to 45 percent of admitted applicants), colleges that enroll fewer than 3,000 students (25 percent), colleges in the New England (38 percent) and Middle States regions (32 percent), and private colleges (26 percent)." (See page 23 of *State of College Admission 2006* at http://www.nacacnet.org.)

[5]For 2007-2008 academic year, MIT offers early action and regular action, Harvard and Yale offer single-choice early action and regular decision, Princeton and UPenn offer single-

In the face of this highly sophisticated admissions process, a highschool senior considering higher education has to decide not only which colleges to apply but also when to apply. His or her decision on which early programs to apply is even further clouded by the lack of information on whether he or she will be eligible for financial aid and if so how much, since colleges do not finalize the aid package until regular decision period.[6] The growing complexity of college admissions game stirred public concern over the competitiveness and the efficiency of the higher education market which is substantiated by a recent study by Avery et.al. (2003). This study finds that early decision applicants gain an admission advantage by applying early that is equivalent of 100 additional SAT points. It also confirms that these applicants tend to come from affluent socioeconomic backgrounds, while students seeking for financial aid postpone their admission to the regular decision period.

Being the first to introduce early action program over 30 years ago, Harvard College recently announced that it will eliminate early admissions beginning in the fall of 2007 with the purpose of making the admissions process simpler and fairer and the hope of being a leading example again, only this time to reverse the course.[7] A week after, Princeton University and University of Virginia, independently, announced that they will also end their early admissions (decision) programs and admit all undergraduates through a single process, beginning next year.[8]

Motivated by these recently sparked controversies about the early admissions programs, we study in this paper college admissions problem, introduced by Gale and Shapley (1962) and reformulated by Roth (1985), in a model with early decision and analyze the strategic issues involved. Our model considers many-to-one matching problems (markets) involving two periods: an early decision period and a regular decision period. There are two finite and disjoint sets of agents, say colleges and students. Each college has a finite overall capacity that limits the number of students it can accept in

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choice early decison and regular decision to name a few. Although some colleges drop the 'single-choice' qualifier in their early admissions programs and call it early action (as done by Harvard), they indicate that they keep the right to rescind their offer of admission to a student who applies elsewhere for early action programs.

[6] The elimination of "the Overlap" process in the early 90s by the Justice Department on antitrust grounds has subsequently increased competition among schools via financial aid bidding. See Hoxby (2000).

[7] "Harvard to Eliminate Early Admission," *Harvard Gazette*, September 14, 2006.

[8] News@Princeton and UVA Today, November 25, 2006.

the two periods, and each student can enroll to at most one school during the whole matching process. In the regular decison period, each college has a preference relation over the set of groups of students which is responsive to its preference over the set of students and each student has a preference relation over the set of colleges and being unmatched. The capacities of colleges together with the preference profiles of colleges and students in the regular decision period constitute a regular decision market.

In the early decision period, each college announces out of its total capacity an early decision quota, which it aims to fill with respect to its early decision preference ordering that is responsive to some restriction of its regular decision preference ordering on a subset of students. On the other side of the market, an admissible preference ordering of each student in the early decision period is the restriction of his or her preference ordering in the regular decision period on a *singleton* subset of colleges. The quotas of colleges together with the preference profiles of colleges and students in the early decision period define an early decision market. Clearly, for each regular decision market, there is a set of induced early decision markets.[9]

An allocation in the early decision period is a many-to-one *early decision matching* where no college is assigned more students than its early decision quota and no student is assigned more than one college. Given a binding early decision matching, an allocation in the regular decision period is a many-to-one *regular decision matching* where all the assignments realized in the early decision are preserved, no college is assigned more students than its overall capacity and no student is assigned more than one college. We call any assignment that did not exist in the early decision period and realize in the regular decision period as a *regular assignment.* We assume that any student rejected from a college in the early decision period can still apply to the same college in the regular decision period.[10]

A matching in the early decision period is stable if no student prefers remaining unassigned to his or her assignment, no college prefers having a student slot vacant rather than filling it with one of its assignments, and

[9] Many colleges and universities have priority categories for athletes, alumni children, and minorities. We assume that the total capacity of each college in our model is net of its priority quota.

[10] As pointed out by Avery et.al. (2003), "...historically most colleges rejected 5 percent or fewer of their early applicants in December. Some, such as Cornell, Georgetown, MIT, and Tufts, have automatically deferred to the regular pool all early applicants who are not admitted in December."

there exists no unmatched college-student pair such that the college prefers the student to one of its assignments or keeping a vacant slot (if any) or the student prefers the college to his or her assignment. Given a matching realized in the early decision period, a matching in the regular decision period is stable if no student having a regular assignment prefers remaining unassigned to his or her assignment, no college prefers having a regularly assigned student slot vacant rather than filling it with one of its regular assignments, and there exists no unmatched college-student pair such that the college prefers the student to one of its regular assignments or keeping a vacant slot (if any) or the student prefers the college to his or her regular assignment.

An early decision matching rule selects a matching for every early decision market, and is stable if it selects a stable matching for every early decision market. A regular decision matching rule selects a matching for every regular decision market, given any matching in any early decision market induced by the associated regular decision market. A regular decision matching rule is stable at an early decision matching rule if it selects a stable matching for every regular decision market, given any realization of the early decision matching rule in any early decision market induced by the associated regular decision market.

An early decision rule and a regular decision rule as an ordered pair form a *matching system*. A matching system is stable if it involves a stable early decision rule at which the regular matching rule in the system is also stable.

We first study manipulation of a matching system via early decision quotas and preferences, and show that there is no matching system that is stable and either nonmanipulable by colleges via early decision quotas or nonmanipulable by colleges or students via early decision preferences.

Next, we analyze the Nash equilibria of the game in which the preferences of colleges and students are common knowledge and each college determines a quota for the early decision period given its total capacity for the two periods. Under college-optimal and student-optimal matching systems, we show that there may not be a pure strategy equilibrium. So, we restrict preferences to ensure the existence of pure strategy equilibria. We prove that when either colleges or students have common preferences over the other set of agents, 'terminating early decision program' becomes a weakly dominant strategy for each college if every student, choosing to act early, applies to his or her top choice college irrespective of the early decision quotas of colleges.[11] Relaxing

[11]Assumption of common preferences for colleges is justified, in page 29 of *State of*

the said restriction on the early decision preferences of students appropriately, we can obtain 'offering as well as not offering an early decision program' as a Nash equilibrium strategy of the early decision quota reporting game.

Our results can be related to those in the literature dealing with manipulation of preferences or quotas under two-sided matching in a single-period. Roth (1982) shows that there is no stable matching rule which is immune to preference manipulation. Mongell and Roth (1990) report a high percentage of truncated preference profiles (single alternative preference) submitted in sorority rush. Roth and Vande Vate (1991) show that in a decentralized one-to-one matching with random matching process, for any strategies of the other players, each player will always have a truncation strategy as a best response. Roth and Rothblum (1999) introduce the truncation of the true preferences as a potentially profitable strategic behavior, instead of changing the order of true preferences, in a low information environment in one-to-one matchings. Sönmez (1999) shows that there is no stable matching rule in hospital-intern markets which is immune to manipulation via early contracting (unraveling) between a hospital and a single intern.[12]

Definitely, the closest papers to ours are by Sönmez (1997) and Konishi and Ünver (2006). Sönmez (1997) shows that in a single-shot hospital-intern market there is no stable matching rule that is nonmanipulable by hospitals via underreporting capacities. Konishi and Ünver (2006) study a capacity manipulation game for hospital-intern markets with a single decision period. They show that under two most widely used matching rules, namely hospital-optimal and intern-optimal stable rules, there may not be a pure strategy equilibrium in general, and whenever a pure strategy equilibrium exists, every hospital weakly prefers this equilibrium outcome to the outcome of any larger capacity profile. Konishi and Ünver (2006) also consider two restrictions on preferences, each of which guarantees the existence of a pure strategy equilibrium. The first restriction requires hospitals to always prefer

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College Admission 2006 at http://www.nacacnet.org, by the common top factors in the college admission decision for all colleges and universities enlisted as: grades in college preparatory courses, admission test scores (such as ACT or SAT), and overall grades. However, arguments for common preferences for students are less appealing because of the students' concerns over the locations of, and the financial aid packages offered by, colleges.

[12] Unraveling was previously studied by Roth and Xing (1994) showing that the instability of matchings realized at the final date of transactions are neither necessary or sufficient for the unraveling to occur. The two potential causes of unraveling are evolving uncertainty and the exercise of market power.

a larger set of acceptable interns to a smaller set. By that, reporting the number of assigned interns is an equilibrium strategy if the matching rule is hospital-optimal, whereas under the intern-optimal matching rule reporting the actual capacity is a weakly dominant strategy. The second restriction requires common preferences of one group of agents (hospitals or interns) over the other group and ensures that reporting the true capacity is always a weakly dominant strategy for colleges.

Although, we benefit from Sönmez (1997) and Konishi and Ünver (2006) both in the exposition and the analysis of our model, we depart from them as well as from the rest of the literature in a significant respect: we are not interested in the strategic incentives of colleges (corresponding to hospitals in some previous studies) in reporting their total capacities. In our model, the total capacity of each college over the two-period admissions process is fixed and common knowledge. Leaving aside the strategic considerations about preferences that we partially address in this study, what colleges rather determine here is the strategic allocation of their total available capacities over the two admission periods. In this regard, our paper aims to obtain some belated insights about the early decision system at a time of possible termination in the entire United States.

The organization of the paper is as follows: Section 2 introduces the model. Section 3 gives results on manipulability of matching systems via preferences and quotas. Section 4 defines an early decision quota game and characterizes restrictions on preferences to ensure the existence of a pure strategy equilibrium. Finally Section 5 concludes.

2 Model

2.1 Basic Structures

We consider many-to-one matching problems (markets) involving two periods: an early decision period and a regular decision period. A matching (college admissions) market is denoted by the list $(C, S, q^R, q^E, R^R, R^E)$.[13] The first two components are non-empty, finite and disjoint sets of colleges $C = c_1, c_2, ..., c_m$ and students $S = s_1, s_2, ..., s_n$. The third component is a

[13] This exposition of a matching market as well as a number of definitions that we shall introduce below simply extend the basic structures in the one-stage matching model of Sönmez (1997), studying capacity manipulation problem in hospital-intern markets.

vector of positive natural numbers $q^R = (q^R_{c_1}, ..., q^R_{c_m})$, where q^R_c is the total capacity of college c during the whole decision process involving both the early decision and the regular decision. The fourth component is a vector of nonnegative natural numbers $q^E = (q^E_{c_1}, ..., q^E_{c_m})$, where $q^E_c \in \{0, 1, ..., q^R_c\}$ is the quota of college c in the early decision period. The fifth component is a list of preference relations $R^R = (R^R_{c_1}, ..., R^R_{c_m}, R^R_{s_1}, ..., R^R_{s_n})$ where R^R_c is the preference relation of college c and R^R_s is the preference relation of student s in the regular decision period. (We will sometimes call these preferences simply as "regular preferences".) The last component in the matching problem, R^E, denotes the preference relations of colleges and students in the early decision period. The lists (q^E, R^E) and (q^R, R^R) for a given society $\langle C, S\rangle$ are called *the early decision market* and *the regular decision market*, respectively.

For any $c \in C$, R^R_c is a binary preference relation that is a linear order on $\Sigma^R_c = 2^S$. Similarly, for any $s \in S$, R^R_s is a binary relation that is a linear order on $\Sigma^R_s = \{\{c_1\}, \{c_2\}, ...\{c_m\}, \emptyset\}$. Let $\mathcal{R}^R_c$ and $\mathcal{R}^R_s$ respectively denote the class of all preference relations for college $c \in C$ and for student $s \in S$, and let P^R_k denote the strict preference relation associated with the preference relation R^R_k for agent $k \in C \cup S$. Define also $\mathcal{R}^R = \prod_{k \in C \cup S} \mathcal{R}^R_k$.

The preference relation R^R_c of college $c \in C$ is said to be *responsive* (Roth, 1985) whenever for all $S' \subset S$ it is true that

i) for all $s \in S \backslash S'$, $S' \cup \{s\} \, P^R_c \, S'$ if and only if $\{s\} P^R_c \emptyset$,

ii) for all $s, s' \in S \backslash S'$, $S' \cup \{s\} \, P^R_c \, S' \cup \{s'\}$ if and only if $\{s\} P^R_c \{s'\}$.

For any college c and any $R^R_c \in \mathcal{R}^R_c$, *projection* of R^R_c over $S \cup \{\emptyset\}$ is denoted by $R^R_c[S]$ and satisfies

i) for all $s \in S$, $\{s\} \, R^R_c[S] \, \emptyset$ if and only if $\{s\} \, R^R_c \, \emptyset$,

ii) for all $s, s' \in S$, $\{s\} \, R^R_c[S] \, \{s'\}$ if and only if $\{s\} \, R^R_c \, \{s'\}$.

