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Ciżkowicz, Piotr and Hołda, Marcin and Rzońca, Andrzej

Warsaw School of Economics, Poland, National Bank of Poland

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Piotr Ciżkowicz*, Marcin Hołda, Andrzej Rzońca*****

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Abstract: The article contains a review of monetary growth models. We analyze the ways in which money is introduced into these models and the models' conclusions about the impact of inflation on investment. We find that the models differ widely with respect to the ways in which they account for money and its functions in the economy as well as with respect to the "technical" assumptions, about e.g. the form of the utility function or the production function. Despite these differences most models fail to adequately capture money's role and are highly sensitive to changes in the assumptions. Moreover, the models differ in their predictions about inflation's impact on capital accumulation, with some models offering conclusions that are not only counterintuitive but also inconsistent with empirical evidence.

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Key words: investment, inflation, monetary models of growth, monetary search theoretic models

*Warsaw School of Economics, pcizko@sgh.waw.pl
**National Bank of Poland, Warsaw School of Economics, mholda@mail.nbp.pl
***Warsaw School of Economics, arzonc@sgh.waw.pl

1. Introductory remarks

Price stability, or – more generally – the care about the value of the domestic currency is the major monetary policy objective of all central banks. It is motivated by the conviction that inflation, above a certain level, is detrimental to economic growth in the long run. The conviction about the high costs of inflation is based not only on strong theoretical grounds (see e.g. Fischer and Modligliani 1980) but also on the results of numerous empirical studies (see. e.g. Levine and Renelt 1992; Fischer 1993; Barro 1995; Bruno and Easterly 1998; Li and Zou 2002). Although one can find studies according to which inflation, as long as it is low, does not have a statistically significant effect on economic performance, one cannot point at any serious studies which would ascertain a positive long-run relationship between economic performance and high inflation.

In the studies about the effects of inflation many mechanisms of its influence on the economy have been isolated, among them the impact of changes in inflation on the size of investment. However, no agreement exists among economists as to the direction, magnitude and role of the particular channels of this impact. One of the possible reasons is the fact that the analysis of this relationship by means of mathematical models of the economy, the fundamental tool in economic growth theory, does not provide consistent conclusions.

In the present article a systematic review of monetary growth models is conducted, with particular emphasis on two areas of differences between these models: the first, connected with the way in which money and its functions in the economy are included in the model, and the second, resulting from the "technical" assumptions about e.g. the form of the utility function or the production function. The aim of the paper is, on the one hand, to summarize the most important ways of introducing money into growth models, and on the other hand, to examine how sensitive the conlusions from these models are to changes in the assumptions. We conduct some kind of Solow test. Solow (1956, p. 65) concluded that

„The art of successful theorizing is to make the inevitable assumptions in such a way that the final results are not very sensitive. A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that the crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if this assumption is dubious, the results are suspect".

Due to the fact that the review does not encompass all monetary growth models available in the literature, the criteria for the choice of models to be analyzed as well as the scope of reasoning conducted on their basis are explained below.

First, only models with an infinite time horizon are presented in the article[1], whereas overlapping-generations (henceforth: OLG) models are left out of the analysis. The rationale is that in OLG models money does not fulfil most of the functions that it serves in the real-world economy (e.g. facilitating transactions and the comparison of the value of goods). Money's principal role in these models is thesaurisation – money is a kind of fixed asset that facilitates intergenerational exchange. Such a narrow role for money leaves little room for studying the influence of changes in the value of money on the real economy. This in turn leads to conclusions drawn from these models being in many cases contradictory to implications of models based on other concepts, directly accounting for the transaction role of money (cf. McCallum 1984), and, above all, contradictory to empirical evidence (cf. Mayer 1996, pp. 161-163).

Second, the analysis concentrates on exogenous growth models, i.e. models in which economic growth results from growth in factor productivity unexplained by variables included in the model. On the one hand, such models allow one to analyze the full spectrum of methods of introducing money into the growth models, and on the other hand, they are solved analytically[2], in contrast to some endogenous growth models in which the growth in factor productivity is explicitly modelled instead of being assumed as given, as well as in contrast to OLG models. It should also be emphasized that the effect of inflation on capital accumulation in exogenous growth models allows one to determine qualitatively the direction of the relationhip between economic growth and inflation in many endogenous growth models. Endogenous growth models may in many cases be reduced to the AK model, i.e. the simplest endogenous growth model in which economic growth is fully determined by capital outlays (although in these models capital is usually defined more broadly than in exogenous growth models – see e.g. Acemoglu, 2009).

Third, the analysis of conclusions from the models presented in the article is limited mainly to the long-run implications, i.e. to the steady-state solution. The generalization of the analysis to the path of reaching the steady state is relatively rare in the literature as in many cases it does not provide results that could be given a clear economic interpretation[3].

The article has the following structure.

The second section contains an analysis of exogenous growth models in which money is introduced on the basis of *ad hoc* assumptions. Considered are all major methods of modelling money based on this approach. Each model is presented according to the same pattern. First, the way in which money is introduced in the model is demonstrated. Both the "technical" formulation of the problem (the way in which particular equations are modified) and the intuitive motivation for the particular specifications are presented. Then the steady-state solution of the model is described as well as the conclusions with respect to the relationship between inflation and capital accumulation. Defined are also the assumptions on which the direction of this relationship critically hinges.

The third section contains a description of exogenous growth models in which the transactions role of money is derived from the characteristics of the exchange conducted between economic agents, and not derived on the basis of ad hoc assumptions. These models have been deemed *monetary search theoretic models*.

The fourth section contains a summary of the main conclusions.

2. Monetary models with money introduced on the basis of *ad hoc* assumptions

2. 1. Money as a substitute for capital

The first exogenous growth model allowing for the impact of inflation on investment was presented by Tobin (1965). He introduced into the Solow (1956)-Swan (1956) model (henceforth: SS model) money issued by the government in order to finance public expenditure. The main channel through which money impacts the real variables in this model is through its impact on the level of household real disposable income (Y_D). In the SS model real disposable income is equal to real output ($Y=F(K,L)$), whereas in Tobin's model it is a sum of two elements[4].

$$Y_D = F(K,L) + \frac{\partial\left(\frac{M}{p}\right)}{\partial t} \tag{1}$$

Where,

- M - money supply
- p - price level

The second expression on the right-hand side of equation (1) reflects the change in time of the real money stock's value due to the interaction of two factors of opposing signs:

$$\frac{\partial\left(\frac{M}{p}\right)}{\partial t}=\frac{\partial\frac{M}{p}}{\partial M}\frac{dM}{dt}+\frac{\partial\frac{M}{p}}{\partial p}\frac{dp}{dt}=\frac{1}{p}\dot{M}-\frac{M}{p^2}\dot{p}=\frac{M}{p}\left(\frac{\dot{M}}{M}-\frac{\dot{p}}{p}\right)=\frac{M}{p}(\mu-\pi) \tag{2}$$

where:

- $\mu=\frac{\dot{M}}{M}$ - growth rate of nominal money supply;
- $\pi=\frac{\dot{p}}{p}$ - growth rate of the price level (inflation).

In a model defined in this way there are two types of assets in which households store their wealth (A): physical capital (K) and the real money stock ($\frac{M}{p}$), hence the real value of wealth per representative household can be defined as follows:

$$a=m+k \tag{3}$$

Money demand is defined as:

$$m=\beta(.)k \tag{4}$$

β being a decreasing function of the opportunity cost of holding wealth in the form of money. This cost is equal to the natural rate of interest (i), that is the difference between the rate of return on capital (r) (equal to the marginal product of capital ($\frac{\partial f}{\partial k}$)) and the rate of return on money equal to $(-\pi)$ [5]:

$$i = r - (-\pi) = r + \pi \tag{5}$$

Thus a rise in inflation means a rise in the opportunity cost of holding wealth in the form of money and causes a reallocation of wealth leading to a decrease in money's share and an increase in capital's share in the asset portfolio[6].

In the steady state, the capital stock is given by the following expression:

$$0 = sy - (1-s)\beta(r+\pi)nk - nk \tag{6}$$

where s stands for the saving rate, and n is the exogenous rate of population growth. It follows from equation (6) that the capital stock *per capita* is an increasing function of inflation, i.e. $\frac{\partial k}{\partial \pi} > 0$. Similarly, the investment rate ($\frac{\dot{K}}{Y}$) is an increasing function of inflation, i.e. $\frac{\partial \frac{\dot{K}}{Y}}{\partial \pi} > 0$. Around the steady state it is equal to:

$$\frac{\dot{K}}{Y} = \frac{s}{1+(1-s)\beta} \quad , \; \beta = \beta(r+\pi) \tag{7}$$

while its derivative with respect to inflation

$$\frac{\partial \frac{\dot{K}}{Y}}{\partial \pi} = \frac{\partial \frac{s}{1+(1-s)\beta}}{\partial \pi} = -\frac{s(1-s)\frac{\partial \beta}{\partial \pi}}{\left(1+(1-s)\beta\right)^2} \tag{8}$$

where: $\frac{\partial \beta}{\partial \pi}$ <0 (see comments concerning eq. 4).