Any responsive preference $R^R_c \in \mathcal{R}^R_c$ is said to be *responsive to* $R^R_c[S]$. (Notice that preferences of students over the individual colleges are trivially responsive. We will usually represent strict preferences of the agents by the ordered, from top to bottom, list of acceptable mates.)

Define for all $q^R \in N^n_+$, $\mathcal{Q}^E_c(q^R) = \{0, 1, ..., q^R_c\}$ for $c \in C$ and $\mathcal{Q}^E(q^R) = \prod_{c \in C} \mathcal{Q}^E_c(q^R)$. Let $\mathcal{Q}^E = \cup_{q^R} \mathcal{Q}^E(q^R)$.

For any $c \in C$, let $\Sigma_c^E \subseteq \Sigma_c^R$. Then for a given $R_c^R \in \mathcal{R}_c^R$, let $\mathcal{R}_c^E(R_c^R)$ denote the class of all preference relations of college c such that any $R_c^E \in \mathcal{R}_c^E(R_c^R)$ is a *restriction* of R_c^R to Σ_c^E. For convenience, introduce, for all $c \in C$, the notation $\mathcal{R}_c^E(R_c^R, q^E) = \mathcal{R}_c^E(R_c^R)$ for all $q^E \in \mathcal{Q}^E(q^R)$ and for all $q^R \in N_+^n$.

At this point, we introduce the following notation that will be useful. For any finite set X, any linear order R defined over $X \cup \{\emptyset\}$, and any positive integer $l \leq |X| + 1$, denote by $Top(R; l)$ the *lth-ranked element, from top,* of $X \cup \{\emptyset\}$ under R.

For any $s \in S$, let $\Sigma_s^E \subseteq \Sigma_s^R$ such that $|\Sigma_s^E| \leq 1$. Then for a given $R^R \in \mathcal{R}^R$, the early decision preference, R_s^E, of any student s is a *restriction* of R_s^R to Σ_s^E. Moreover, we assume that for any $s \in S$, R_s^E is *participatory*; i.e., for given $R^R \in \mathcal{R}^R$, $q^R \in N_+^n$, and $q^E \in \mathcal{Q}^E(q^R)$, we have $Top(R_s^E; 1) \in C$ whenever $R_c^E = R_c^R$ and $q_c^E = q_c^R$ for all c. Note that the assumption of participatory early decision preferences is minimally restrictive in the sense that it requires every student to apply early one of the colleges in the college admissions market when he or she does not distinguish between the early decision and regular decision periods in terms of the preferences and quotas of colleges.[14] Let $\mathcal{R}_s^E(R_s^R)$ denote the class of all such preference relations of student s. Thus, we have constructed (single-choice) early decision preferences. For convenience, introduce, for all $s \in S$, the notation $\mathcal{R}_s^E(R_s^R, q^E) = \mathcal{R}_s^E(R_s^R)$ for all $q^E \in \mathcal{Q}^E(q^R)$, given any $q^R \in N_+^n$.

For all $k \in C \cup S$, define $\mathcal{R}_k^E = \cup_{(R_k^R, q^E)} \mathcal{R}_k^E(R_k^R, q^E)$. For any $k \in C \cup S$, and any $R_k^E \in \mathcal{R}_k^E$, denote by P_k^E the respective strict preference relation. Also define $\mathcal{R}^E(R^R, q^E) = \prod_{k \in C \cup S} \mathcal{R}_k^E(R_k^R, q^E)$ and $\mathcal{R}^E = \cup_{(R^R, q^E)} \mathcal{R}^E(R^R, q^E)$.

For given C and S, define $\mathcal{E}^R = N_+^n \times \prod_{k \in C \cup S} \mathcal{R}_k^R$, the class of all matching problems in the regular decision period. For any $(q^R, R^R) \in \mathcal{E}^R$ and $q^E \in \mathcal{Q}^E(q^R)$, define also $\mathcal{E}^E(q^R, R^R, q^E) = \{q^E\} \times \prod_{k \in C \cup S} \mathcal{R}_k^E(R_k^R, q^E)$, the class of all matching problems in the early decision period induced by (q^R, R^R, q^E). Let $\mathcal{E}^E = \cup_{(q^R, R^R)} \cup_{q^E \in \mathcal{Q}^E(q^R)} \mathcal{E}^E(q^R, R^R, q^E)$.

[14] However, we do not require students to apply early to their top choice colleges under participatory preferences since they may be willing to compromise under an early decision plan where they can only submit a single choice.

2.2 Matching Systems

A matching μ^E in the early decision period for a given profile, q^E, of early decision quotas is a function from the set $C \cup S$ into $2^{C \cup S}$ such that:

i) for all $s \in S$, $|\mu^E(s)| \leq 1$ and $\mu^E(s) \subseteq C$;

ii) for all $c \in C$, $|\mu^E(c)| \leq q_c^E$ and $\mu^E(c) \subseteq S$;

iii) for all $(c, s) \in C \times S$, $\mu^E(s) = \{c\}$ if and only if $s \in \mu^E(c)$.

We denote the set of all matchings for a given q^E by $\mathcal{M}^E(q^E)$ and the set of all matchings in the early decision period by $\mathcal{M}^E$. Given a preference relation R_s^E of student $s \in S$, we say that s prefers matching μ_1^E to matching μ_2^E if and only if it prefers $\mu_1^E(s)$ to $\mu_2^E(s)$. We do the same for each college.

Given a matching μ^E realized in the early decision period and a capacity vector q^R, we define a matching μ^R in the regular decision period as a function from the set $C \cup S$ into $2^{C \cup S}$ such that:

i) for all $s \in S$, $|\mu^R(s)| \leq 1$, and $\mu^E(s) \subseteq \mu^R(s) \subseteq C$;

ii) for all $c \in C$, $|\mu^R(c)| \leq q_c^R$, and $\mu^E(c) \subseteq \mu^R(c) \subseteq S$;

iii) for all $(c, s) \in C \times S$, $\mu^R(s) = \{c\}$ if and only if $s \in \mu^R(c)$.

We notice that the function μ^R preserves the matchings achieved under μ^E in the early decision period, i.e. early decisions are binding. Here, we denote the set of all matchings in the regular decision period for a given (q^R, μ^E) by $\mathcal{M}^R(q^R, \mu^E)$ and the set of all matchings in the regular decision period by $\mathcal{M}^R$. Given a preference relation R_s^R of student $s \in S$, we say that s prefers matching μ_1^R to matching μ_2^R in the regular decision period if and only if s prefers $\mu_1^R(s)$ to $\mu_2^R(s)$. We do the same for each college.

For a given preference profile $R \in \mathcal{R}^E \cup \mathcal{R}^R$, let $A(R_c)$ denote the set of acceptable students for college c, i.e., $A(R_c) = \{s \in S : s\, P_c\, \emptyset\}$, where P denotes the strict preference profile associated with R. Similarly, let $A(R_s)$ denote the set of acceptable colleges for student s, i.e., $A(R_s) = \{c \in C : c\, P_s\, \emptyset\}$.

The acceptable choice of a college c from a group of students $T \subseteq S$ in the early decision market (q^E, R^E) is defined as

$$Ch_c^E(R_c^E, q_c^E, T) = \{T' \subseteq T \cap A(R_c^E) : |T'| \leq q_c^E,\ T' \, R_c^E \, T''$$
$$\text{for all } T'' \subseteq T \text{ such that } |T''| \leq q_c^E\}.$$

Similarly, for a given realization μ^E in the early decision period, the acceptable choice of a college c from a group of students $T \subseteq S \backslash \mu^E(c)$ available for assignment in the regular decision market (q^R, R^R) is defined as

$$Ch_c^R(R_c^R, q_c^R, \mu^E, T) = \{T' \subseteq T \cap A(R_c^R) : |T'| \leq q_c^R - |\mu^E(c)|,$$
$$T' \cup \mu^E(c) \, R_c^R \, T'' \cup \mu^E(c) \text{ for all } T'' \subseteq T$$
$$\text{such that } |T''| \leq q_c^R - |\mu^E(c)|\}.$$

In the early decision period, for a given q^E, a matching $\mu^E \in \mathcal{M}^E(q^E)$ is blocked by student $s \in S$ if $\emptyset \, P_s^E \mu^E(s)$, and blocked by college $c \in C$ if $\mu^E(c) \neq Ch_c^E(R_c^E, q_c^E, \mu^E(c))$. A matching μ^E is said to be acceptable to a college that does not block it in the early decision. Similarly, μ^E is said to be acceptable to a student who does not block it in the early decision. A matching μ^E is blocked by a college-student pair $(c, s) \in C \times S$ if $\{c\} \, P_s^E \, \mu^E(s)$ and $\mu^E(c) \neq Ch_c^E(R_c^E, q_c^E, \mu^E(c) \cup \{s\})$. A matching μ^E is stable if it is not blocked by a student, a college, or a college-student pair.

We denote the set of stable matchings in the early decision market (q^E, R^E) by $\mathcal{S}^E(q^E, R^E)$. In this set, there exists a matching $\mu_C^E(q^E, R^E)$, called *the college-optimal stable matching in the early decision period*, such that

$$\mu_C^E(q^E, R^E)(c) \, R_c^E \, \mu^E(c)$$

for all $c \in C$ and for all $\mu^E \in \mathcal{S}^E(q^E, R^E)$.

Analogously, there is a *student-optimal stable matching in the early decision period*, $\mu_S^E(q^E, R^E)$, that every student likes as well as any other stable matching.

Given an early decision matching μ^E, a regular decision matching $\mu^R \in \mathcal{M}^R(q^R, \mu^E)$ is blocked by student $s \in S$ in the regular decision period if $\emptyset \, P_s^R \, \mu^R(s) \backslash \mu^E(s)$. A matching $\mu^R \in \mathcal{M}^R(q^R, \mu^E)$ is blocked by college $c \in C$ in the regular decision period if $\mu^R(c) \backslash \mu^E(c) \neq Ch_c^R(R_c^R, q_c^R, \mu^E, \mu^R(c) \backslash \mu^E(c))$. A matching μ^R is said to be acceptable to a college that does not block it in the regular decision. Similarly, μ^R is said to be acceptable to a student who does not block it in the regular decision. A matching $\mu^R \in \mathcal{M}^R(q^R, \mu^E)$ is blocked by a college-student pair $(c, s) \in C \times S$ in the regular decision period if $\mu^E(s) = \emptyset$, $\{c\} P_s^R \mu^R(s)$ and $\mu^R(c) \backslash \mu^E(c) \neq Ch_c^R(R_c, q_c^R, \mu^E, \{s\} \cup \mu^R(c) \backslash \mu^E(c))$. A matching μ^R is stable if it is not blocked by a student, a college, or a college-student pair. Given the early decision μ^E, we denote the set of stable matchings in the regular decision market (q^R, R^R) by $\mathcal{S}^R((q^R, R^R), \mu^E)$. This set contains a matching $\mu_C^R((q^R, R^R), \mu^E)$, called *the college-optimal stable matching in the regular decision period*, such that

$$\mu_C((q^R, R^R), \mu^E)(c) \, R_c^R \, \mu^R(c)$$

for all $c \in C$ and for all $\mu^R \in \mathcal{S}^R((q^R, R^R), \mu^E)$.

Analogously, there is a *student-optimal stable matching in the regular decision period*, $\mu_S((q^R, R^R), \mu^E)$, that every student likes as well as any other stable matching.[15]

We say, as similar in Roth and Sotomayor (1990), that for given $(q^E, R^E) \in \mathcal{E}^E$, college c and student s are *achievable* for one another in the early decision if there is some stable matching in $\mathcal{S}^E(q^E, R^E)$ at which they are matched. Likewise, we define achievability in the regular decision.

A matching rule in the early decision period is a function $\varphi^E : \mathcal{E}^E \to \mathcal{M}^E$ such that for all $(q^E, R^E) \in \mathcal{E}^E$, we have $\varphi^E(q^E, R^E) \in \mathcal{M}^E(q^E)$. Let $\bar{\varphi}^E$ denote the set of all matching rules in the early decision period.

A matching rule in the regular decision period is a function $\varphi^R : \mathcal{E}^R \times \mathcal{M}^E \to \mathcal{M}^R$ such that for all $(q^R, R^R) \in \mathcal{E}^R$, for all $q^E \in \mathcal{Q}^E(q^R)$, and for all $\mu^E \in \mathcal{M}^E(q^E)$, we have $\varphi^R((q^R, R^R), \mu^E) \in \mathcal{M}^R(q^R, \mu^E)$. Let $\bar{\varphi}^R$ denote the set of all matching rules in the regular decision period.

A matching rule φ^E in the early decision period is stable if $\varphi^E(q^E, R^E) \in \mathcal{S}^E(q^E, R^E)$ for all $(q^E, R^E) \in \mathcal{E}^E$.