The result obtained by Tobin can be interpreted as follows. Given a level of real wealth, the level of capital in the economy depends on the proportion in which wealth is divided into capital and the real money stock. If the rate of return on the real money stock falls, a part of wealth is reallocated to the more profitable asset, which is capital. Due to the fact that the opportunity cost of holding wealth in the form of money is equal to the nominal interest rate, that is to the sum of the marginal product of capital and inflation, a rise in inflation causes a decrease in the real money stock and an increase in the capital stock. In the model the stock of wealth is constant by assumption, thus higher inflation will lead to the growth in the capital stock. This counterintuitive conclusion is purely a result of the model's assumptions. In reality, the stock of wealth is an endogenous variable, dependent on the choice made by the household while allocating income to consumption and savings. In order for the change in the portfolio structure to have an effect on the capital stock in a way consistent with Tobin's model, the assumption about the saving rate being independent of inflation is necessary[7]. Only then is an increase in the capital's share in the asset portfolio the only way to avoid an inflation-driven decrease in wealth. Levhari and Patinkin (1968) have shown that if the above assumption is relaxed and the saving rate is a function of the rates of return on assets included in the portfolio (money and capital), that is:

$$s = s(r, -\pi), \quad \frac{\partial s}{\partial r} > 0, \; \frac{\partial s}{\partial \pi} < 0 \tag{9}$$

then the Tobin effect need not occur. Similarly, Dornbusch and Frenkel (1973), assuming the following consumption function:

$$c = c(a, \pi), \quad \frac{\partial c}{\partial a} > 0 \tag{10}$$

have demonstrated that assuming a positive impact of inflation on consumption, Tobin's effect is reversed (that is higher inflation means a lower capital stock on the balanced-growth path).

Sensitivity to changes in the assumptions is but one of the criticisms addressed at Tobin's model. Much more important objections concern the fact that the model does not account, even in an *ad hoc* manner, for money's function in the exchange process. This leads to at least two contradictions in the model's results.

First, as Johnson (1966) and Levhari and Patinkin (1968) have noticed, the solution of the model stands in contradiction to economic intuition. In Tobin's model the steady-state levels of capital, output and consumption are lower than in the SS model, if only the real money stock is different from zero. Due to the fact that Tobin's only modification to the SS model was the introduction of money into the economy, this would mean that money (regardless of its role) lowers household welfare in the long run. This would be inconsistent with the broadly accepted and rooted in experience view that welfare would be lower in an economy without money than in the one with money.

Second, the model does not explain, why households were to hold a part of their wealth in money (particularly, as the elimination of money would increase welfare – see preceding paragraph). In order for the model to converge to the balanced-growth path, the nominal interest rate ($i = r + \pi$) must be greater than 0, thus the rate of return on money must be lower than the rate of return on capital, that is $-\pi < r$. If money were only a store of value, households would have no reason to hold a part of their wealth in money as capital would always yield a higher return. Therefore, in order for households to be inclined to hold any part of their wealth in the form of money, it is necessary to make additional assumptions concerning the role of money, assumptions which in Tobin's model are not explicitly presented. The addition of such assumptions[8] causes the model to give ambiguous results with respect to the direction in which inflation impacts capital accumulation on the balanced-growth path.

2.2. Money as a direct source of utility

One of the first attempts to eliminate the assumptions that formed the basis for the critique of Tobin's model was made by Sidrauski (1967). He proposed a monetary growth model that is a modification of the Ramsey-Cass-Koopmans (henceforth: RCK) model. In the RCK model, unlike in the SS model, the saving rate is not exogenous but reflects the intertemporal decisions made by households to maximize the whole consumption path. The change introduced by Sidrauski consisted in modifying the household's utility function in such a way that it would take account of the utility of the stream of services provided by the real money[9] stock held by households. Monetary models in which money is included in the utility function are called *Money-in-Utility-Function* (henceforth: MIUF) models. Assuming the stream of services to be proportional to the real money stock, the household's temporary utility function (that is the utility of consumption in a given period) can be formulated in a MIUF model as follows:

$$U_t = u(c_t, m_t) \tag{11}$$

The function U_t is increasing in both arguments and concave. If one assumes that $\lim_{m \to 0} \frac{\partial u(c,m)}{\partial m} = \infty$, then the real money stock will always be greater than zero.

The total utility function maximized by the household is defined in the following way:

$$W = \int_0^\infty u(c_t, m_t) \cdot e^{-\delta t} dt \tag{12}$$

Where δ is the houseold's time discount rate. Function (12) is maximized subject to two constraints. The first one states that at every moment wealth is distributed between capital and the real money stock:

$$a = m + k \tag{13}$$

The second is a law of motion for wealth:

$$\dot{a} = f(k) - c + \tau - n \cdot a - \pi \cdot m \tag{14}$$

Where τ stands for government transfers. According to equation (14) the accumulation of wealth in a given period is increased by savings ($f(k) - c$) and government transfers, and decreased by the (negative when inflation is above zero) return on the real money stock ($\pi \cdot m$). In turn, the element ($n \cdot a$) stands for the size of wealth accumulation that, given the rate of growth in the number of households (*n*), is necessary to keep the wealth per household constant.

For the above model the current-value Hamiltonian[10] is:

$$H = u(c, m) + \lambda\left(f(k) - c + \tau - na - \pi m\right) + \gamma(a - m - k) \tag{15}$$

In the steady state the following first-order conditions (henceforth: FOCs) must be satisfied:

$$\frac{\partial f(k)}{\partial k} = \delta + n \tag{16}$$

and

$$\frac{\partial u(c,m)}{\partial c}\left(\pi+\delta+n\right)=\frac{\partial u(c,m)}{\partial m} \tag{17}$$

Condition (16) states that in the steady state the marginal product of capital is equal to the sum of the time discount rate and the rate of growth in the number of households. Households accumulate capital until the obtained rate of return compensates for the cost of deferred consumption. Thus this result is consistent with the standard RCK model (without money – see e.g. Romer 2000). The second of the conditions sets the marginal utility of money equal to the marginal cost of holding money. The system of equations (16) and (17) describes the model's steady-state solution for the variables *(k,m)* where the variable c is presented as a function of the variable k with the use of the budget constraint in the steady state: $c = f(k) - nk$ [11].

In order to determine the direction of inflation's impact on capital accumulation in the steady state one can use the implicit function theorem[12], which (given some fairly general assumptions) lets one find the derivative of any endogenous variable of a system with respect to any exogenous variable[13]. The derivative of capital with respect to the inflation rate is equal to:

$$\frac{\partial k}{\partial \pi}=\frac{0}{\dfrac{\partial^2 f}{\partial k^2}\left(\dfrac{\partial^2 u}{\partial c\partial m}\left(\pi+\delta+n\right)-\dfrac{\partial^2 u}{\partial m^2}\right)}=0 \tag{18}$$

The fact that level of capital in the steady state does not depend on the inflation rate implies that inflation also does not have an effect on the level of output and consumption in the steady state. Money is superneutral in this model, that is the values of real variables are independent both of the money stock and the the growth rate of money supply[14]. The only variable that is negatively impacted by higher inflation is the real money stock[15].

In the MIUF model money is superneutral only in the steady state. As Fischer (1979) and Asako (1983)[16] have demonstrated, whether inflation has an effect on the level and the rate of capital accumulation outside the steady state depends on the form of the utility function. For a constant relative risk aversion (CRRA) utility function of the form

$$u(c,m)=\frac{(c^{\alpha}m^{\beta})^{1-\theta}}{1-\theta},\ \alpha,\beta,\theta>0;\ \ \alpha+\beta\leq 1 \tag{19}$$

the speed with which the economy reaches the steady state will be independent of the inflation level only for $\theta\to 1$. For $\theta>1$ consumption and the real money stock will be substitutes in the utility function (i.e. $\frac{\partial^2 u}{\partial c\partial m}<0$). As a result, higher inflation, by increasing the cost of holding money, will simultaneously cause consumption growth, and thus a fall in savings and a fall in capital accumulation. For $\theta<1$ consumption and the real money stock are complementary goods in the utility function (i.e. $\frac{\partial^2 u}{\partial c\partial m}>0$), which means that higher inflation, by increasing the cost of holding money, will at the same time cause a fall in consumption, and thus a rise in savings and capital accumulation. Thus the model does not give a clear-cut answer to the question of the direction of inflation's impact on capital accumulation in a situation in which the economy is outside the steady state.

The critique of the MIUF model as proposed by Sidrauski concerns mainly two issues:

First, even though the model solves the problem, present in Tobin's model, of a lack of rationale for households' holding a positive real money stock, it does so in an *ad hoc* manner. The fact that the real money stock is always positive is a result of the (presented above) assumptions about the utility function's form. The model does not contain a clearly-defined mechanism describing the role money plays in the transaction process.

Second, the money superneutrality effect present in the MIUF model is rather a special case than a rule for monetary growth models. As demonstrated earlier, minor changes to the assumptions, done even within this model's framework, cause the superneutrality effect to disappear outside the steady state. A similar situation occurs in the case of conclusions for the steady-state: a "technical" change in the assumptions leads to a qualitative change in the conclusions about the role of money in the modelled economy. Below two models are presented that are the result of modifications to the assumptions of Sidrauski's model. The effect of both modifications is that money is no longer superneutral in the model.

2.2.1. Money as a direct source of utility when the supply of labour is elastic

In Sidrauski's model the supply of labour is inelastic, equal to the number of households and depends only on the exogenous population growth rate. In the model first proposed by Brock (1974) leisure is another (in addition to consumption and the real money stock) argument of the utility function. Households allocate their time to labour which will let them increase their consumption and in effect their utility, and to leisure which is a direct source of utility.

If by $x_t = 1 - l_t$ we denote the amount of leisure chosen by the household, then its total utility function will have the following form:

$$W = \int_0^\infty u(c_t, m_t, x_t) \cdot e^{-\delta t} dt \tag{20}$$

As a result of the modification introduced, the model's steady-state solution is defined by three conditions:

$$\frac{\partial f(k,l)}{\partial k} = \delta + n\,, \tag{21}$$

$$\frac{\partial u(c,m,x)}{\partial c}\left(\pi + \delta + n\right) = \frac{\partial u(c,m,x)}{\partial m} \tag{22}$$

$$\frac{\partial f(k,l)}{\partial l}\frac{\partial u(c,m,x)}{\partial c} = \frac{\partial u(c,m,x)}{\partial x} \tag{23}$$

The first two conditions are equivalent to conditions defining the solution of the standard Sidrauski model. On the other hand, condition (23) stipulates that the labour input is increased by the household until the increase in consumption utility obtained due to an additional unit of labour is equal to the loss of utility due to a decrease in leisure[17].

Money superneutrality is retained in this model when the relationhip between the marginal utility of leisure and the marginal utility of consumption is independent of the real money stock. This condition is fulfilled if the utility function is separable with respect to money, i.e. it takes the general form:

$$u(c,m,x) = v(c,x)w(m) \tag{24}$$

Given this assumption, the following equality holds:

$$\frac{\dfrac{\partial^2 u(c,m,x)}{\partial x \partial m}}{\dfrac{\partial^2 u(c,m,x)}{\partial c \partial m}} = \frac{\dfrac{\partial v(c,x)}{\partial x}\dfrac{\partial w(m)}{\partial m}}{\dfrac{\partial v(c,x)}{\partial c}\dfrac{\partial w(m)}{\partial m}} = \frac{\dfrac{\partial v(c,x)}{\partial x}}{\dfrac{\partial v(c,x)}{\partial c}} \tag{25}$$

which, as implied by condtion (23), means that the marginal product of labour does not change in response to changes in the real money stock. In effect, a rise in inflation and the associated reduction in the real money stock does not change households' supply of labour. Otherwise, that is if the utility function is not separable with respect to money, a change in the inflation rate can change the supply of labour and other real quantities.

In the steady state the derivative of capital with respect to the inflation rate is equal to:

$$\frac{\partial k}{\partial \pi} = -\frac{\frac{\partial^2 f}{\partial k \partial l}\frac{\partial u}{\partial c}\left(\frac{\partial f}{\partial l}\frac{\partial^2 u}{\partial c \partial m} - \frac{\partial^2 u}{\partial x \partial m}\right)}{\det A} \tag{26}$$

where

$$A = \begin{bmatrix} \frac{\partial^2 f}{\partial k^2} & \frac{\partial^2 f}{\partial k \partial l} & 0 \\ 0 & -\frac{\partial^2 u}{\partial m \partial x} + \frac{\partial^2 u}{\partial c \partial x}(\pi + \delta + n) & \frac{\partial^2 u}{\partial m^2} - \frac{\partial^2 u}{\partial c \partial m}(\pi + \delta + n) \\ \frac{\partial u}{\partial c}\frac{\partial^2 f}{\partial k \partial l} & -\frac{\partial^2 u}{\partial c \partial x}\frac{\partial f}{\partial l} + \frac{\partial u}{\partial c}\frac{\partial^2 f}{\partial l^2} + \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial c \partial m}\frac{\partial f}{\partial l} - \frac{\partial^2 u}{\partial x \partial m} \end{bmatrix} \tag{27}$$

The sign of the derivative (26), which determines the direction of the impact of a change in the inflation rate on the capital stock in the steady state, is not unambiguously determined and depends on the assumed utility and production functions. Assuming that

(1) labour and capital are complementary inputs in the production process ($\frac{\partial^2 f}{\partial k \partial l} > 0$),

(2) the production function is concave ($\frac{\partial^2 f}{\partial k^2}\frac{\partial^2 f}{\partial l^2} - \left(\frac{\partial^2 f}{\partial k \partial l}\right)^2 \geq 0$), which means that it exhibits non-increasing returns to scale[18], and

(3) leisure and consumption are not complementary goods in the utility function ($\frac{\partial^2 u}{\partial c \partial x} \leq 0$),

the sign of the derivative (26) will depend on the mutual relationship between the partial derivative of, respectively, the marginal utility of consumption and leisure with respect to the real money stock.

Given such assumptions, four cases are possible, depending on the form of the utility function (Wang and Yip 1992):

- consumption and money are complementary goods, whereas leisure and money are substitutes:

$$\frac{\partial^2 u(c,m,x)}{\partial c \partial m} > 0, \quad \frac{\partial^2 u(c,m,x)}{\partial x \partial m} < 0; \tag{28}$$

in this case inflation, by reducing the real money stock, reduces consumption and increases leisure; both effects lead to a decrease in labour supply, which, through the assumption about the complementarity of inputs ($\frac{\partial^2 f}{\partial k \partial l} > 0$), leads to a decrease in the steady-state capital stock;

- consumption and leisure are substitutes, whereas leisure and money – complements:

$$\frac{\partial^2 u(c,m,x)}{\partial c \partial m} < 0, \quad \frac{\partial^2 u(c,m,x)}{\partial x \partial m} > 0 \tag{29}$$

in this case higher inflation in the steady state would increase the labour supply and act in line with the Tobin effect, that is towards increasing the capital stock in the steady state;

- consumption and money as well as leisure and money are substitutes

$$\frac{\partial^2 u(c,m,x)}{\partial c \partial m} < 0, \quad \frac{\partial^2 u(c,m,x)}{\partial x \partial m} < 0 \tag{30}$$

or consumption and money, as well as leisure and money are complements

$$\frac{\partial^2 u(c,m,x)}{\partial c \partial m} > 0, \quad \frac{\partial^2 u(c,m,x)}{\partial x \partial m} > 0; \tag{31}$$

for these two cases, the direction of inflation's impact on steady-state capital level depends on whether the effect of a change in the money supply on the marginal utility of leisure is weaker or stronger than the effect on the marginal utility of consumption (cf. Danthine 1985).

A clear economic interpretation can be given to the signs of the partial derivatives in equations (28)-(31) only in two out of the four cases:

- The complementarity between consumption and money means that money allows for a decrease in transaction costs (expressed in units of consumption) that are connected with the purchase of goods or that the purchase of consumption goods is impossible without money.
- The complementarity between leisure and money means that money allows for a reduction in the labour input devoted to purchasing, and thus, at a given consumption level, for an increase in the amount of leisure. .

However, the foregoing interpretations are not a result of the model's assumptions as according to these assumptions money gives utility to households not because it plays a specific role in transactions, but because it is simply assumed so. Models in which this kind of interpretation follows from assumptions about the role of money are discussed further in the article. Furthermore, assumption (2) is not satisfied by many production functions. The Cobb-Douglas production function, for instance, is concave if the sum of elasticities of output with respect to capital and labour is less than one, and convex otherwise.

2.2.2. Money in the production function

Another modification of Sidrauski's model, which consists in introducing money into the production function, was proposed by Fischer (1974)[19]. He motivates this change by the assumption that holding real money balances facilitates transactions. Here money is treated as an additional input, besides capital and labour, hence the name of this class of monetary growth models – *Money-in-Production-Function*, henceforth: MIPF).

In this model the production function takes the following form:

$$y = f(k,m)\,,\ \frac{\partial f}{\partial k}, \frac{\partial f}{\partial m} > 0 \tag{32}$$

and the total utility function maximized by the household[20]:

$$W = \int_0^\infty u(c_t) \cdot e^{-\delta t}\, dt \tag{33}$$

Due to the introduction of money into the production function, in the steady state the marginal product of capital, in addition to the condition

$$\frac{\partial f(k,m)}{\partial k} = \delta + n \tag{34}$$

must also satisfy the condition

$$\frac{\partial f(k,m)}{\partial k} = \frac{\partial f(k,m)}{\partial m} - \pi \tag{35}$$

In the steady state capital and the real money stock are held in such a relation that the increase in output due to a one-unit increase in capital is equal to the increase in output due to one-unit increased in the real money stock.

The derivative of capital in the steady state with respect to the inflation rate is equal to

$$\frac{\partial k}{\partial \pi} = -\frac{\dfrac{\partial^2 f}{\partial k \partial m}}{\dfrac{\partial^2 f}{\partial m^2}\dfrac{\partial^2 f}{\partial k^2} - \left(\dfrac{\partial^2 f}{\partial k \partial m}\right)^2} \tag{36}$$

Thus a change in the inflation rate causes a change in the steady-state level of capital. However the direction of this impact is not unambiguous and depends of the form of the production function. An increase in inflation leads to a decrease in the steady-state level of capital if:

(1) capital and money are complementary inputs in the production process ($\frac{\partial^2 f}{\partial k \partial m} > 0$), in other words, the use of money in transactions allows every additional unit of capital to be used more effectively in the production process.