A matching rule φ^R in the regular decision period is stable at an early decision matching rule φ^E if $\varphi^R((q^R, R^R), \varphi^E(q^E, R^E)) \in \mathcal{S}^R((q^R, R^R), \varphi^E(q^E, R^E))$ for all $(q^R, R^R) \in \mathcal{E}^R$, for all $q^E \in \mathcal{Q}^E(q^R)$, and for all $R^E \in \prod_{k \in C \cup S} \mathcal{R}_k^E(R_k^R, q^E)$.

[15] To find the college-optimal and student-optimal stable matchings in the two decision periods, we respectively use the well-known college-proposing and student-proposing deferred acceptance algorithms by Gale and Shapley (1962).

For any $\varphi^E \in \bar{\varphi}^E$ that is used in the early decision period and for any $\varphi^R \in \bar{\varphi}^R$ that is used in the regular decision period, the ordered pair (φ^E, φ^R) is called a *matching system*. Let $\vec{\varphi}$ denote the matching system (φ^E, φ^R).

A matching system $\vec{\varphi}$ is stable if (i) φ^E is stable, and (ii) φ^R is stable at φ^E.

Let $\vec{\varphi}_C$ be a matching system such that $\varphi^E_C(q^E, R^E) = \mu^E_C(q^E, R^E)$ and $\varphi^R_C((q^R, R^R), \varphi^E_C(q^E, R^E)) = \mu^R_C((q^R, R^R), \mu^E_C(q^E, R^E))$ for all $q^E \in \mathcal{Q}^E(q^R)$ and for all $R^E \in \prod_{k \in C \cup S} \mathcal{R}^E_k(R^R_k, q^E)$. We call $\vec{\varphi}_C$ as *the college-optimal stable matching system.*

Let $\vec{\varphi}_S$ be such that $\varphi^E_S(q^E, R^E) = \mu^E_S(q^E, R^E)$ and $\varphi^R_S((q^R, R^R), \varphi^E_S(q^E, R^E)) = \mu^R_S((q^R, R^R), \mu^E_S(q^E, R^E))$ for all $q^E \in \mathcal{Q}^E(q^R)$ and for all $R^E \in \prod_{k \in C \cup S} \mathcal{R}^E_k(R^R_k, q^E)$. We call $\vec{\varphi}_S$ as *the student-optimal stable matching system.*

3 Manipulation via Preferences and Quotas

A matching system $\vec{\varphi}$ is *nonmanipulable by agent $k \in C \cup S$ via early decision preferences* if

$$\varphi^R((q^R, R^R), \varphi^E(q^E, R^E))(k) \; R^R_k \; \varphi^R((q^R, R^R), \varphi^E(q^E, \hat{R}^E_k, R^E_{-k}))(k)$$

for all $(q^R, R^R) \in \mathcal{E}^R$, for all $q^E \in \mathcal{Q}^E(q^R)$, for all $R^E \in \prod_{i \in C \cup S} \mathcal{R}^E_i(R^R_i, q^E)$, and for all $\hat{R}^E_k \in \mathcal{R}^E_k(R^R_k, q^E)$.

Let $K(\vec{\varphi})$ denote the set of all agents in $C \cup S$ by every of which $\vec{\varphi}$ is nonmanipulable via early decision preferences. A matching system $\vec{\varphi}$ is *(individually) nonmanipulable by colleges via early decision preferences* if $K(\vec{\varphi}) \supseteq C$. Likewise, a matching system $\vec{\varphi}$ is *(individually) nonmanipulable by students via early decision preferences* if $K(\vec{\varphi}) \supseteq S$. A matching system $\vec{\varphi}$ is *(individually) nonmanipulable via early decision preferences* if $K(\vec{\varphi}) = C \cup S$.

A matching system $\vec{\varphi}$ is *nonmanipulable by college $c \in C$ via early decision quotas* if

$$\varphi^R((q^R, R^R), \varphi^E(q^E, R^E))(c) \; R^R_c \; \varphi^R((q^R, R^R), \varphi^E(\hat{q}^E_c, q^E_{-c}, R^E))(c)$$

for all $(q^R, R^R) \in \mathcal{E}^R$, for all $q^E \in \mathcal{Q}^E(q^R)$, for all $R^E \in \mathcal{R}^E(R^R, q^E)$, and for all $\hat{q}^E_c \in \mathcal{Q}^E_c(q^R)$. A matching system is *(individually) nonmanipulable via early decision quotas* if it is nonmanipulable by any college via early decision quotas.

Finally, a matching system is *(individually) nonmanipulable* if it is (individually) nonmanipulable via early decision preferences and (individually) nonmanipulable via early decision quotas.

Theorem 1. *Suppose there are at least two colleges and one student. Then there exists no matching system that is stable and nonmanipulable by colleges via early decision quotas.*

The above result is no longer true when there exists a unique college in the market. It is known by Roth (1985) that in a single-stage decision process, for each matching market there exists a unique stable matching which is college-optimal, i.e. a matching that assigns to each college the highest ranked achievable students allowed by its quota. Since the preferences of students are participatory, every student that is achievable in the regular decision period will apply early to the single college in the market if it reports its total capacity and adopts its regular decision preference in the early decision period. Hence, in our two-stage matching model, manipulating the quota in the early decision can make for a college no difference except for the timing of the admission of some students already achievable in the regular decision.

Below, we consider whether colleges have incentives to manipulate their preferences in the early decision period.

Theorem 2. *Suppose there are at least two colleges and two students. Then there exists no matching system that is stable and nonmanipulable by colleges via early decision preferences.*

One can also claim that the proposition in Theorem 2 remains to hold when the number of students in the market is one. For example, when $C = \{c_1, c_2\}$, $S = \{s_1\}$, $q^R_{c_1} = 1$, $q^R_{c_2} = 1$, $q^E_{c_1} = 1$, $q^E_{c_2} = 0$,

$$P^R_{c_1} = \{s_1\}, \emptyset,$$

$$P^E_{c_1} = \emptyset,$$

$$\hat{P}^E_{c_1} = P^R_{c_1},$$

$$P^R_{c_2} = P^E_{c_2} = \{s_1\}, \emptyset,$$

$$P^R_{s_1} = \{c_2\}, \{c_1\}, \emptyset,$$

$$P^E_{s_1} = \{c_1\}, \emptyset,$$

we have $S^E(q^E, R^E) = \{\mu_1\}$, $S^R(q^R, R^R, \mu_1) = \{\mu_2\}$, $S^E(q^E, \hat{R}^E_{c_1}, R^E_{-c_1}) = \{\mu_3\}$, $S^R(q^R, R^R, \mu_3) = \{\mu_3\}$, where

$$\mu_1 = \begin{pmatrix} c_1 & c_2 \\ \emptyset & \emptyset \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} c_1 & c_2 \\ \emptyset & \{s_1\} \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} c_1 & c_2 \\ \{s_1\} & \emptyset \end{pmatrix}.$$

Therefore, $\varphi^E(q^E, R^E) = \mu_1$, $\varphi^R((q^R, R^R), \mu_1) = \mu_2$, $\varphi^E(q^E, \hat{R}^E_{c_1}, R^E_{-c_1}) = \mu_3$, and $\varphi^R((q^R, R^R), \mu_3) = \mu_3$. Hence,

$$\varphi^R((q^R, R^R), \varphi^E(q^E, \hat{R}^E_{c_1}, R^E_{-c_1}))(c_1) \; P^R_{c_1} \; \varphi^R((q^R, R^R), \varphi^E(q^E, R^E))(c_1).$$

That is, college c_1 can manipulate the matching system (φ^E, φ^R) via early decision preferences, completing the proof for the case of two colleges and one student. Finally, we can include colleges whose top choice is admitting no student in both the early decision period and the regular decision period to generalize this proof to cases with one student and at least two colleges. However, we should remark in this setup that in a matching market involving a single student, the unacceptability of the student in the early decision period for any college who indeed considers him or her acceptable under its regular decision preference is not 'rational' at all, though admissible, to be assumed in the first place.

On the other hand, we would no longer have the negative result in Theorem 2 when there exists only one college in the market. Manipulation via early decision preferences, like the previously studied manipulation via early decision quotas, can in this case affect only the timing of the admission of some students already achievable in the regular decision.

Theorem 3. *Suppose there are at least two colleges and one student. Then there exists no matching system that is stable and nonmanipulable by students via early decision preferences.*

When there is a unique college in the market, under participatory preferences every student that is achievable in the regular decision period will apply early to this college if it admits students early. Hence, the only case a student can manipulate his or her preference ordering in the early decision period arises when the college does not accept students early, which is obviously immaterial for the student.

Theorem 4. *Suppose there are at least two colleges and two students. Then there exists no matching system that is stable and nonmanipulable.*

In the following section, we will study the strategic incentives of colleges in an early decision quota game.

4 Early Decision Quota Game

We consider an early decision quota game played by colleges for a given regular decision market (C, S, R^R, q^R), where R^R and q^R are commonly known.[16] Each college is asked to announce out of its total capacity an early decision quota, i.e. the strategy of college c is $q_c^E \in \mathcal{Q}_c^E(q^R)$. We assume that for each possible announcement of the vector q^E, the preferences of colleges and students in the early decision period, denoted by $R^E(q^E)$, is also common knowledge. Suppose that a matching system $\vec{\varphi}$ is used to determine the matchings in the early decision market and the regular decison market. College c's pereferences over reported early decision quotas are represented by a binary relationship $\succeq_c^{\vec{\varphi}}$ over $\mathcal{Q}^E(q^R)$ such that for all q'^E, $q''^E \in \mathcal{Q}^E(q^R)$ we have $q'^E \succeq_c^{\vec{\varphi}} q''^E$ if and only if

$$\varphi^R((q^R, R^R), \varphi^E(q'^E, R^E(q'^E)))\, R_c^R\, \varphi^R((q^R, R^R), \varphi^E(q''^E, R^E(q''^E))).$$

An early decision quota reporting game under matching system $\vec{\varphi}$ is described by a strategic form game $(C, (\mathcal{Q}_c^E(q^R), \succeq_c^{\vec{\varphi}})_{c\in C})$.

Define college c's *best response correspondence* under matching system $\vec{\varphi}$ by $\beta_c^{\vec{\varphi}} : \mathcal{Q}_{-c}^E(q^R) \to \mathcal{Q}_c^E(q^R)$ such that for any $q_{-c}^E \in \mathcal{Q}_{-c}^E(q^R)$,

$$\beta_c^{\vec{\varphi}}(q_{-c}^E) = \{\tilde{q}_c^E \in \mathcal{Q}_c^E(q^R) : (\tilde{q}_c^E, q_{-c}^E) \succeq_c^{\varphi} (q'^E_c, q_{-c}^E) \text{ for all } q'^E_c \in \mathcal{Q}_c^E(q^R)\}.$$

A *pure strategy (Nash) equilibrium* of the game $(C, (\mathcal{Q}_c^E(q^R), \succeq_c^{\vec{\varphi}})_{c\in C})$ is a strategy profile $q^E \in \mathcal{Q}^E(q^R)$ such that $q_c^E \in \beta_c^{\vec{\varphi}}(q_{-c}^E)$ for all $c \in C$.

The following two theorems respectively show that a pure strategy equilibrium may not exist under the college-optimal and student-optimal matching systems.

[16] A similar game in a single-shot hospital-intern market is considered by Konishi and Ünver (2006).

Theorem 5. *The early decision quota reporting game under the college-optimal matching system may not have a pure strategy equilbrium.*

Theorem 6. *The early decision quota reporting game under the student-optimal matching system may not have a pure strategy equilibrium.*

Konishi and Ünver (2006) have a similar pair of results showing the nonexistence of a pure strategy equilibrium of the game of capacity manipulation in a single-period hospital-intern market. They argue that hospitals can improve their position by restricting the available capacity under the hospital-optimal matching rule. The reason is that larger capacities of hospitals make interns better off by giving them more alternatives to choose from. However, when capacities are limited, an intern cannot play one offer against the other and has to choose among rather limited set of offers which may not include his/her favorite hospital. Konishi and Ünver (2006) also remark that hospitals, which are getting their least preferable acceptable interns under the intern-optimal matching rule, can do better by swapping interns through some reduction in their quotas. In our model, similar incentives are at work. However, given the fixed total capacity in the regular decision period, colleges are able to improve their position not by limiting the total number of incoming students but by spreading admissions process across periods. This essentially increases their "bargaining power" without necessarily reducing the overall intake of the students. If we use the analogy between the college admissions market and the marriage market remarked by Roth (1985) by viewing each vacant position in a college as an individual player who has the same preferences as the college that it belongs to, spreading admissions across periods increases the bargaining power of a college by making the supply side of the market "thinner". Obviously whether this strategy will always improve the position of colleges depends on the preferences of students adopted in the early decision market.