(2) the production function is concave ($\frac{\partial^2 f}{\partial k^2}\frac{\partial^2 f}{\partial m^2} - \left(\frac{\partial^2 f}{\partial k \partial m}\right)^2 \geq 0$), which means that it exhibits non-increasing returns to scale.

Hence the modification of Sidrauski's model, consisting in the introduction of money as an additional production input does not eliminate the ambiguity about inflation's impact on capital accumulation.

2.3. Money as an indirect source of utility

In the models presented in the preceding subsection households maintain positive real money balances due to the associated **direct** increase in utility. In these models it is assumed that money provides services that either directly increase household utility or that are a necessary condition for production. Money is thus held by households due to the utility derived from it. The source of this utitlity is undefined, however, in particular it is not associated with money's principal role in the economy, namely facilitating the exchange of goods.

An attempt to eliminate this simplification are models which explicitly try to account for money's main role in the economy, that is facilitating the exchange of goods. In these models it is explicitly assumed that

- money allows for a reduction in costs connected with the exchange of goods or
- barter (exchange without money) is impossible.

As a result, in such models money is held by households because it allows them (by lowering the costs of exchange or by facilitating transactions) to increase utility **indirectly**.

2.3.1. Money as a tool for reducing transaction costs

In Transaction-Costs[21] (henceforth: TC) models it is assumed that the use of money in the exchange of goods redcues the costs connected with that exchange. By eliminating the condition of double coincidence of wants it allows one to reduce the quantity of real resources or the time needed to find a counterparty to the barter transaction. It needs to be emphasized, however, that these models do not permit a direct analysis of the differences in the functioning of the economy with and without money.

In models based on this concept transaction costs can be expressed either:

- in units of consumption or of output (real quantities), which is in line with the intuitive interpretation according to which the consumer or the firm must pay an intermediary for finding a counterparty to the transaction; models of this type are called Shopping-Costs (henceforth: SC) models;
- in units of labour input (time), which can be interpreted as the cost of leisure time devoted to finding a counterparty to the transaction; models of this type are called Shopping-Time (henceforth: ST) models.

2.3.1.1. Shopping-Costs models

The assumption according to which money allows for a reduction in transaction costs expressed in units of real quantities is included in SC models by introducing a transaction costs function $v(m)$, which defines what part of consumption or output is assigned for exchange. The larger the money balances held by the household, the lower the transaction costs it incurs ($\frac{dv}{dm} < 0$). However, each additional unit of money allows for an ever smaller cost reduction than the previous unit ($\frac{d^2 v}{dm^2} > 0$). In the special case of so-called full liquidity, that is when all transactions are concluded with the use of money, the following equality holds: $v = \frac{dv}{dm} = \frac{d^2 v}{dm^2} = 0$.

In a model in which transaction costs lower consumption, raising the real money stock increases (indirectly) the household's utility as it allows to increase consumption with the capital and labour inputs unchanged. This is due to the fact that an increase in the money stock allows the household to save that part of consumption which has thus far been used to cover the transaction costs. The appropriate model is based on the MIUF-model specification presented in section 2.2, the only difference being that money is not an argument of the utility function, but is introduced through the wealth accumulation equation, which modifies the second of the constraints on the maximization of the houdehold's total utility function:

$$\dot{a} = f(k) - c \cdot (1 + v(m)) + \tau - n \cdot a - \pi \cdot m \tag{37}$$

The steady-state solution of the model is described by two conditions:

$$\frac{\partial f(k)}{\partial k} = \delta + n \tag{38}$$

$$-\frac{dv(m)}{dm} \cdot c = \pi + \delta + n \tag{39}$$

The left side of condition (39) determines by how many units of consumption the transaction costs will decrease as a result of the household holding an additional unit of money. The household will increase real money balances until the profit from increasing them by a unit becomes equal to the loss incurred due to keeping that unit in the portfolio. The interpretation of this condition is thus identical to that of condition (17) in the MIUF model.

As a result, the solutions of the model above and the MIUF model provide qualitatively identical conclusions about inflation's impact on real variables: in both cases money is superneutral. In particular, inflation does not influence the steady-state level of capital: the derivative of capital with respect to inflation is equal to (obviously, one may also derive this fact directly from the equation 38):

$$\frac{\partial k}{\partial \pi} = \frac{0}{-\frac{d^2 v}{dm^2} \cdot c \cdot \frac{\partial^2 f}{\partial k^2}} = 0 \tag{40}$$

Compared to the specification in which money is a direct source of utility defining money as a medium of exchange facilitating a reduction in transaction costs connected to the purchase of consumer goods does not change the results of the model with respect to inflation's impact on capital accumulation. The intuitive explanation of the lack of differences between these two specifications is as follows: in the MIUF model money provides transaction services that **directly** (by assumption) increase the utility level. In the SC model the role of money is in principle identical, except that money provides transaction services that facilitate a reduction in costs, which leads to an increase in consumption and, **indirectly**, utility.

The differences between these two specifications of money's functions become apparent in the case where the assumption about the inelasticity of labour supply is relaxed.

The first condition that must be satisfied in the steady state has an identical form for both models:

$$\frac{\partial f(k,l)}{\partial k} = \delta + n \tag{41}$$

The second condition, despite having a slightly different form, is equivalent to condition (22) in the MIUF model with an elastic labour supply.

$$-\frac{dv(m)}{dm}\cdot c = \pi + \delta + n \tag{42}$$

The models differ in the third condition. In the SC model it becomes:

$$\frac{\partial f(k,l)}{\partial l}\frac{\partial u(c,x)}{\partial c}\frac{1}{(1+v(m))} = \frac{\partial u(c,x)}{\partial x} \tag{43}$$

whereas in the MIUF model:

$$\frac{\partial f(k,l)}{\partial l}\frac{\partial u(c,m,x)}{\partial c} = \frac{\partial u(c,m,x)}{\partial x} \tag{44}$$

In both cases this condition sets the marginal utility of leisure equal to the marginal utility of consumption created by a one-unit increase in the labour input (which means that the household increases the supply of labour until the increase in the utility of output produced with the use of an additional unit of labour becomes equal to the loss of utility due to a one-unit decrease in leisure). But whereas in the MIUF model the assumed form of the utility function (more precisely, the assumption about the signs of the partial derivatives of the marginal utility of consumption, and the marginal utility of leisure, with respect to the real money stock) was crucial for the determination of inflation's impact on real variables, in the SC model this assumption is no loger significant. A change in the real money stock does not have an effect on either the marginal utility of consumption or the marginal utility of leisure. However, it has an effect on the level of consumption through the function of transaction costs. Intuitively, in the SC model the household's felicity due to an additional unit of consumption or leisure is independent of the money stock held, but the larger the money stock held, the more the household can consume given the labour supply, or the more leisure it can obtain given the level of consumption.

In the steady state the derivative of capital with respect to infation is equal to

$$\frac{\partial k}{\partial \pi} = -\frac{-\dfrac{\partial^2 f}{\partial k \partial l}\dfrac{\partial u}{\partial c}\dfrac{\partial f}{\partial l}\dfrac{1}{\left(1+v(m)\right)^2}\cdot\dfrac{dv}{dm}}{\det A} \tag{45}$$

where

$$A = \begin{bmatrix} \dfrac{\partial^2 f}{\partial k^2} & \dfrac{\partial^2 f}{\partial k \partial l} & 0 \\ 0 & 0 & -c\dfrac{d^2 v}{dm^2} \\ \dfrac{\partial u}{\partial c}\dfrac{\partial^2 f}{\partial k \partial l}\dfrac{1}{1+v(m)} & \left(\dfrac{\partial u}{\partial c}\dfrac{\partial^2 f}{\partial l^2} - \dfrac{\partial^2 u}{\partial c \partial x}\dfrac{\partial f}{\partial l}\right)\dfrac{1}{1+v(m)} + \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial u}{\partial c}\dfrac{\partial f}{\partial l}\dfrac{1}{\left(1+v(m)\right)^2}\dfrac{dv}{dm} \end{bmatrix} \tag{46}$$

The numerator of expression (44) is positive if

(1) capital and labour are complementary inputs ($\frac{\partial^2 f}{\partial k \partial l} > 0$), which means that each additional unit of capital (labour) allows to produce more output, the larger the labour (capital) stock used in production.

In order for the denominator of this expression to be positive as well, additionally two conditions must be satisfied:

(2) the production function must be concave ($\frac{\partial^2 f}{\partial k^2}\frac{\partial^2 f}{\partial l^2}-\left(\frac{\partial^2 f}{\partial k \partial l}\right)^2 \geq 0$), which means that it exhibits non-increasing returns to scale, and

(3) leisure and consumption cannot be complementary goods in the utility function ($\frac{\partial^2 u}{\partial c \partial x} \leq 0$).

The above equation shows that whereas in the MIUF model the effect of a rise in inflation on the steady-state level of capital was ambiguous and dependent on the assumptions about the signs of the partial derivatives of the marginal utility of consumption and the marginal utility of leisure with respect to the real money stock, in the SC model the direction of the effect of a rise in inflation on capital accumulation in the steady state is independent of these assumptions. If conditions (1-3) are satisfied, higher inflation unambiguously lowers the steady-state level of capital. The intuitive explanation is as follows: inflation, by reducing the money stock, causes an increase in transaction costs connected with consumer purchases (a rise in *v(m)*), which in effect reduces consumption and the labour supply. This leads, through the assumption about the complementarity of production inputs ($\frac{\partial^2 f}{\partial k \partial l} > 0$), to a decline in the steady-state capital stock.