Another interesting finding of Konishi and Ünver (2006) is that a hospital's capacity underreport makes all other hospitals weakly better off and all interns weakly worse off. But, this observation does not carry over our framework that involves two stages of admissions. It is apparent from the examples exhibited in Theorems 5 and 6 that in the early decision quota reporting game, a college's quota report does not have a monotonous effect on the welfare of the other colleges and students under college-optimal and student-optimal matching systems. For example in Table 2 in the Appendix, if c_1 reduces its early decision quota at $q^E = (3,0)$, c_2 is first worse off ($\{s_3, s_5\} P^R_{c_2} \{s_2, s_3\}$) then better off ($\{s_1, s_2, s_3\} P^R_{c_2} \{s_2, s_3\}$). Similarly, in Table 4, if c_1 decreases its early decision quota at $q^E = (3,1)$, c_2 is first worse off ($\{s_4, s_5\} P^R_{c_2} \{s_3, s_5\}$) then better off ($\{s_2, s_3, s_5\} P^R_{c_2} \{s_3, s_5\}$). In terms of students' welfare, we observe that in Table 2, if c_1 reduces its early decision quota at $q^E = (3,1)$, not all sudents become weakly worse off. While s_1 is better off, s_2 is initially better off as c_1 gradually reduces its quota from 3 to 1 but then worse off. Similiarly, in Table 4, if c_1 reduces its early decision quota at $q^E = (3,1)$, not all students become weakly worse off. While s_1 is better off, s_2 is worse off.

Given the nonexistence results in Theorems 5 and 6 under the full domain of preferences, we will now consider certain restrictions on preferences to guarantee the existence of pure strategy equilibria. Dealing with a similar problem in a single-stage hospital-intern market, Konishi and Ünver (2006) consider two types of restrictions. In one of them, hospitals' preferences satisfy strong monotonicity in population, if and only if any hospital strictly prefers among any two groups of acceptable interns of distinct sizes the one that is more populated. With this domain restriction, Konishi and Ünver (2006) show that in the capacity-reporting game, under the hospital-optimal matching rule reporting the number of interns that the matching rule assigns is an equilibrium strategy whereas under the intern-optimal matching rule reporting the actual capacity is a weakly dominant strategy. In our model, we say that colleges' preferences $R \in \mathcal{R}^E \cup \mathcal{R}^R$ satisfy *strong monotonicity in population*, if and only if for any $c \in C$ and for any $S', S'' \subseteq A(R_c)$, we have $|S'| > |S''| \Rightarrow S' P_c S''$. We should immediately note that the preferences of colleges in the market examples considered in the proofs of Theorems 5 and 6 already satisfy strong monotonicity in population. Thus, this type of restriction does not seem to be promising for our purpose.

The second type of restriction that Konishi and Ünver (2006) consider is the common preferences for one group over the agents of the opposite group. With such preferences, they are able to show that reporting the true capacity is always a weakly dominant strategy. Below, we will show that this kind of domain restriction will also allow us to achieve a set of interesting existence results.

4.1 Common Preferences for Colleges

A preference profile $R \in \mathcal{R}^E \cup \mathcal{R}^R$ satisfies *common preferences for colleges over individual students* if and only if for any $c, c' \in C$ and for any $s, s' \in S$ we have $\{s\} \, P_c \, \{s'\} \Leftrightarrow \{s\} \, P_{c'} \, \{s'\}$.[17]

Let $\mathcal{R}^{R-CC} \subset \mathcal{R}^R$ and $\mathcal{R}^{E-CC} \subset \mathcal{R}^E$ be the domains of such profiles of common preference relations in the regular decision period and the early decision period, respectively.

Now, pick $q^R \in N_+^n$, $q^E \in \mathcal{Q}^E(q^R)$, $R^R \in \mathcal{R}^{R-CC}$, and $R^E \in \mathcal{R}^E(R^R, q^E) \cap \mathcal{R}^{E-CC}$. Rename and reorder acceptable students by the preference ordering P_c^E of college c as $s_1, s_2, ..., s_{l^E(c)}$. That is, college c has the preference ordering

$$P_c^E = \{s_1\}, \{s_2\}, ..., \{s_{l^E(c)}\}, \emptyset.$$

With an abuse of notation, assume that $P_c^E = \emptyset$ corresponds to $l^E(c) = 0$. Let $\hat{l}^E = \max\{l^E(c) : c \in C\}$. Define $Ch_s^E(R_s^E, C')$ be the acceptable college, if any, in $C' \subseteq C$ for student s with respect to the preference relation R_s^E, i.e., $Ch_s^E(R_s^E, C') = C' \cap A(R_s^E)$. If $\hat{l}^E > 0$, consider a matching $\hat{\mu}^E$ in the early decision period generated by the following *serial-dictatorship:*[18]

Step 1: Let $C^{E,1} = \{c \in C : q_c^E \neq 0 \text{ and } s_1 \in A(R_c^E)\}$ and $q_c^{E,1} = q_c^E$ for all $c \in C$. Set $\hat{\mu}^E(s_1) = Ch_{s_1}^E(R_{s_1}^E, C^{E,1})$.

$\vdots \qquad\qquad\qquad \vdots$

Step t: For all $c \in C$, let $q_c^{E,t} = q_c^{E,t-1} - 1$ if $\hat{\mu}^E(s_{t-1}) = \{c\}$, and $q_c^{E,t} = q_c^{E,t-1}$ otherwise. Let $C^{E,t} = \{c \in C : q_c^{E,t} \neq 0 \text{ and } s_t \in A(R_c^E)\}$. Set $\hat{\mu}^E(s_t) = Ch_{s_t}^E(R_{s_t}^E, C^{E,t})$.

The above algorithm stops after $\hat{l}^E$ steps, and $\hat{\mu}^E$ becomes the matching of the early decision market. Notice that for all $s \in S \backslash \{s_1, s_2, ..., s_{\hat{l}^E}\}$, we have $\hat{\mu}^E(s) = \emptyset$, trivially. If $\hat{l}^E = 0$, set $\hat{\mu}^E(s) = \emptyset$ for all $s \in S$.

[17] This definition of common preferences for colleges as well as the definition of common preferences for students given in the next subsection slightly weaken a pair of definitions in Konishi and Ünver (2006) who require also a common set of acceptable students (colleges) under the common preferences for colleges (students).

[18] The serial dictatorship rules that we use throughout the paper extend those in Konishi and Ünver (2006) to our two-stage framework. For the other uses of serial dictatorship in one-sided matching markets, see Svensson (1994), Abdulkadiroğlu and Sönmez (1998), and Papai (2000).

Now, we consider a similar matching in the regular decision period. Rename and reorder acceptable students by the preference ordering P_c^R of college c as $s_1, s_2, ..., s_{l^R(c)}$. That is, college c has the preference ordering

$$P_c^R = \{s_1\}, \{s_2\}, ..., \{s_{l^R(c)}\}, \emptyset.$$

Assume that $P_c^R = \emptyset$ corresponds to $l^R(c) = 0$. Let $\hat{l}^R = \max\{l^R(c) : c \in C\}$. Define $Ch_s^R(R_s^R, C')$ be the most preferable acceptable college in $C' \subseteq C$ for student s with respect to the preference relation R_s^R, i.e., $Ch_s^E(R_s^R, C') = \{c \in C' \cap A(R_s^R) : \{c\}R_s^R\{c'\}$ for any $c' \in C'\}$. Given the matching $\hat{\mu}^E$, consider a matching $\hat{\mu}^R$ in the regular decision period, generated by the following *serial-dictatorship* if $\hat{l}^R > 0$:

Step 1: Let $q_c^{R,1}(\hat{\mu}^E) = q_c^R - |\hat{\mu}^E(c)|$ for all $c \in C$, and $C^{R,1}(\hat{\mu}^E) = \{c \in C : q_c^{R,1}(\hat{\mu}^E) \neq 0$ and $s_1 \in A(R_c^R)\}$. Set $\hat{\mu}^R(s_1) = Ch_{s_1}^R(R_{s_1}^R, C^{R,1}(\hat{\mu}^E))$ if $\hat{\mu}^E(s_1) = \emptyset$, and $\hat{\mu}^R(s_1) = \hat{\mu}^E(s_1)$ otherwise.

$\vdots \qquad \qquad \vdots$

Step t: For all $c \in C$, let $q_c^{R,t}(\hat{\mu}^E) = q_c^{R,t-1}(\hat{\mu}^E) - 1$ if $\hat{\mu}^R(s_{t-1}) \backslash \hat{\mu}^E(s_{t-1}) = \{c\}$, and $q_c^{R,t}(\hat{\mu}^E) = q_c^{R,t-1}(\hat{\mu}^E)$ otherwise. Let $C^{R,t}(\hat{\mu}^E) = \{c \in C : q_c^{R,t}(\hat{\mu}^E) \neq 0$ and $s_t \in A(R_c^E)\}$. Set $\hat{\mu}^R(s_t) = Ch_{s_t}^R(R_{s_t}^R, C^{R,t}(\hat{\mu}^E))$ if $\hat{\mu}^E(s_t) = \emptyset$, and $\hat{\mu}^R(s_t) = \hat{\mu}^E(s_t)$ otherwise.

The above algorithm stops after $\hat{l}^R$ steps, and $\hat{\mu}^R$ becomes the matching of the regular decision market. Notice that for all $s \in S \backslash \{s_1, s_2, ..., s_{\hat{l}^R}\}$, we have $\hat{\mu}^R(s) = \hat{\mu}^E(s) = \emptyset$, trivially. If $\hat{l}^R = 0$, set $\hat{\mu}^R(s) = \hat{\mu}^E(s) = \emptyset$ for all $s \in S$.

Lemma 1. *Consider any college admissions market with common preferences for colleges in both periods. Then $(\hat{\mu}^E, \hat{\mu}^R)$ is the unique pair of stable matchings in the early and regular decision periods.*

Early decision program requires every student to apply early only to at most one college and to commit himself or herself to that college in case of an acceptance. It is interesting to find that under this program, 'accepting no student early' becomes a weakly dominant strategy if colleges have common preferences and every student, choosing to act early, applies to his or her top-ranked college.

Theorem 7. *Consider a regular decision market with common preferences for colleges. Assume that at all possible announcements of the quota profile colleges adopt their regular preferences in the early decision market and each student, choosing to act under early decision, applies early to the top-ranked college under his or her regular preference ordering. Then, in the early decision quota game under the stable matching system, reporting zero quota is a weakly dominant strategy for each college.*

The above theorem assumes that any college can be matched in the early decision period only with students that rank it at the top in their preference orderings. However, any such student, if not already in the list of students that this particular college accepts in the regular decision period when it announces zero quota for the early decision period, must be undesirable for the college since the matchings are determined by the serial dictatorship of students under the common preferences for colleges. Hence, it is true for each college that reporting the early decision quota as zero is a weakly dominant strategy. The below example highlights our reasoning why a college can become worse off by setting a positive early-decision quota.

Example 1. Consider the regular decision market (C, S, q^R, R^R) with $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3\}$, $q^R_{c_1} = 1$, $q^R_{c_2} = 1$, and the following preference profile for colleges and students:

$$P^R_{c_1} = P^R_{c_2} = \{s_1\}, \{s_2\}, \{s_3\}, \emptyset,$$
$$P^R_{s_1} = P^R_{s_2} = \{c_1\}, \{c_2\}, \emptyset,$$
$$P^R_{s_3} = \{c_2\}, \{c_1\}, \emptyset.$$

We have $\mathcal{Q}^E_c(q^R) = \{0, 1\}$ for all $c \in C$. Let $P^E_{c_1}(q^E) = P^E_{c_2}(q^E) = P^R_{c_1}$ for all $q^E \in \mathcal{Q}^E_c(q^R)$, and let $P^E_s(q^E) = \{Top(R^R_s; 1)\}, \emptyset$ for all $q^E \in \mathcal{Q}^E_c(q^R)$ and for all $s \in S$. Pick any $q^E_{c_1} \in \mathcal{Q}^E_c(q^R)$. We have $\hat{\varphi}^E((q^E_{c_1}, 0), R^E(q^E_{c_1}, 0))(c_2) = \emptyset$, and $\hat{\varphi}^R(((1,1), R^R), \hat{\varphi}^E((q^E_{c_1}, 0), R^E(q^E_{c_1}, 0)))(c_2) = \{s_2\}$. On the other hand, when c_2 reports its total capacity and the early decision quota profile changes to $(q^E_{c_1}, 1)$, we have $\hat{\varphi}^R(((1,1), R^R), \hat{\varphi}^E((q^E_{c_1}, 1), R^E(q^E_{c_1}, 1)))(c_2) = \hat{\varphi}^E((q^E_{c_1}, 1), R^E(q^E_{c_1}, 1))(c_2) = \{s_3\}$. Since $(q^E_{c_1}, 0) \succeq^{\vec{\varphi}}_{c_2} (q^E_{c_1}, 1)$, college c_2 becomes worse off by not setting its early decision quota as zero.