Alternatively, one may assume that the transaction costs are incurred during goods production and not during their consumption. The real money stock allows then to reduce the cost (expressed in terms of units of real output) of "finding" a buyer for the goods produced. In effect, the quantity of output, given the capital level, is equal to $f(k)[1-v(m)]$ (where $v(m)$ is a function of transaction costs, defined in the same way as in the previous model).

The steady-state solution of the model is given by the following FOCs:

$$\frac{\partial f(k)}{\partial k} \cdot (1 - v(m)) = \delta + n \tag{47}$$

$$\frac{\partial f(k)}{\partial k} \cdot \left(1 - v(m)\right) = -f(k) \cdot \frac{dv(m)}{dm} - \pi \tag{48}$$

The interpretation of these conditions is identical as in the case of the MIPF model discussed in section 2.2.2. However, as the transaction costs connected with the sale of the goods produced are directly included in the model a more intuitive interperetation of the formulated constraints is possible compared to the assumption about the presence of money in the utility function. The expression $\left(1 - v(m)\right)$ defines the size of transaction costs per unit of output at a given level of the real money stock, whereas $\left(-f(k) \cdot \frac{dv(m)}{dm}\right)$ defines the profit per unit of output due to a reduction (caused by a one-unit increase in the real money stock) in transaction costs.

In effect, the interpretation of the expression which defines the impact of a change in the inflation rate on capital accumulation in the steady-state is more intuitive (although still ambiguous). The derivative of capital with respect to inflation is equal to

$$\frac{\partial k}{\partial \pi} = -\frac{-\frac{dv}{dm}\frac{\partial f}{\partial k}}{(1-v) \cdot \frac{\partial^2 f}{\partial k^2} \cdot \frac{d^2 v}{dm^2} \cdot f + \left(\frac{dv}{dm} \cdot \frac{\partial f}{\partial k}\right)^2} \tag{49}$$

The sign of the numerator on the right side of equation (49) is positive. The direction of the impact of a change in the inflation rate on capital accumulation is thus determined by the sign of the denominator. The first element of the sum in the denominator measures the direct (negative) effect of a change in the real money stock and capital. The second element measures the (positive) indirect effect. If the economy is in a state close to full liquidity (i.e.

the majority of transactions are concluded with the use of money) then an additional quantity of money reduces transaction costs to a very small extent, that is $\frac{dv}{dm}$ is small. In this situation higher inflation lowers the capital stock as the first effect dominates the second one. But if an assumption were made that $\frac{d^2m}{dm^2} = 0$, that is each additional unit of money were to reduce transaction costs to the same extent as the preceding unit, the impact of inflation on the capital level would be positive.

As a result, the impact of a change in the inflation rate on the steady-state level of capital is ambiguous in this model. However, once money's role in transactions is accounted for explicitly, a deeper interpretation can be given to the conditions necessary for inflation to have, respectively, a negative or a positive effect on capital accumulation.

2.3.1.2.Shopping-time models

The other approach that allows one to directly account for the role of money in the transaction process is based on the assumption according to which the purchase of consumption goods without money requires the household to give up a part of its leisure. The difference between the amount of leisure time needed to buy the same number of consumption goods across different levels of the real money stock can be an approximation of the costs of achieving the double coincidence of wants that is characteristic of barter[22].

Formally this condition is introduced into the model by assuming that the purchase of goods requires the use of transaction services which consist in "localising" an agent inclined to exchange goods. The production technology of these services is defined by a function of two variables: real money stock and household's leisure. The function defines, how much leisure time needs to be devoted to concluding a transaction at given levels of consumption and the real money stock. Money and leisure are susbstitutes in the production of transaction services: a higher level of the real money stock decreases the amount of time needed to produce transaction services necessary to purchase a given amount of consumption goods[23]. The transaction services (ψ) are expressed in units defined in such a way that consumption in the amount of c requires $\psi = c$ services. The transaction services production function is defined as:

$$\psi = \psi(m,n^s) = c,\ \frac{\partial\psi}{\partial m} \geq 0, \frac{\partial\psi}{\partial n^s} \geq 0, \frac{\partial^2\psi}{\partial m^2} \leq 0, \frac{\partial^2\psi}{\partial n^{s\,2}} \leq 0 \tag{50}$$

where n^s stands for the time devoted to purchasing consumption goods. On the basis of (50) the time devoted to making a purchase can be expressed as a function of consumption and the real money stock.

$$n^s = g(c,m),\ \frac{\partial \mathrm{g}}{\partial \mathrm{c}} > 0,\ \frac{\partial \mathrm{g}}{\partial \mathrm{m}} < 0, \frac{\partial^2 \mathrm{g}}{\partial \mathrm{c}^2} > 0, \frac{\partial^2 \mathrm{g}}{\partial \mathrm{m}^2} > 0, \frac{\partial^2 \mathrm{g}}{\partial \mathrm{c}\partial \mathrm{m}} \leq 0 \tag{51}$$

The condition $\frac{\partial \mathrm{g}}{\partial \mathrm{m}} \leq 0$ means that each additional money unit lets transaction costs decline, however to an extent smaller than the preceding unit ($\frac{\partial^2 \mathrm{g}}{\partial \mathrm{m}^2} > 0$). If $m \geq c$ money makes it possible to conclude all transactions, and the leisure required to make purchases equals zero, hence $g = \frac{\partial g}{\partial c} = \frac{\partial g}{\partial m} = \frac{\partial^2 g}{\partial c^2} = \frac{\partial^2 g}{\partial m^2} = \frac{\partial^2 g}{\partial c \partial m} = 0$. It follows from the condition $\frac{\partial^2 g}{\partial c \partial m} \leq 0$ that a higher money stock decreases the marginal transaction costs of consumption or leaves them unchanged (Feenstra 1986).

In ST models the utility obtained by a household depends not only on the level of consumption, but also on leisure, the latter being equal to $x = 1 - l - n^s = 1 - l - g(c,m)$.

Thus, in the steady state the following conditions must be satisfied:

$$\frac{\partial f(k,l)}{\partial k} = \delta + n\,, \tag{52}$$

$$-\frac{\partial g(c,m)}{\partial m} \cdot \frac{\partial f(k,l)}{\partial l} = \delta + \pi + n \text{ oraz} \tag{53}$$

$$\frac{\partial f(k,l)}{\partial l}\frac{\partial u(c,l)}{\partial c} - \frac{\partial u(c,l)}{\partial x}\frac{\partial g(c,m)}{\partial c} \cdot \frac{\partial f(k,l)}{\partial l} = \frac{\partial u(c,l)}{\partial x} \tag{54}$$

Condtion (53) sets the marginal cost of holding money equal to the marginal income from holding it, that is to the output produced by a unit of labour saved thanks to a reduction in transaction costs (given the consumption level) due to a one-unit increase in the real money stock. On the other hand, condition (54) sets the marginal income of labour equal to its marginal cost:

- The first multiplicand on the left-hand side of the equals sign implies that an increase in the labour input leads to an increase in output, which causes an increase in utitility as a result of increased consumption. The second multiplicand shows how much this increase in utility is reduced as a result of shorter leisure time, a part of which (with the real money stock unchanged) needs to be allocated to producing transaction services that allow for the consumption of an additional unit of output.
- The right-hand side of the condition defines the utility derived from an additional unit of leisure.

Thus these condtions describe the same relations between the paths of particular variables as the FOCs in the MIUF model with an elastic labour supply, however, similarly to the case of the SC model, they can be given a much more intuitive interpretation.

In the steady state the impact of changes in inflation on capital accumulation is described by the expression:

$$\frac{\partial k}{\partial \pi} = -\frac{\frac{\partial^2 f}{\partial k \partial l}\left(\frac{\partial g}{\partial m}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial f}{\partial l}\frac{\partial^2 v}{\partial c \partial x} + \frac{\partial f}{\partial l}\frac{\partial^2 v}{\partial x^2}\frac{\partial g}{\partial c}\right) - \frac{\partial^2 g}{\partial c \partial m}\frac{\partial f}{\partial l}\frac{\partial v}{\partial x}\right)}{\det A} \tag{55}$$

Where

$$A^T = \begin{bmatrix} \frac{\partial^2 f}{\partial k^2} & -\frac{\partial g}{\partial m}\frac{\partial^2 f}{\partial l \partial k} & \frac{\partial^2 f}{\partial l \partial k}\left(\frac{\partial u}{\partial c} - \frac{\partial g}{\partial c}\frac{\partial u}{\partial x}\right) \\ \frac{\partial^2 f}{\partial l \partial k} & -\frac{\partial g}{\partial m}\frac{\partial^2 f}{\partial l^2} & \frac{\partial^2 f}{\partial l^2}\left(\frac{\partial u}{\partial c} - \frac{\partial g}{\partial c}\frac{\partial u}{\partial x}\right) - \frac{\partial f}{\partial l}\left(\frac{\partial^2 u}{\partial c \partial x} - \frac{\partial g}{\partial c}\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial^2 u}{\partial x^2} \\ 0 & -\frac{\partial^2 g}{\partial m^2}\frac{\partial f}{\partial l} & \frac{\partial g}{\partial m}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial f}{\partial l}\frac{\partial^2 v}{\partial c \partial x} + \frac{\partial f}{\partial l}\frac{\partial^2 v}{\partial x^2}\frac{\partial g}{\partial c}\right) - \frac{\partial^2 g}{\partial c \partial m}\frac{\partial f}{\partial l}\frac{\partial v}{\partial x} \end{bmatrix} \tag{56}$$

The numerator of expression (55) is positive if capital and labour are complements in the production process ($\frac{\partial^2 f}{\partial k \partial l} > 0$). In order for the denominator to be positive as well, the following conditions must also be met:

(1) the production function must be concave ($\frac{\partial^2 f}{\partial k^2}\frac{\partial^2 f}{\partial l^2} - \left(\frac{\partial^2 f}{\partial k \partial l}\right)^2 \geq 0$), which means that it exhibits non-increasing returns to scale, and

(2) leisure and consumption cannot by complementary goods in the utility function ($\frac{\partial^2 u}{\partial c \partial x} \leq 0$).