Restricting each student to apply early to his or her top-ranked college only when there is at most one college in the early decision market, we obtain the prospect of unparticipating in the early decision market as an equilibrium strategy for colleges, which is stated by the following corollary to Theorem 7.

Corollary 1. *Consider a regular decision market with common preferences for colleges. Assume that colleges adopt their regular preferences in the early decision market, and each student, choosing to act under early decision, applies early only to the top-ranked college under his or her regular preference ordering when there is at most one college in the early decision market. Then, every college's reporting zero quota is a Nash equilibrium of the early decision quota game under the stable matching system.*

The assumed early-decision preferences of students driving the above results are in line with the constant recommendation of the College Board, many college guides, and counsellars that students should apply under early decision only to a clear-cut first-choice college. The actual response of students to this recommendation is highlighted by the survey reported in Avery et.al. (2003). According to this survey, which is conducted during 1997-2000, 98 percent of a total of 48 sample students applied to an early decision program, as a 'strong or weak first choice', at either Princeton, Yale or Wesleyan.

However, it is, of course, a possibility that a student who wants to enjoy his or her senior year in the highschool without any pressure may instead be applying early to a college that is not his or her top choice, while at the same time easier to be admitted. Such compromising behaviour of students makes more sense in real-life situations where the applicant pool is very large and the information about the true preference profile as well as the matching process is not completely available to all applicants.

Interestingly, the existence of compromising students in the applicant pool explains why some lower-ranked colleges may not agree to terminate the single-choice early decision programs. For instance, a college that is considered to be low-ranked by almost all students in the regular decision period and nonetheless highly preferred in the early decision period by some compromising students who are high-ranked under the common regular preferences of colleges, may agressively enter the early decision market. The below example elaborates this idea.

Example 2. Consider the regular decision market (C, S, q^R, R^R) with $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3\}$, $q^R_{c_1} = 2$, $q^R_{c_2} = 2$, and the following regular preferences for colleges and students:

$$P^R_{c_1} = P^R_{c_2} = \{s_1\}, \{s_2\}, \{s_3\}, \emptyset,$$

$$P^R_{s_1} = P^R_{s_2} = \{c_1\}, \{c_2\}, \emptyset,$$

$$P^R_{s_3} = \{c_2\}, \{c_1\}, \emptyset.$$

We have $\mathcal{Q}^E_c(q^R) = \{0, 1, 2\}$ for all $c \in C$. Let $P^E_c = P^R_c$ for all $c \in C$. Also, let $P^E_s(q^E) = \{Top(R^R_s; 1)\}, \emptyset$ for all $s \in \{s_1, s_3\}$ and $P^E_{s_2}(q^E) = \{Top(R^R_{s_2}; 2)\}, \emptyset$ for all $q^E \in \mathcal{Q}^E(q^R)$.

We will show that the early decision quota profile $(0, 0)$ is no longer a Nash equilibrium. We have $\hat{\varphi}^E((0, 0), R^E(0, 0))(c_2) = \emptyset$, and $\hat{\varphi}^R(((2, 2), R^R), \hat{\varphi}^E((0, 0), R^E(0, 0)))(c_2) = \{s_3\}$. College c_2, which is the lowest-ranked college by the majority of students, has an incentive to enter the early decision market, since if it unilaterally deviates and announces $q^E_{c_2} = 2$, we have $\hat{\varphi}^R(((2, 2), R^R), \hat{\varphi}^E((0, 2), R^E(0, 2)))(c_2) = \hat{\varphi}^E((0, 2), R^E(0, 2))(c_2) = \{s_2, s_3\}$. Thus, it is not true that $(0, 0) \succeq^{\vec{\hat{\varphi}}}_{c_2} (0, 2)$. Here, one can easily check that the Nash equilibria of this game are $(0, 1)$, $(1, 1)$, $(2, 1)$, $(0, 2)$, $(1, 2)$, and $(2, 2)$, which all yield the same matching outcome in the regular decision period.

4.2 Common Preferences for Students

A preference profile $R \in \mathcal{R}^E \cup \mathcal{R}^R$ satisfies *common preferences for students over individual colleges* if and only if for any $s, s' \in S$ and for any $c, c' \in C$ we have $\{c\}\, P_s \,\{c'\} \Leftrightarrow \{c\}\, P_{s'} \,\{c'\}$.

Let $\mathcal{R}^{R-CS} \subset \mathcal{R}^R$ and $\mathcal{R}^{E-CS} \subset \mathcal{R}^E$ be the domains of such profiles of common preference relations in the regular decision period and the early decision period, respectively.

Now, pick $q^R \in N^n_+$, $q^E \in \mathcal{Q}^E(q^R)$, $R^R \in \mathcal{R}^{R-CS}$, and $R^E \in \mathcal{R}^E(R^R, q^E) \cap \mathcal{R}^{E-CS}$. Define $S^E(R^E) = \{s \in S : A(R^E_s) \neq \emptyset\}$. Rename the acceptable college by the preference ordering P^E_s of student $s \in S^E(R^E)$ as c_1; i.e.,

$$P^E_s = \{c_1\}, \emptyset.$$

If $S^E(R^E) \neq \emptyset$, consider a matching $\tilde{\mu}^E$ in the early decision period, generated by the following *serial-dictatorship:*

Single Step: Set $\tilde{\mu}^E(c_1) = Ch^E_{c_1}(R^E_{c_1}, q^E_{c_1}, S^E(R^E))$.

Notice that for all $c \in C\backslash\{c_1\}$, we have $\tilde{\mu}^E(c) = \emptyset$, trivially. If $S^E(R^E) = \emptyset$, set $\tilde{\mu}^E(c) = \emptyset$ for all $c \in C$.

Now, we consider a similar matching in the regular decision period. Rename and reorder acceptable colleges by the preference ordering P^R_s of student s as $c_1, c_2, ..., c_{l^R(s)}$. That is, student s has the preference ordering

$$P^R_s = \{c_1\}, \{c_2\}, ..., \{c_{l^R(s)}\}, \emptyset.$$

With an abuse of notation, assume that $P^R_s = \emptyset$ corresponds to $l^R(s) = 0$. Let $\tilde{l}^R = \max\{l^R(s) : s \in S\}$. Given the matching $\tilde{\mu}^E$, consider a matching $\tilde{\mu}^R$ in the regular decision period generated by the following *serial-dictatorship* if $\tilde{l}^R > 0$:

Step 1: Let $S^{R,1}(q^E, q^R) = \{s \in S\backslash \cup^{\tilde{l}^E}_{\tau=1} \tilde{\mu}^E(c_\tau) : c_1 \in A(R^E_s)\}$. Set $\tilde{\mu}^R(c_1) = \tilde{\mu}^E(c_1) \cup Ch^R_{c_1}(R^R_{c_1}, q^{R,1}_{c_1}, \tilde{\mu}^E, S^{R,1}(q^E, q^R))$.

$\vdots \qquad\qquad \vdots$

Step t: Let $S^{R,t}(q^E, q^R) = \{s \in S\backslash \left(\left(\cup^{t-1}_{\tau=1} \tilde{\mu}^R(c_\tau)\right) \cup \left(\cup^{\tilde{l}^E}_{\tau=t} \tilde{\mu}^E(c_\tau)\right)\right) : c_t \in A(R^R_s)\}$. Set $\tilde{\mu}^R(c_t) = \tilde{\mu}^E(c_t) \cup Ch^R_{c_t}(R^R_{c_t}, q^{R,t}_{c_t}, \tilde{\mu}^E, S^{R,t}(q^E, q^R))$.

The above algorithm stops after $\tilde{l}^R$ steps, and $\tilde{\mu}^R$ becomes the matching of the regular decision market. Notice that for all $c \in C\backslash\{c_1, c_2, ..., c_{\tilde{l}^R}\}$, we have $\tilde{\mu}^R(c) = \tilde{\mu}^E(c) = \emptyset$. If $\tilde{l}^R = 0$, set $\tilde{\mu}^R(c) = \tilde{\mu}^E(c) = \emptyset$ for all $c \in C$.

Lemma 2. *Consider any college admissions market with common preferences for students in both periods. Then $(\tilde{\mu}^E, \tilde{\mu}^R)$ is the unique pair of stable matchings in the early and regular decision periods.*

We will below show that any feasible quota profile becomes an equilibrium strategy if students have common preferences over the individual colleges and each student, choosing to act under early decision, applies to his or her top-ranked college irrespective of the quota announcements. As the matchings are determined by the serial dictatorship of colleges, no college except for the top-ranked one with respect to the common preferences of students can ever accept any student in the early decision period. Given this fact, it should also be clear that the top-ranked college becomes completely impartial to any early quota decision, since it is already the first in selecting students in the regular decision period.

Theorem 8. *Consider a regular decision market with common preferences for students. Assume that at all possible announcements of the quota profile, each student, choosing to act under early decision, applies early only to the top-ranked college under his or her regular preference ordering. Then, any feasible quota profile is a Nash equilibrium of the early decision quota game under the stable matching system.*

Since colleges in the above result are completely indifferent over all feasible early decision quota profiles, we can safely assume that they may just consider to terminate their early decision plans as a weakly dominant strategy.

Like in Corollary 1, we can now restrict each student in the early decision program to apply to his or her top-ranked college only when there exists at most one college in the early decision market. The below corollary to Theorem 8 thus obtains reporting no quota as an equilibrium strategy.

Corollary 2. *Consider a regular decision market with common preferences for students. Assume that each student, choosing to act under early decision, applies early only to the top-ranked college under his or her regular preference ordering when there is at most one college in the early decision market. Then, every college's reporting zero quota is a Nash equilibrium of the early decision quota game under the stable matching system.*

Finally, we relax the restriction on students' preferences that gives rise to the above pair of results. When the single choice of some students in the early decision market does not coincide with the top-ranked college under their common regular preference ordering, some colleges may have an incentive to offer an early decision program in order to admit some students compromising in the early stage of the matching process. This observation is highlighted by the following simple example.

Example 3. Consider the regular decision market (C, S, q^R, R^R) with $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3\}$, $q^R_{c_1} = 2$, $q^R_{c_2} = 2$, and the following regular preferences for colleges and students:

$$P^R_{c_1} = \{s_1\}, \{s_2\}, \{s_3\}, \emptyset,$$

$$P^R_{c_2} = \{s_3\}, \{s_2\}, \{s_1\}, \emptyset,$$

$$P^R_s = \{c_1\}, \{c_2\}, \emptyset, \text{ for all } s \in S.$$

We have $\mathcal{Q}^E_c(q^R) = \{0, 1, 2\}$ for all $c \in C$. Let $P^E_c(q^E) = P^R_c$ for all $c \in C$, and $P^E_s(q^E) = \{Top(R^R_s; 2)\}, \emptyset$ for all $s \in S$ and for all $q^E \in \mathcal{Q}^E(q^R)$. (Here, we have kept on assuming common early decision preferences for compromising students to simply obtain the unique stable matching by the serial dictatorship of colleges.)

We will show that the early decision quota profile $(0, 0)$ is no longer a Nash equilibrium. We have $\hat{\varphi}^E((0, 0), R^E(0, 0))(c_2) = \emptyset$, and $\hat{\varphi}^R(((2, 2), R^R), \hat{\varphi}^E((0, 0), R^E(0, 0)))(c_2) = \{s_3\}$. College c_2, which is the lowest-ranked college by all students, has an incentive to enter the early decision market, since if it unilaterally deviates and announces $q^E_{c_2} = 2$, it can select the set of students $\hat{\varphi}^R(((2, 2), R^R), \hat{\varphi}^E((0, 2), R^E(0, 2)))(c_2) = \hat{\varphi}^E((0, 2), R^E(0, 2))(c_2) = \{s_2, s_3\}$. Thus, it is not true that $(0, 0) \succeq^{\vec{\varphi}}_{c_2} (0, 2)$. Here, one can easily check that the Nash equilibria of this game are $(0, 2)$, $(1, 2)$, and $(2, 2)$, which all yield the same matching outcome in the regular decision period.

5 Conclusions

Harvard College's recent announcement of eliminating its early action program brought the prospect of abolition of all early admissions programs in the United States under scrutiny. The main argument underlying Harvard's decison was that early admissions programs favor already advantaged students with high income and thus hurt low-income students who generally prefer to wait until regular admissions where they can apply to and compare different financial aid packages. Princeton University and University of Virginia who had both used to offer early decision programs immediately followed the suit, sharing the concerns of Harvard. However, some colleges and universities that offer early action programs argued that it is not clear how elimination of non-binding early admissions programs such as early action will result in admission of more low-income students, as voiced by Richard Levin, the President of Yale University.[19]

[19] See http://www.yale.edu/opa/president/statements/20060912.html.