The conditions under which a rise in inflation has a negative impact on capital accumulation are thus identical to those in the SC model. Similar is also the intuitive explanation of this relationship: inflation, by leading to a reduction in the real money stock, forces households to devote a larger amount of their leisure to the production of transaction services (an increase in $g(c,m)$), the essence of which lies in finding an agent inclined to exchange goods. This in turn reduces the supply of labour allocated to producing consumption goods. As a result of the assumption about the complementarity of production inputs ($\frac{\partial^2 f}{\partial k \partial l} > 0$), a decrease in the labour supply leads to a decrease in capital outlays.

2.3.2. Money as a tool solving the financial constraint

In the models presented in subsection 2.3.1. it was assumed that holding money allows households to reduce costs connected with concluding transactions. The approach proposed by Clower (1967) and developed by Grandmont and Younes (1972) and Lucas (1980) is based on a different assumption. According to this approach money is held by households because the exchange of goods is impossible without it[24]. Similarly to the case of models discussed earlier in this article, this assumption does not allow one to analyze differences in the functioning of the economy with and without money, it is rather a way of "imposing" on the modelled economy the necessity to use money in the process of goods exchange. This assumption is introduced into the model by widening the standard set of constraints the household is faced with while maximizing utility by a so-called financial constraint or cash-in-advance (henceforth: CIA) constraint. This constraint means that in order to purchase a certain amount of goods, the household must possess in the period preceding the transaction a real money balance equal to or greater than the value of these goods[25]. In contrast to ST and SC models, money cannot be replaced in the exchange process either by time or by real goods. Holding money does not provide utility directly, as in the MIUF model, but indirectly, by facilitating consumption and production.

The precise form of the constraint depends on the assumption about the kind of goods that can be exchanged solely with the use of money (Walsh 2003, p. 101). The constraint may apply to the purchase of consumption goods, investment goods or both simultaneously[26].

In one of the first monetary growth models with the CIA constraint, proposed by Stockman[27] (1981), the constraint applies to the purchase of all goods (i.e. both consumption and investment goods) and has the following form:

$$m_t = c_t + \dot{k}_t \tag{57}$$

This means that both households and firms need to hold a real money stock equal to the value of consumption and investment.

In the steady state the following FOCs must be met:

$$\frac{du(c)}{dc} = \lambda + \gamma \tag{58}$$

$$\frac{\partial f(k)}{\partial k} = \delta + n + \delta(\delta + n + \pi) \tag{59}$$

$$\frac{\lambda}{\gamma} = \delta + n + \pi \tag{60}$$

where λ is the shadow price of assets and γ stands for the shadow price of capital (both calculated in units of utility).

The above conditions indicate that in the steady state both the marginal product of capital and the marginal utility of consumption depend on the inflation level.

For a solution described in this way the effect of inflation on the capital stock can be represented as

$$\frac{\partial k}{\partial \pi} = \frac{\frac{\partial f}{\partial k}}{(\delta + \mu + n + 1)\frac{\partial^2 f}{\partial k^2}} < 0 \tag{61}$$

A rise in inflation thus means a decrease in the steady-state level of capital. This is due to the fact that the accumulation of an additional unit of capital in period *t* requires the holding of an additional unit of the real money stock at the beginning of this period. A rise in inflation raises the cost connected with holding this unit of money while also raising the cost of the inititated investment. In effect, it reduces the return on the investment (which accounts for the cost of holding money) and reduces the demand for capital. In other words, if the household decides to decrease consumption and increase investment today in order to increase its consumption in the future, this means that its future income will increase. This income can however be exchanged into future consumption only by increasing the money stock held. With higher inflation holding money becomes costlier, hence the net return on the investment (expressed either in units of consumption or utility) declines. This in turn leads to lower investment and, in effect, to lower capital outlays in the steady-state. In the model inflation is a *de facto* tax imposed on consumption goods which discourages capital accumulation.

In this model the negative impact on capital accumulation is a result of the complementarity of money and capital (Orphanides and Solow 1990): an increase in capital accumulation requires a rise in the capital stock. But while in the MIPF model the direction of the relationship between money and the marginal product of capital was a result of making an arbitrary assumption (about the sign of the partial derivative of the marginal product of capital with respect to money), in the model with the CIA constraint it has an indirect nature that can be justified on the grounds of economic reasoning.

The effect described above hinges critically on the assumptions made about the role of money in financing investment goods. A model in which the assumption[28] that the purchase of investment goods must be subject to the CIA constraint is relaxed was proposed by Abel (1985, 1987)[29]. The CIA constraint refers here only to consumption, hence it has the form:

$$m_t = c_t \tag{62}$$

In the steady state the following conditions must be satisfied:

$$\frac{\partial f(k)}{\partial k} = \delta + n \tag{63}$$

$$\frac{du(c)}{dc} = \lambda\left(\frac{\partial f(k)}{\partial k} + (1+\pi)\right) \tag{64}$$

In effect, changes in the level of inflation do not have an impact on the steady-state level of capital, that is

$$\frac{\partial k}{\partial \pi} = \frac{0}{\frac{\partial^2 f}{\partial k^2}\frac{d^2 u}{dc^2}} = 0 \tag{65}$$

This is thus a (qualitatively) identical result as in the MIUF model and the SC model with a function of transaction costs that lower consumption. The consequence of the assumptions made is however not only the lack of inflation's influence on the steady-state capital stock but also on the level of household welfare. Higher inflation does mean lower real money stock in the steady state, but in contrast to the MIUF and SC models it does not cause a (direct or indirect, respectively) decline in the total utility obtained by the household. Consumption, which is the only determinant of utility in this model, remains at an unchanged level despite a rise in inflation.

In the MIUF and SC models, the relaxation of the assumption about the inelasticity of labour supply led to a change in the direction of inflation's impact on capital accumulation. A model with the CIA constraint imposed only on consumption, in which both the capital stock and the labour supply are elastic, was proposed by Gomme (1997)[30].

As a result of such a modification the following FOCs must bust be satisfied in the steady state:

$$\frac{\partial f(k)}{\partial k} = \delta + n \tag{66}$$

$$\frac{\partial f(k,l)}{\partial l}\frac{\partial u(c,x)}{\partial c} = \frac{\partial u(c,x)}{\partial x}(1+\delta+\pi) \tag{67}$$

Condition (67) sets the marginal utility of leisure equal to the marginal utility of consumption generated by a one-unit increase in the labour input. In contrast to the MIUF and SC models inflation has a direct impact on this relationship.

The impact of inflation on capital accumulation in the steady state can be expressed as:

$$\frac{\partial k}{\partial \pi} = -\frac{\frac{\partial u}{\partial x}\frac{\partial^2 f}{\partial k \partial l}}{\det A} \tag{68}$$

where:

$$A = \begin{bmatrix} \frac{\partial^2 f}{\partial k^2} & \frac{\partial^2 f}{\partial k \partial l} \\ \frac{\partial^2 f}{\partial k \partial l}\frac{\partial u}{\partial c} & \frac{\partial^2 f}{\partial l^2}\frac{\partial u}{\partial c} - \frac{\partial f}{\partial l}\frac{\partial^2 u}{\partial c \partial x} + \frac{\partial^2 u}{\partial x^2}(1+\delta+\pi) \end{bmatrix} \tag{69}$$

The numerator of expression (68) is positive if capital and labour are complements in the production process ($\frac{\partial^2 f}{\partial k \partial l} > 0$). The denominator is positive when additionally the following conditions are fulfilled:

(1) the production function is concave ($\frac{\partial^2 f}{\partial k^2}\frac{\partial^2 f}{\partial l^2} - \left(\frac{\partial^2 f}{\partial k \partial l}\right)^2 \geq 0$) and

(2) leisure and consumption are not complements in the utility function (then $\frac{\partial^2 u}{\partial c \partial x} \leq 0$).

The above model points to a negative effect of inflation on the steady-state capital stock given assumptions that must also be satisified in other models with an elastic labour supply in order for this effect to be achieved. The intuitive interperation is as follows: a rise in inflation raises the cost of holding money and this decreases the real labour income as according to the CIA constraint a unit of money earned in the current period cannot be spent until the next period. In effect, the supply of labour declines and, through the assumption about the complementarity of production inputs, the capital stock declines as well. Moreover, the rise in inflation leads to a reduction in the money stock which, with the CIA constraint binding, additionally limits consumption, and thus the labour and capital inputs.