Many institutions, in fact, have strong incentives to continue their early admissions programs. As argued by Avery et.al. (2003) it is unlikely that the conduct of Harvard will be followed by liberal arts colleges that rely on early decision much more than larger universities because small miscalculations about class size can have much more serious consequences than at larger institutions.[20] Another reason why some colleges and universities are reluctant to switch back to single-date admissions programs is that they can manipulate by means of early decision programs the admission and matriculation rates, which in turn determine the rankings of these institutions that students take into account when applying.[21]

Among several rationales behind the adoption of early admissions programs, our paper tackles one involved in the intertemporal quota allocation problem. To this aim, we have modeled college admissions under early decision in a two-period many-to-one matching framework. Our first set of results show that every stable matching system is vulnerable to manipulation by students and colleges via early decision preferences and by colleges via early decision quotas. Next, we have considered an early decision quota game to be played by colleges and studied its equilibria. We have established that a pure strategy equilibrium of this game may not exist. However, when colleges or students have common preferences over the other set of agents, we have found that 'terminating early decision plan' can become a weakly dominant strategy (or an equilibrium strategy) for every college if each student, choosing to act under early decision, applies to his or her first choice college, as strongly recommended by the College Board, many college guides, college admission officers, and counsellars. This result is overwhelmingly striking in that the *raison d'être* of the system ceases to exist if students behave according to the presciptions of the designers or executers of the early decision system.

However, we have also showed that some colleges may have incentives to offer early decision plans when some students compromise under early decision. Moreover, the dynamic nature of our model uncovered the instrumental role played by early decision programs in increasing the bargaining power of colleges (*vis à vis* students) by making the supply side of the market "thinner".

We believe that our model can be extended in several dimensions to cap-

[20] See page 274 of Avery et.al. (2003).

[21] See Avery et.al. (2004).

ture the other raised concerns about and motives for early admissions programs. In this regard, future research may profitably study the early decision and early action programs in comparison, taking into consideration the issues of fairness and manipulability of rankings as well as the manipulability of quotas, capacities and preferences.

Appendix

Proof of Theorem 1. We first prove the theorem for two colleges and one student. Let the matching system $(\varphi^E, \varphi^R) \in \bar{\varphi}^E \times \bar{\varphi}^R$ be stable, $C = \{c_1, c_2\}$, $S = \{s_1\}$, $q^R_{c_1} = 1$, $q^R_{c_2} = 1$, $q^E_{c_1} = 0$, $q^E_{c_2} = 0$, $\hat{q}^E_{c_1} = 1$,

$$P^R_{c_1} = P^E_{c_1} = \{s_1\}, \emptyset,$$

$$P^R_{c_2} = P^E_{c_2} = \{s_1\}, \emptyset,$$

$$P^R_{s_1} = \{c_2\}, \{c_1\}, \emptyset,$$

$$P^E_{s_1} = \{c_1\}, \emptyset.$$

We have $S^E(q^E, R^E) = \{\mu_1\}$, $S^R(q^R, R^R, \mu_1) = \{\mu_2\}$, $S^E(\hat{q}^E_{c_1}, q^E_{c_2}, R^E) = \{\mu_3\}$, and $S^R(q^R, R^R, \mu_2) = \{\mu_3\}$, where

$$\mu_1 = \begin{pmatrix} c_1 & c_2 \\ \emptyset & \emptyset \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} c_1 & c_2 \\ \emptyset & \{s_1\} \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} c_1 & c_2 \\ \{s_1\} & \emptyset \end{pmatrix}.$$

Therefore, $\varphi^E(q^E, R^E) = \mu_1$, $\varphi^R((q^R, R^R), \mu_1) = \mu_2$, $\varphi^E(\hat{q}^E_{c_1}, q^E_{c_2}, R^E) = \mu_3$, and $\varphi^R((q^R, R^R), \mu_2) = \mu_3$. Hence,

$$\varphi^R((q^R, R^R), \varphi^E(\hat{q}^E_{c_1}, q^E_{c_2}, R^E))(c_1) \; P^R_{c_1} \; \varphi^R((q^R, R^R), \varphi^E(q^E, R^E))(c_1).$$

That is, college c_1 can manipulate the matching system (φ^E, φ^R) via early decision quotas when its quota in the early decision is $q^E_{c_1} = 0$, by accepting student compromising in the early decision with the announcement $\hat{q}^E_{c_1} = 1$. This completes the proof for the case of two colleges and one student. Finally, we can include colleges whose top choice is admitting no student and students whose top choice is staying unassigned in both the early decision period and the regular decision period to generalize this proof to cases with at least two colleges and one student. ∎

Proof of Theorem 2. We first prove the theorem for two colleges and two students. Let the matching system $(\varphi^E, \varphi^R) \in \bar{\varphi}^E \times \bar{\varphi}^R$ be stable, $C = \{c_1, c_2\}$, $S = \{s_1, s_2\}$, $q^R_{c_1} = 1$, $q^R_{c_2} = 2$, $q^E_{c_1} = 1$, $q^E_{c_2} = 0$,

$$P^R_{c_1} = \{s_1\}, \{s_2\}, \emptyset,$$

$$P^E_{c_1} = \{s_1\}, \emptyset,$$
$$P^R_{c_2} = P^E_{c_2} = \{s_1\}, \{s_2\}, \emptyset,$$
$$\hat{P}^E_{c_1} = P^R_{c_1},$$
$$P^R_{s_1} = \{c_2\}, \{c_1\}, \emptyset,$$
$$P^E_{s_1} = \emptyset,$$
$$P^R_{s_2} = \{c_2\}, \{c_1\}, \emptyset,$$
$$P^E_{s_2} = \{c_1\}, \emptyset.$$

We have $S^E(q^E, R^E) = \{\mu_1\}$, $S^R(q^R, R^R, \mu_1) = \{\mu_2\}$, $S^E(q^E, \hat{R}^E_{c_1}, R^E_{-c_1}) = \{\mu_3\}$, $S^R(q^R, R^R, \mu_3) = \{\mu_4\}$ where

$$\mu_1 = \begin{pmatrix} c_1 & c_2 \\ \emptyset & \emptyset \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} c_1 & c_2 \\ \emptyset & \{s_1, s_2\} \end{pmatrix},$$

$$\mu_3 = \begin{pmatrix} c_1 & c_2 \\ \{s_2\} & \emptyset \end{pmatrix}, \quad \mu_4 = \begin{pmatrix} c_1 & c_2 \\ \{s_2\} & \{s_1\} \end{pmatrix}.$$

Therefore, $\varphi^E(q^E, R^E) = \mu_1$, $\varphi^R((q^R, R^R), \mu_1) = \mu_2$, $\varphi^E(q^E, \hat{R}^E_{c_1}, R^E_{-c_1}) = \mu_3$, and $\varphi^R((q^R, R^R), \mu_3) = \mu_4$. Hence,

$$\varphi^R((q^R, R^R), \varphi^E(q^E, \hat{R}^E_{c_1}, R^E_{-c_1}))(c_1) \; P^R_{c_1} \; \varphi^R((q^R, R^R), \varphi^E(q^E, R^E))(c_1).$$

That is, college c_1 can manipulate the matching system (φ^E, φ^R) via early decision preferences, completing the proof for the case of two colleges and two students. Finally, we can include colleges whose top choice is admitting no student and students whose top choice is staying unassigned in both the early decision period and the regular decision period to generalize this proof to cases with at least two colleges and two students. ∎

Proof of Theorem 3. We first prove the theorem for two colleges and one student. Let the matching system $(\varphi^E, \varphi^R) \in \bar{\varphi}^E \times \bar{\varphi}^R$ be stable, $C = \{c_1, c_2\}$, $S = \{s_1\}$, $q^R_{c_1} = 1$, $q^R_{c_2} = 1$, $q^E_{c_1} = 0$, $q^E_{c_2} = 1$,

$$P^R_{c_1} = P^E_{c_1} = \{s_1\}, \emptyset,$$

$$P^R_{c_2} = P^E_{c_2} = \{s_1\}, \emptyset,$$

$$P^R_{s_1} = \{c_1\}, \{c_2\}, \emptyset,$$

$$P^E_{s_1} = \{c_2\}, \emptyset,$$

$$\hat{P}^E_{s_1} = \{c_1\}, \emptyset.$$

We have $S^E(q^E, R^E) = \{\mu_1\}$, $S^R(q^R, R^R,\ \mu_1) = \{\mu_1\}$, $S^E(q^E, \hat{R}^E_{s_1}, R^E_{-s_1}) = \{\mu_2\}$, and $S^R(q^R, R^R, \mu_2) = \{\mu_3\}$, where

$$\mu_1 = \begin{pmatrix} c_1 & c_2 \\ \emptyset & \{s_1\} \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} c_1 & c_2 \\ \emptyset & \emptyset \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} c_1 & c_2 \\ \{s_1\} & \emptyset \end{pmatrix}.$$

Therefore, $\varphi^E(q^E, R^E) = \mu_1$, $\varphi^R((q^R, R^R), \mu_1) = \mu_1$, $\varphi^E(q^E, \hat{R}^E_{s_1}, R^E_{-s_1}) = \mu_2$, and $\varphi^R((q^R, R^R), \mu_2) = \mu_3$. Hence,

$$\varphi^R((q^R, R^R), \varphi^E(q^E, \hat{R}^R_{s_1}, R^E_{-s_1}))(s_1)\ P^R_{s_1}\ \varphi^R((q^R, R^R), \varphi^E(q^E, R^E))(s_1).$$

That is, student s_1 can manipulate the matching system (φ^E, φ^R) via early decision preferences, completing the proof for the case of two colleges and one student. Finally, we can include colleges whose top choice is admitting no student and students whose top choice is staying unassigned in both the early decision period and the regular decision period to generalize this proof to cases with at least two colleges and one student. ∎

Proof of Theorem 4. A direct corollary to Theorems 1-3. ∎

Proof of Theorem 5. Consider the regular decision market (C, S, q^R, R^R) with $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, $q^R_{c_1} = 3$, $q^R_{c_2} = 3$, and

$$P^R_{c_1} = \{s_1\}, \{s_2\}, \{s_4\}, \{s_3\}, \{s_5\}, \emptyset,$$

$$P^R_{c_2} = \{s_4\}, \{s_5\}, \{s_1\}, \{s_3\}, \{s_2\}, \emptyset,$$

with $S' P_c^R S''$ for any $c \in C$ and for any $S', S'' \subseteq S$ such that $|S'| > |S''|$,

$$P_s^R = \{c_2\}, \{c_1\}, \emptyset \text{ for } s \in \{s_1, s_2, s_3\},$$

$$P_s^R = \{c_1\}, \{c_2\}, \emptyset \text{ for } s \in \{s_4, s_5\}.$$

We have $\mathcal{Q}_c^E(q^R) = \{0, 1, 2, 3\}$ for all $c \in C$. Let $P_c^E(q^E) = P_c^R$ for all $q^E \in \mathcal{Q}^E(q^R)$ and for all $c \in C$.

Moreover, let $P_s^E((0,0)) = \emptyset$ for $s \in S$, $P_{s_4}^E((0,1)) = \{c_2\}, \emptyset$, $P_s^E((0,2)) = \{c_2\}, \emptyset$ for $s \in \{s_4, s_5\}$, $P_s^E((0,3)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_4, s_5\}$, $P_{s_1}^E((1,0)) = \{c_1\}, \emptyset$ $P_s^E((2,0)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2\}$, $P_s^E((3,0)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2, s_4\}$, $P_{s_1}^E((1,1)) = \{c_1\}, \emptyset$, $P_{s_4}^E((1,1)) = \{c_2\}, \emptyset$, $P_{s_1}^E((1,2)) = \{c_1\}, \emptyset$, $P_s^E((1,2)) = \{c_2\}, \emptyset$ for $s \in \{s_4, s_5\}$, $P_{s_2}^E((1,3)) = \{c_1\}, \emptyset$, $P_s^E((1,3)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_4, s_5\}$, $P_s^E((2,1)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2\}$, $P_{s_4}^E((2,1)) = \{c_2\}, \emptyset$, $P_s^E((3,1)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2, s_4\}$, $P_{s_5}^E((3,1)) = \{c_2\}, \emptyset$, $P_s^E((2,2)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2\}$, $P_s^E((2,2)) = \{c_2\}, \emptyset$ for $s \in \{s_4, s_5\}$, $P_s^E((2,3)) = \{c_1\}, \emptyset$ for $s \in \{s_2, s_4\}$, $P_s^E((2,3)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_3, s_5\}$, $P_s^E((3,2)) = \{c_1\}, \emptyset$ for $s \in \{s_2, s_3, s_4\}$, $P_s^E((3,2)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_5\}$, $P_s^E((3,3)) = \{c_1\}, \emptyset$ for $s \in \{s_4, s_5\}$, $P_s^E((3,3)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_2, s_3\}$. For any $q^E \in \mathcal{Q}^E(q^R)$ and for any $s \in S$, if $P_s^E(q^E)$ is not defined in the above list, assume $P_s^E(q^E) = \emptyset$.