3. Monetary search models

In the models presented thus far the use of money by economic agents in order to conclude transactions was a consequence of making certain arbitrary assumptions, such as e.g. the CIA constraint which eliminates other goods (and hence barter) from the transaction process. In these models money constituted a direct or indirect source of utility, which provided a rationale for its existence in the economy. The process of goods exchange was not modelled explicitly, however, i.e. the model did not describe the mechanism of household transactions. As a result, the change in the allocation of wealth between particular goods made by households in order to maximize utility is described in the model's categories as a change from allocation A to allocation B, without taking into account the process of exchanging assets which permits such a change.

The models of monetary search theory take a different approach to money's functions. The starting point in these models is the exchange process itself, described by the mechanism of bilateral adjustments between the transaction counterparties. This approach lets one describe the role of money more realistically as a good facilitating the exchange and improving the effectiveness of resource allocation in the economy[31].

The functioning of the market in these models is based on the following assumptions. In the market there is a large number of agents producing different kinds of consumption goods[32]. It is assumed that each agent specializes in the production of one of the consumption goods. In most specifications it is assumed that goods cannot be stored and must be consumed in the period in which they were produced. The agents seek to exchange the goods due to differences in preferences. A given agent does not derive utility from the consumption of the good it produces. Formally, the agent producing good i (henceforth: agent i) consumes goods from the interval $[i^*-\alpha/2, i^*+\alpha/2]$, where the i^* index is chosen randomly from a unit circle, and i is not an element of this interval[33]. When a pair of agents i and j meet, the good produced by agent j lies in the interval of goods consumed by agent i with the probability α [34]. With the same probability the good produced by agent i lies in the interval of goods consumed by agent j. Four situations are thus possible:

- single coincidence of wants (two cases): agent i wants to consume good j, but agent j does not want to consume good i, that is $j \in [i^*-\alpha/2, i^*+\alpha/2]$ and $i \notin [j^*-\alpha/2, j^*+\alpha/2]$, or agent i does not want to consume good j, but agent j wants to consume good i, that is $j \notin [i^* - \alpha/2, i^* + \alpha/2]$ and $i \in [j^* - \alpha/2, j^* + \alpha/2]$; each of these situations occurs with the probability $\alpha(1-\alpha)$;

- the situation of a double coincidence of wants: agent i wants to consume good j, and simultaneously agent j wants to consume good i, that is $j \in [i^* - \alpha/2, i^* + \alpha/2]$ and $i \in [j^* - \alpha/2, j^* + \alpha/2]$; such a situation occurs with the probability α^2;
- lack of coincidence of wants: agent i does not want to consume good j, and agent j does not want to consume good i, that is $j \notin [i^* - \alpha/2, i^* + \alpha/2]$ and $i \notin [j^* - \alpha/2, j^* + \alpha/2]$; such a situation occurs with the probability $(1-\alpha)^2$;

In effect, a transaction between agents would be concluded only in a situation of a double coincidence of wants, but the lower the value of α, the slimmer the chance that a transaction between two randomly drawn agents will be concluded. However, if a good exists that is not a consumption good for either of the agents but can be exchanged costlessly into a consumption good in the next transaction, then a transation will take place also in a situation of a single coincidence of wants. Fiat money is such a good, which in itself is worthless (i.e. is not a source of utility and is not a production factor) but carries value as a medium of exchange[35].

The constraints resulting from the dual coincidence of wants are not, however, a sufficient condition for the existence of exchange based on paper money. Only the occurrence of communication costs between the agents and the costs of storing and disseminating information about transactions made by agents makes (combined with the dual coincidence of wants) exchange with the use of paper money worthwile[36].

Most monetary search models do not allow one to analyze the impact of inflation on capital accumulation as in these models capital is not treated as a special good. The few exceptions are models proposed by Shi (1999) and Aruoba, Waller and Wright (2007)[37].

In the model proposed by Shi (1999) it is assumed, in contrast to the majority of models of this type, that the economy is composed of *H* households, and *H>2*. The *h*-th household consumes only good *h* and produces only good *h*+1. A given household may accumulate the kind of goods that it consumes, and these accumulated goods become capital that is used in the production process[38].

Exchange occurs only with the use of money and in a situation of a single coincidence of wants, that is when household *h* meets household *h-1* or when it meets household *h*+1. In both cases exchange is conducted by means of money. The probability that a single coincidence of wants occurs is equal to $\alpha \equiv 1/H$. Barter is not possible as the assumed specification of hosueholds' tastes precludes the possibility of occurrence of a dual coincidence of wants. The conditions of exchange between two agents in a situation of a single coincidence of wants are laid out in negotations.

Each household is comprised of an infinite number of agents. Such an assumption ensures that the level of consumption, capital, money stock and labour inputs of a given agent are not random variables, which is the case when individual agents that are transaction counterparties are randomly paired. In effect, in this way a given agent's individual risk connected with the random result of the exchange process is eliminated: each agent being a member of a household achieves the same level of consumption and utility, irrespective of the result of the exchange process that it participated in. As a result, despite the fact that individual households consume and produce different kinds of goods, their levels of consumption, capital accumulation and labour inputs may be equal. This in turn allows one to analyze the model in terms of a representative household.

Each of the agents in a household either sells goods (exchanges goods produced for money) or buys them (exchanges money for consumption goods), or optimizes the utility of leisure[39]. At the beginning of each period *t* the household decides how many agents will engage in one of the above three activities. Then it divides the capital stock and the money stock equally between the producing and the buying agents respectively. It also sets the desired quantities of the capital stock and of money for the period *t+1* as well as the negotiation strategy for the agents. Then the agents conduct trade exchange with randomly chosen agents of other households. The obtained stocks of money and consumption goods are accumulated with a view to dividing them between the agents belonging to the household.

For a process of exchange defined in this way, in the steady state a higher growth rate of the money supply and, in effect, higher inflation, causes a rise in the number of agents engaged in buying goods. Intuitively, because higher inflation causes a rise in the cost of holding money, the rational way for a household to behave is to aim to accelerate the exchange of the money stock held into consumption goods. This can be achieved by increasing the number of concluded transactions. With the probability of a single coincidence of wants being constant, this requires increasing the number of agents exchanging money for goods in the market. Increasing the number of transactions made in the market causes the rate of return on capital to grow. This is because capital is used only in a situation in which the producer concludes an exchange transaction. On the other hand, a higher rate of return on capital makes households increase capital accumulation. Thus in the model there is an additional channel, absent in neoclassical models, through which inflation may have an impact on capital accumulation. As emphasized by Shi (1999) the prerequisite for the existence of this channel is a non-Walrasian exchange process[40]: as the single coincidence of wants does not occur in every instance, not every agent is able to make an exchange transaction; the chances of concluding a transaction depend on the number of agents in the market who are engaged in the transaction. If the exchange process were Walrasian (just as in neoclassical models), each agent would be "attributed" to a transaction acceptable to that agent, and the chances of concluding the transaction would not depend on the number of agents in the market (which in turn depends in the model on the level of expected inflation).

The direction of inflation's impact on capital accumulation in the presented model is inasmuch surprising as both consumption and capital goods must be purchased by means of money. This type of constraint was present in the Stockman (1981) model described earlier and it resulted in a negative, not positive, effect of inflation on the steady-state capital stock.

The importance of the described channel of inflation's impact on investment is often widely criticized. The model's author himself emphasizes (Shi 2006) that one should not draw the conclusion that inflation always raises the level of capital in the steady state. First, too high inflation ensures that the model will not reach the state of monetary equilibrium, that is one in which money is used for transations. In such a situation exchange is conducted purely by means of barter, which means that agents abandon money and return to natural economy. Such a reaction of the model is much more consistent with economic intuition: when inflation is too high economic agents abandon money as a medium of exchange, which is a real-world phenomenon observed in high-inflation economies. Second, the positive impact of inflation on capital accumulation is a result of assumptions, described in the previous sections, about the functioning of the market and the price-setting mechanism. A change to these assumptions causes also a qualitative change in the inflation-capital accumulation relationship:

- Molico and Zhang (2006) have proposed a model in which they relax the assumption about households having an infinitely large number of agents[41]. In place of this assumption they assume that an infinite number of agents function in the economy, each of which specializes in the production and consumption of one type of goods. The result of such a change is that inflation has no impact on capital accumulation whatsoever because the number of agents buying goods and hence the number of transactions concluded in the market is not, as in Shi's (1999) model, endogenous.
- In the model proposed by Aruoba, Waller and Wright (2007) the economy is composed of two markets: a decentralized one whose mechanism of functioning has been described in this section, and a centralized one with a Walrasian exchange process. Capital is produced and exchanged in the centralized market, but it constitutes a production input also in the decentralized market[42]. Similarly to the Molico and Zhang (2006) model, such a framework makes the number of transactions concluded in the market cease to be the result of decisions made by economic agents, and so it eliminates the mechanism of inflation's impact on capital accumulation described by Shi. For markets defined in this way Aruoba, Waller and Wright (2007) analyze the model's results depending on the assumed price-setting mechanism. In the case in which prices are set through negotiations inflation has no effect on the steady-state level of capital. In the case where both parties to a transaction are price-takers inflation's impact on capital accumulation is negative. The intuitive interpretation of inflation's negative impact in the second case is the same as in neoclassical models: inflation is a tax imposed on economic activity both in the decentralized and the centralized market. The lack of inflation's impact on capital accumulation in the case of the negotiation-based price-setting mechanism is explained by the so-called *holdup problem*[43], peculiar to this price mechanism, which causes accumulation, regardless of the level of inflation, to be lower than it would follow from

the level of the marginal product of capital. The holdup problem “crowds out” the negative effects of inflation on capital accumulation.