Under the college-optimal matching system $\vec{\varphi}_C$, Table 1 below gives the outcome allocations in the early decision period for each pure strategy in $\mathcal{Q}^E(q^R)$. The first entry in each cell is the assignment of c_1 and the second entry is the assignment of c_2. Those students who are not assigned to either c_1 or c_2 remain unassigned in the early decision period.

Table 1.

	$q_{c_2}^E = 0$	$q_{c_2}^E = 1$	$q_{c_2}^E = 2$	$q_{c_2}^E = 3$
$q_{c_1}^E = 0$	$\emptyset, \emptyset$	$\emptyset, \{s_4\}$	$\emptyset, \{s_4, s_5\}$	$\emptyset, \{s_1, s_4, s_5\}$
$q_{c_1}^E = 1$	$\{s_1\}, \emptyset$	$\{s_1\}, \{s_4\}$	$\{s_1\}, \{s_4, s_5\}$	$\{s_2\}, \{s_1, s_4, s_5\}$
$q_{c_1}^E = 2$	$\{s_1, s_2\}, \emptyset$	$\{s_1, s_2\}, \{s_4\}$	$\{s_1, s_2\}, \{s_4, s_5\}$	$\{s_2, s_4\}, \{s_1, s_3, s_5\}$
$q_{c_1}^E = 3$	$\{s_1, s_2, s_4\}, \emptyset$	$\{s_1, s_2, s_4\}, \{s_5\}$	$\{s_2, s_3, s_4\}, \{s_1, s_5\}$	$\{s_4, s_5\}, \{s_1, s_2, s_3\}$

In Table 2, we give the outcome allocations in the regular decision period for each pure strategy $q^E \in \mathcal{Q}^E(q^R)$ and the corresponding matching $\varphi_C^E(q^E, R^E(q^E))$ at early decision.

Table 2.

	$q_{c_2}^E = 0$	$q_{c_2}^E = 1$	$q_{c_2}^E = 2$	$q_{c_2}^E = 3$
$q_{c_1}^E = 0$	$\{s_4, s_5\}, \{s_1, s_2, s_3\}$	$\{s_2, s_5\}, \{s_1, s_3, s_4\}$	$\{s_2, s_3\}, \{s_1, s_4, s_5\}$	$\{s_2, s_3\}, \{s_1, s_4, s_5\}$
$q_{c_1}^E = 1$	$\{s_1, s_4, s_5\}, \{s_2, s_3\}$	$\{s_1, s_5\}, \{s_2, s_3, s_4\}$	$\{s_1, s_2\}, \{s_3, s_4, s_5\}$	$\{s_2, s_3\}, \{s_1, s_4, s_5\}$
$q_{c_1}^E = 2$	$\{s_1, s_2, s_4\}, \{s_3, s_5\}$	$\{s_1, s_2, s_5\}, \{s_3, s_4\}$	$\{s_1, s_2\}, \{s_3, s_4, s_5\}$	$\{s_2, s_4\}, \{s_1, s_3, s_5\}$
$q_{c_1}^E = 3$	$\{s_1, s_2, s_4\}, \{s_3, s_5\}$	$\{s_1, s_2, s_4\}, \{s_3, s_5\}$	$\{s_2, s_3, s_4\}, \{s_1, s_5\}$	$\{s_4, s_5\}, \{s_1, s_2, s_3\}$

According to the preferences of colleges, their best response correspondences are:

$$\beta_{c_1}^{\vec{\varphi}_C}(0) = \{2,3\},\ \beta_{c_1}^{\vec{\varphi}_C}(1) = \{3\},\ \beta_{c_1}^{\vec{\varphi}_C}(2) = \{3\},\ \beta_{c_1}^{\vec{\varphi}_C}(3) = \{2\},$$

$$\beta_{c_2}^{\vec{\varphi}_C}(0) = \{2,3\},\ \beta_{c_2}^{\vec{\varphi}_C}(1) = \{3\},\ \beta_{c_2}^{\vec{\varphi}_C}(2) = \{2\},\ \beta_{c_2}^{\vec{\varphi}_C}(3) = \{3\}.$$

Therefore the game has no pure strategy equilibrium. ∎

Proof of Theorem 6. We prove by an example. Consider the regular decision market (C, S, q^R, R^R) with $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, $q_{c_1}^R = 3$, $q_{c_2}^R = 3$, and the following preference profile for colleges and students:

$$P_{c_1}^R = \{s_1\}, \{s_2\}, \{s_5\}, \{s_3\}, \{s_4\}, \emptyset,$$

$$P_{c_2}^R = \{s_2\}, \{s_5\}, \{s_1\}, \{s_4\}, \{s_3\}, \emptyset, \text{ with } \{s_2, s_3\}\, P_{c_2}^R\, \{s_1, s_4\},$$

with $S'\, P_c^R\, S''$ for any $c \in C$ and for any $S', S'' \subseteq S$ such that $|S'| > |S''|$,

$$P_s^R = \{c_2\}, \{c_1\}, \emptyset \text{ for } s \in \{s_1, s_3, s_5\},$$

$$P_{s_2}^R = \{c_1\}, \{c_2\}, \emptyset \text{ for } s \in \{s_2, s_4\}.$$

We have $\mathcal{Q}_c^E(q^R) = \{0, 1, 2, 3\}$ for all $c \in C$. Let $P_c^E(q^E) = P_c^R$ for all $q^E \in \mathcal{Q}^E(q^R)$ and for all $c \in C$.

Moreover, let $P_s^E((0,0)) = \emptyset$ for $s \in S$, $P_{s_2}^E((0,1)) = \{c_2\}, \emptyset$, $P_s^E((0,2)) = \{c_2\}, \emptyset$ for $s \in \{s_2, s_5\}$, $P_s^E((0,3)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_2, s_5\}$, $P_{s_1}^E((1,0)) = \{c_1\}, \emptyset$ $P_s^E((2,0)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2\}$, $P_s^E((3,0)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2, s_5\}$, $P_{s_1}^E((1,1)) = \{c_1\}, \emptyset$, $P_{s_2}^E((1,1)) = \{c_2\}, \emptyset$, $P_{s_2}^E((1,2)) = \{c_1\}, \emptyset$, $P_s^E((1,2)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_5\}$, $P_{s_2}^E((1,3)) = \{c_1\}, \emptyset$, $P_s^E((1,3)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_4, s_5\}$, $P_s^E((2,1)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2\}$, $P_{s_5}^E((2,1)) = \{c_2\}, \emptyset$, $P_s^E((3,1)) = \{c_1\}, \emptyset$ for $s \in \{s_1, s_2, s_3\}$, $P_{s_5}^E((3,1)) = \{c_2\}, \emptyset$, $P_s^E((2,2)) = \{c_1\}, \emptyset$ for $s \in \{s_2, s_3\}$, $P_s^E((2,2)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_5\}$, $P_s^E((2,3)) = \{c_1\}, \emptyset$ for $s \in \{s_2, s_4\}$, $P_s^E((2,3)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_3, s_5\}$, $P_s^E((3,2)) = \{c_1\}, \emptyset$ for $s \in \{s_2, s_3, s_4\}$, $P_s^E((3,2)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_5\}$, $P_s^E((3,3)) =$

$\{c_1\}, \emptyset$ for $s \in \{s_2, s_4\}$, $P_s^E((3,3)) = \{c_2\}, \emptyset$ for $s \in \{s_1, s_3, s_5\}$. For any $q^E \in \mathcal{Q}^E(q^R)$ and for any $s \in S$, if $P_s^E(q^E)$ is not defined in the above list, assume $P_s^E(q^E) = \emptyset$.

Under the student-optimal matching system $\vec{\varphi}_S$, Table 3 below (which should be read as the previous tables) gives the outcome allocations of the early admissions game for each pure strategy q^E.

Table 3.

	$q_{c_2}^E = 0$	$q_{c_2}^E = 1$	$q_{c_2}^E = 2$	$q_{c_2}^E = 3$
$q_{c_1}^E = 0$	$\emptyset, \emptyset$	$\emptyset, \{s_2\}$	$\emptyset, \{s_2, s_5\}$	$\emptyset, \{s_1, s_2, s_5\}$
$q_{c_1}^E = 1$	$\{s_1\}, \emptyset$	$\{s_1\}, \{s_2\}$	$\{s_2\}, \{s_1, s_5\}$	$\{s_2\}, \{s_1, s_4, s_5\}$
$q_{c_1}^E = 2$	$\{s_1, s_2\}, \emptyset$	$\{s_1, s_2\}, \{s_5\}$	$\{s_2, s_3\}, \{s_1, s_5\}$	$\{s_2, s_4\}, \{s_1, s_3, s_5\}$
$q_{c_1}^E = 3$	$\{s_1, s_2, s_5\}, \emptyset$	$\{s_1, s_2, s_3\}, \{s_5\}$	$\{s_2, s_3, s_4\}, \{s_1, s_5\}$	$\{s_2, s_4\}, \{s_1, s_3, s_5\}$

In Table 4, we give the outcome allocations in the regular decision period for each pure strategy q^E and the corresponding matching $\varphi_S^E(q^E, R^E(q^E))$ at early decision.

Table 4.

	$q_{c_2}^E = 0$	$q_{c_2}^E = 1$	$q_{c_2}^E = 2$	$q_{c_2}^E = 3$
$q_{c_1}^E = 0$	$\{s_2, s_4\}, \{s_1, s_3, s_5\}$	$\{s_3, s_4\}, \{s_1, s_2, s_5\}$	$\{s_3, s_4\}, \{s_1, s_2, s_5\}$	$\{s_3, s_4\}, \{s_1, s_2, s_5\}$
$q_{c_1}^E = 1$	$\{s_1, s_2, s_4\}, \{s_3, s_5\}$	$\{s_1, s_4\}, \{s_2, s_3, s_5\}$	$\{s_2, s_4\}, \{s_1, s_3, s_5\}$	$\{s_2, s_3\}, \{s_1, s_5, s_5\}$
$q_{c_1}^E = 2$	$\{s_1, s_2, s_4\}, \{s_3, s_5\}$	$\{s_1, s_2, s_4\}, \{s_3, s_5\}$	$\{s_2, s_3, s_4\}, \{s_1, s_5\}$	$\{s_2, s_4\}, \{s_1, s_3, s_5\}$
$q_{c_1}^E = 3$	$\{s_1, s_2, s_5\}, \{s_3, s_4\}$	$\{s_1, s_2, s_3\}, \{s_4, s_5\}$	$\{s_2, s_3, s_4\}, \{s_1, s_5\}$	$\{s_2, s_4\}, \{s_1, s_3, s_5\}$

According to the preferences of colleges, their best response correspondences are:

$$\beta_{c_1}^{\vec{\varphi}_S}(0) = \{3\}, \beta_{c_1}^{\vec{\varphi}_S}(1) = \{3\}, \beta_{c_1}^{\vec{\varphi}_S}(2) = \{2, 3\}, \beta_{c_1}^{\vec{\varphi}_S}(3) = \{1\},$$

$$\beta_{c_2}^{\vec{\varphi}_S}(0) = \{2, 3\}, \beta_{c_2}^{\vec{\varphi}_S}(1) = \{1\}, \beta_{c_2}^{\vec{\varphi}_S}(2) = \{3\}, \beta_{c_2}^{\vec{\varphi}_S}(3) = \{3\}.$$

Therefore, the game has no pure strategy equilibrium. ∎

Proof of Lemma 1. Here, we extend the proof of Lemma 2 in Konishi and Ünver (2006). Let $(C, S, q^R, q^E, R^R, R^E)$ be a college admissions market, where $q^R \in N_+^n$, $q^E \in \mathcal{Q}^E(q^R)$, $R^R \in \mathcal{R}^{R-CC}$, and $R^E \in \mathcal{R}^E(R^R, q^E) \cap \mathcal{R}^{E-CC}$. We will show that $(\hat{\mu}^E, \hat{\mu}^R)$ is the unique pair of stable matchings in the early and regular decision periods.

Consider any stable matching μ^E in the early decision period. If $\hat{l}^E = 0$, then $\mu^E(s) = \emptyset$ for all $s \in S$. If $\hat{l}^E > 0$, we have $\mu^E(s_1) = Ch^E_{s_1}(R^E_{s_1}, C^{E,1})$, for otherwise the pair $(s_1, Ch^E_{s_1}(R^E_{s_1}, C^{E,1}))$ would block μ^E if $Ch^E_{s_1}(R^E_{s_1}, C^{E,1}) \neq \emptyset$; and s_1 would be better off by staying unmatched if $Ch^E_{s_1}(R^E_{s_1}, C^{E,1}) = \emptyset$. If $\hat{l}^E > 1$, first note that $C^{E,t} \supseteq C^{E,t+1}$ for any $t \leq \hat{l}^E - 1$, since $C^{E,t}$ monotonically shrinks (weakly) by construction. Then, given $\mu^E(s_1)$, we have $\mu^E(s_2) = Ch^E_{s_2}(R^E_{s_2}, C^{E,2})$, for otherwise the pair $(s_2, Ch^E_{s_2}(R^E_{s_2}, C^{E,2}))$ would block μ^E if $Ch^E_{s_2}(R^E_{s_2}, C^{E,2}) \neq \emptyset$; and s_2 would be better off by staying unmatched if $Ch^E_{s_2}(R^E_{s_2}, C^{E,2}) = \emptyset$. Similarly, for any $t \leq \hat{l}^E$, $\mu^E(s_t) = Ch^E_{s_t}(R^E_{s_t}, C^{E,t})$. Thus, we must have $\mu^E = \hat{\mu}^E$.

Now given $\hat{\mu}^E$, consider any stable matching μ^R in the regular decision period. If $\hat{l}^R = 0$, then $\mu^R(s) = \hat{\mu}^E(s) = \emptyset$ for all $s \in S$. Now consider the case in which $\hat{l}^R > 0$. Clearly, $\mu^R(s) = \hat{\mu}^E(s)$ for all $s \in \{s_1, s_2, ..., s_{\hat{l}^R}\}$ such that $\hat{\mu}^E(s) \neq \emptyset$. Moreover, we have $\mu^R(s_1) = Ch^R_{s_1}(R^R_{s_1}, C^{R,1}(\hat{\mu}^E))$ if $\hat{\mu}^E(s_1) = \emptyset$, for otherwise the pair $(s_1, Ch^R_{s_1}(R^R_{s_1}, C^{R,1}(\hat{\mu}^E)))$ would block μ^R if $Ch^R_{s_1}(R^R_{s_1}, C^{R,1}(\hat{\mu}^E)) \neq \emptyset$; and s_1 would be better off by staying unmatched if $Ch^R_{s_1}(R^R_{s_1}, C^{R,1}(\hat{\mu}^E)) = \emptyset$. If $\hat{l}^R > 1$, first note that $C^{R,t}(\hat{\mu}^E) \supseteq C^{R,t+1}(\hat{\mu}^E)$ for any $t \leq \hat{l}^R - 1$, since $C^{R,t}(\hat{\mu}^E)$ monotonically shrinks (weakly) by construction. Then, given $\mu^R(s_1)$, we have $\mu^R(s_2) = Ch^R_{s_2}(R^R_{s_2}, C^{R,2}(\hat{\mu}^E))$ if $\hat{\mu}^E(s_2) = \emptyset$, for otherwise the pair $(s_2, Ch^R_{s_2}(R^R_{s_2}, C^{R,2}(\hat{\mu}^E)))$ would block μ^R if $Ch^R_{s_2}(R^R_{s_2}, C^{R,2}(\hat{\mu}^E)) \neq \emptyset$; and s_2 would be better off by staying unmatched if $Ch^R_{s_2}(R^R_{s_2}, C^{R,2}(\hat{\mu}^E)) = \emptyset$. Similarly, for any $t \leq \hat{l}^R$, $\mu^R(s_t) = Ch^R_{s_t}(R^R_{s_t}, C^{R,t}(\hat{\mu}^E))$ if $\hat{\mu}^E(s_t) = \emptyset$. Thus, we must have $\mu^R = \hat{\mu}^R$. ∎

Proof of Theorem 7. Let (C, S, q^R, R^R) be a regular decision market such that $R^R \in \mathcal{R}^{R-CC}$. Let the early decision preference of each college $c \in C$ be $R^E_c(q^E) = R^R_c$ for any $q^E \in \mathcal{Q}^E(q^R)$. Assume that for all $s \in S$, $Top(R^E_s(q^E); 1) \in \{Top(R^R_s; 1), \emptyset\}$ for all $q^E \in \mathcal{Q}^E(q^R)$. Pick a college $c \in C$. Consider any $q'^E \in \mathcal{Q}^E(q^R)$. Let $\vec{\hat{\varphi}}$ be the unique stable matching system in $(C, S, q^R, (q^E_c, q'^E_{-c}), R^R, R^E(q^E_c, q'^E_{-c}))$ where $q^E_c = 0$. We have $\hat{\varphi}^E(0, q'^E_{-c}, R^E(0, q'^E_{-c}))(c) = \emptyset$. Let $\hat{\varphi}^R((q^R, R^R), \hat{\varphi}^E(0, q'^E_{-c}, R^E(0, q'^E_{-c})))(c) = \{s_k, s_l, ..., s_r\}$, where $k < l < ... < r \leq n$. With the report $q'^E_c \in \mathcal{Q}^E_c(q^R)$ of college c, we have $\hat{\varphi}^E(q'^E, R^E(q'^E))(c) \backslash \{s_k, s_l, ..., s_r\} \subseteq \{s_{r+1}, s_{r+2}, ..., s_n\}$. We consider two cases: if $\hat{\varphi}^E(q'^E, R^E(q'^E))(c) \backslash \{s_k, s_l, ..., s_r\} = \emptyset$, then we have $\hat{\varphi}^R((q^R, R^R), \hat{\varphi}^E(q'^E, R^E(q'^E)))(c) = \{s_k, s_l, ..., s_r\}$ by the construction of $\hat{\varphi}^R$. Oppositely, if $\hat{\varphi}^E(q'^E, R^E(q'^E))(c) \backslash \{s_k, s_l, ..., s_r\} \neq \emptyset$, then it must be true that $|\{s_k, s_l, ..., s_r\}| = q_c^R$, and $\hat{\varphi}^R((q^R, R^R), \hat{\varphi}^E(q'^E, R^E(q'^E)))(c) = T \subset \{s_k, s_l, ..., s_r\} \cup \{s_{r+1}, s_{r+2}, ..., s_n\}$ such that $|T| = q_c^R$ by the construction of $\hat{\varphi}^R$. Thus, we have $(0, q'^E_{-c}) \succeq_c^{\vec{\varphi}} q'^E$ in both cases by the responsiveness of colleges' preferences. ∎

Proof of Lemma 2. Here, we extend the proof of Lemma 3 in Konishi and Ünver (2006). Let $(C, S, q^R, q^E, R^R, R^E)$ be a college admissions market, where $q^R \in N_+^n$, $q^E \in \mathcal{Q}^E(q^R)$, $R^R \in \mathcal{R}^{R-CS}$, and $R^E \in \mathcal{R}^E(R^R, q^E) \cap \mathcal{R}^{E-CS}$. We will show that $(\tilde{\mu}^E, \tilde{\mu}^R)$ is the unique pair of stable matchings in the early and regular decision periods.

Consider any stable matching μ^E in the early decision period. If $S^E(q^E) = \emptyset$, then $\mu^E(c) = \emptyset$ for all $c \in C$. If $S^E(q^E) \neq \emptyset$, we have $\mu^E(c_1) = Ch_{c_1}^E(R_{c_1}^E, q_{c_1}^E, S^E(q^E))$. Otherwise, if $Ch_{c_1}^E(R_{c_1}^E, q_{c_1}^E, S^E(q^E)) \neq \emptyset$, then for some $s \in Ch_{c_1}^E(R_{c_1}^E, q_{c_1}^E, S^E(q^E))$, pair (s, c_1) would block μ^E; and if $Ch_{c_1}^E(R_{c_1}^E, q_{c_1}^E, S^E(q^E)) = \emptyset$, then c_1 would be better off by staying unmatched and deviating from μ^E. Notice also that for all $c \in C \backslash \{c_1\}$, we have $\mu^E(c) = \emptyset$, trivially. Thus, we must have $\mu^E = \tilde{\mu}^E$.

Now given $\tilde{\mu}^E$, consider any stable matching μ^R in the regular decision period. If $\tilde{l}^R = 0$, then $\mu^R(c) = \tilde{\mu}^E(c) = \emptyset$ for all $c \in C$. Now consider the case in which $\tilde{l}^R > 0$. Clearly, $\mu^R(c) \supseteq \tilde{\mu}^E(c)$ for all $c \in C$. Moreover, we have $\mu^R(c_1) \backslash \tilde{\mu}^E(c_1) = Ch_{c_1}^R(R_{c_1}^R, q_{c_1}^{R,1}, \tilde{\mu}^E, S^{R,1}(q^E, q^R))$. Otherwise, if $Ch_{c_1}^R(R_{c_1}^R, q_{c_1}^{R,1}, \tilde{\mu}^E, S^{R,1}(q^E, q^R)) \neq \emptyset$, then for some $s \in Ch_{c_1}^R(R_{c_1}^R, q_{c_1}^{R,1}, \tilde{\mu}^E, S^{R,1}(q^E, q^R))$ pair (s, c_1) would block μ^R; and if $Ch_{c_1}^R(R_{c_1}^R, q_{c_1}^{R,1}, \tilde{\mu}^E, S^{R,1}(q^E, q^R)) = \emptyset$, then c_1 would be better off by staying unmatched and deviating from μ^R. If $\tilde{l}^R > 1$, first note that $S^{E,t}(q^E, q^R) \supseteq S^{E,t+1}(q^E, q^R)$ for any $t \leq \tilde{l}^E - 1$, since $S^{E,t}(q^E, q^R)$ monotonically shrinks (weakly) by construction. Then, given $\mu^R(c_1)$, we have $\mu^R(c_2) \backslash \tilde{\mu}^E(c_2) = Ch_{c_2}^R(R_{c_2}^R, q_{c_2}^{R,2}, \tilde{\mu}^E, S^{R,2}(q^E, q^R))$. Otherwise, if $Ch_{c_2}^R(R_{c_2}^R, q_{c_2}^{R,2}, \tilde{\mu}^E, S^{R,2}(q^E, q^R)) \neq \emptyset$, then for some $s \in Ch_{c_2}^R(R_{c_2}^R, q_{c_2}^{R,2}, \tilde{\mu}^E, S^{R,2}(q^E, q^R))$ pair (s, c_2) would block μ^R; and if $Ch_{c_2}^R(R_{c_2}^R, q_{c_2}^{R,2}, \tilde{\mu}^E, S^{R,2}(q^E, q^R)) = \emptyset$, then c_2 would be better off by staying unmatched and deviating from μ^R. Similarly, for any $t \leq \tilde{l}^R$, $\mu^R(c_t) \backslash \tilde{\mu}^E(c_t) = Ch_{c_t}^R(R_{c_t}^R, q_{c_t}^{R,t}, \tilde{\mu}^E, S^{R,t}(q^E, q^R))$. Thus, we must have $\mu^R = \tilde{\mu}^R$. ∎

Proof of Theorem 8. Let (C, S, q^R, R^R) be a regular decision market such that $R^R \in \mathcal{R}^{R-CS}$. Let the early decision preferences of colleges be $(R_c^E(q^E))_{c \in C}$, for any $q^E \in \mathcal{Q}^E(q^R)$. Assume that for all $s \in S$, $Top(R_s^E(q^E); 1) \in \{Top(R_s^R; 1), \emptyset\}$ for all $q^E \in \mathcal{Q}^E(q^R)$. Consider any $q^E \in$

$\mathcal{Q}^E(q^R)$. Let $\vec{\tilde{\varphi}}$ be the unique stable matching system in $(C, S, q^R, q^E, R^R, R^E(q^E))$. Rename and reorder acceptable colleges by the preference ordering P_s^R of student $s \in S$ as $c_1, c_2, ..., c_{l^R(s)}$. Let $\tilde{l}^R = \max\{l^R(s) : s \in S\}$. If $\tilde{l}^R = 0$, the proof is trivially complete. Suppose $\tilde{l}^R > 0$. Let $\bar{S} = \{s \in S : l^R(s) > 0\}$. By assumption, $Top(R_s^E(q^E); 1) \in \{c_1, \emptyset\}$ for all $s \in \bar{S}$ and for all $q^E \in \mathcal{Q}^E(q^R)$. We have $q_c^E \in \beta_c^{\vec{\tilde{\varphi}}}(q_{-c}^E)$ for any college c such that $c \notin A(R_s^R)$ for any $s \in \bar{S}$. So, pick any college $c_t \in \{c_1, c_2, ..., c_{\tilde{l}^R}\}$, and consider any feasible strategy $q_{c_t}^{\prime E} \in \mathcal{Q}_{c_t}^E(q^R)$. Clearly, we have $\tilde{\varphi}^R((q^R, R^R), \tilde{\varphi}^E((q_{c_t}^{\prime E}, q_{-c_t}^E), R^E(q_{c_t}^{\prime E}, q_{-c_t}^E)))(c_t) = \tilde{\varphi}^R((q^R, R^R), \tilde{\varphi}^E(q^E, R^E(q^E)))(c_t)$. (No college except for the top-ranked one with respect to the common preferences of students can ever accept any student in the early decision period. Given this, c_1 is also impartial to any early quota decision.) Therefore, $q_{c_t}^E \in \beta_{c_t}^{\vec{\tilde{\varphi}}}(q_{-c_t}^E)$. ∎

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