4. Summary

In most monetary growth models the role of money is based on *ad hoc* assumptions. These models do not explain which goods and why are treated as money by economic agents, and which are not. Moreover, the specifications of functions that include money are fixed by assumption and do not permit the inclusion of factors that in the real-world influence the way in which money facilitates transactions, such as changes in monetary-policy parameters, progress in information technology, etc. (Shi 2006).

It is difficult to indicate which of the ways of introducing money into the growth models based on *ad hoc* assumptions best reflects its actual role in the economy. Despite the differences between particular models, in many cases they are equivalent. The first to point this out was Brock (1974) who provided the rationale for the MIUF specification. He showed that the SC model with transaction costs expressed in consumption units can be reduced to the form equivalent with the MIUF model. On the other hand Wang and Yip (1992) have demonstrated that between the MIUF, TC[44] and CIA concepts *qualitative equivalence*[45] exists if only appropriate restrictions are imposed on the parameters of the constraints and on the partial derivatives of the utility and the production functions. However, between some concepts neither functional nor qualitative equivalence exists. If in a model with a CIA constraint the constraint is imposed both on consumption and investment, there is no equivalence between this model and models based on other concepts.

Depending on the model, a rise in inflation may in the long run either increase, decrease or leave the capital outlays unchanged. In all specifications in which inflation affects capital accumulation, inflation constitutes a tax imposed on income from labour, capital or money (or on more than one of these goods simultaneously). The direction of this tax's impact on capital outlays depends on the mutual relationhip (the degree of substitutionality or complementarity) between particular real quantities and money in the production function, the utility function or the budget constraint. An empirical assessment of whether these assumptions hold encounters serious difficulties.

Monetary search models are a new and interesting approach to the problem of accounting for money in growth models. However due to the high degree of complexity of these models the inclusion of even the fundamental relationships from other areas of macroeconomic theory makes them impossible to be solved analytically. As a result, also in the case of these models the direction of inflation's impact on real variables, including investment, depends on assumptions that are difficult to verify empirically. In consequence they also do not allow one to identify the most important channels through which inflation affects capital accumulation.

None of the models presented in this article passed the Solow test (described in the introductory remarks) satisfactorily. This applies at least to these models' conclusions about inflation's effect on investment, and more broadly, on economic performance. One cannot use them to explain the view, broadly confirmed by empirical studies, that inflation is not conducive to economic growth. For such an explanation it is necessary to include in the analysis market imperfections such as the asymmetry of information. However, a review of the impact of these imperfections on how inflation affects economic performance deserves a separate article.

The weaknesses of monetary growth models exposed in the review presented in this article are not without significance for the practice of monetary policymaking. Some of the structural econometric models used in central banks to forecast future inflation and the effects of monetary policy are theoretically grounded exactly in these models. In effect, these econometric models' predictions about the effects of changes in inflation on investment (or more broadly, economic growth) may diverge from the actual relationship between these two variables.

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[1] The infinite time-horizon assumption for utility maximization can be justified e.g. by the existence of strong intergenerational ties, i.e. the endeavours of each generation to ensure maximum utility not only for the current generation, but also for the next one (see e.g. Barro 1974).

[2] A comparison of the advantages and disadvantages of exogenous and endogenous growth models can be found in Barro and Sala-i-Martin (2004) and Acemoglu (2009).

[3] Compare e.g. the analysis of the dynamics of reaching the steady state conducted by Bond, Wang and Yip (1996) and the comments on the results.

[4] The following notation has been adopted:

- If a capital letter is used to represent a variable in a model, a small letter is used to mark the values of that variable divided by the labour force. The exception to this rule is the notation used to denote the real money stock divided by the labor stock: *m* instead of $\frac{m}{p}$.
- A dot over a variable signifies the derivative of that variable with respect to time.

Moreover, in most cases the subscript *t* has been omitted for variables that are a function of time.

[5] In order to demonstrate that the rate of return on the real money stock is equal to $-\pi$, one needs to notice that the change in wealth per unit of time caused by the change in the price level is equal to:

$$\frac{\partial\left(\frac{M}{p}\right)}{\partial p}\frac{dp}{dt} = -\frac{M}{p^2}\frac{dp}{dt} = -\frac{M}{p}\frac{\dot{p}}{p} = -\frac{M}{p}\pi$$

It follows from the above that the growth in wealth per unit of real money stock (that is the rate of return) is equal to:

$$\frac{\frac{\partial\left(\frac{M}{p}\right)}{\partial p}\frac{dp}{dt}}{\frac{M}{p}} = \frac{-\frac{M}{p}\pi}{\frac{M}{p}} = -\pi$$

[6] Hence $\beta = \beta(r+\pi)$, while $\beta'(.) < 0$

[7] In some studies it is stated that the assumption necessary to achieve the effect described byTobin's model is an exogenous (that is independent of all variables in the model) saving rate (compare e.g. Shi 1999). That is not true: Dornbusch and Frenkel (1973) have demonstrated, for instance, that if the constant saving rate assumption is replaced in Tobin's model by the life-cycle hypothesis, which makes consumption (alongside with savings, the latter being the difference between income and consumption) an increasing function of wealth, that is

$c=c(a), \quad \frac{dc}{da} > 0$,

the effect obtained by Tobin holds.

[8] One of the modifications to Tobin's model, proposed by Levhari and Patinkin, is described further in this section.

[9] Patinkin (1965, chapter 4) was the first to present an outline of a model with money in the utility function, but his model did not include capital accumulation (quoted after Walsh 2003, p. 44).

[10] The description of the methods of dynamic optimization employed in his article can be found in Chiang (2002, chapters 7-10).

[11] This convention of presenting the model's solution will be maintained also for the other models. In the case of models with elastic labour supply, the solution will be presented for the set of variables (*k*, *m* ,*l*), where *l* stands for the time devoted to household work.

[12] The use of this theorem in economic analysis is discussed in Chiang (1996).

[13] Inflation in the models described can be regarded as an exogenous variable as it depends only on the exogenous growth rate of money supply and the rate of growth in the number of households.

[14] The derivatives of capital, consumption and output with respect to the growth rate of money supply are equal to their corresponding derivatives with respect to inflation.

[15] It is worth emphasizing that due to the real money stock being an argument of the utility function, higher inflation causes a decline in household welfare on the balanced growth path. Sidrauski's model shows, then, that although in the long run inflation does not have an impact on the level or growth of capital, output and consumption, it entails a lower level of welfare.

[16] The fact that superneutrality may concern only the steady state was already indicated by Sidrauski (1967).

[17] The third condition is a derivative of the Hamiltonian with respect the variable l_t, hence the right side of equation (23) could be $\frac{\partial u(c,m,x)}{\partial l} = \frac{\partial u(c,m,x)}{\partial x}\cdot\frac{\partial x}{\partial l} = -\frac{\partial u(c,m,x)}{\partial x}$. The derivative defined in this way denotes household's marginal disutility of labour.

[18] A function exhibits decreasing (increasing) returns to scale if a one-per-cent increase in all inputs causes output to increase by less (more) than one per cent. Concavity of the production function is a sufficient condtion for the existence of non-increasing returns to scale as long as the assumption *f*(0)=0 is satisfied.

[19] This idea appeared also in earlier studies (e.g. Levhari and Patinkin 1968), but no analytic results were presented of a model with money's role defined in this way.

[20] One may assume that money is still an argument of the utility function but this will not change the conclusions about the direction of the relationship between capital and the inflation rate in the steady state (cf. Orphanides and Solow 1990).

[21] The first models with the role of money defined in this way were presented by Saving (1971), Dornbusch and Frenkel (1973) and Dutton, Gramm and Brock (1974). Compare with Kimbrough (1986).

[22] One of the first ST models was presented by Saving (1971). Brock (1974) proposed this concept as a justification of the the MIUF-based approach. See also models proposed by McCallum (1983) and Kimbrough (1986).

[23] A detailed definition of transaction services, including e.g. the marginal rate of substitution of leisure and money in these services' production function depends on the assumptions made *ad hoc* in the particular models.

[24] This constraint has been defined by Clower in the following way: *„Money buys goods, goods buy money, but goods do not buy goods"* (Clower, 1967).

[25] Walsh (2003, p. 105) emphasizes that money is not a source of utility for the household, the equality constraint is binding, hence there is no need to formulate a constraint with an inequality.
[26] It may also apply only to a part of the goods in each of these categories. For instance, Lucas and Stokey (1983, 1987) assume that the CIA constraint applies only to selected consumption goods, the so-called money goods. For purchases of other goods, the so-called credit goods, holding money is not required. In order for the purchase of money goods to be worthwile their marginal utility must be greater than or equal to the sum of the marginal utility of credit goods and the inflation rate. Inflation is thus similar to a tax imposed on money goods, and its growth causes the substitution of money goods with credit goods. In the approach presented by Lucas and Stokey it is assumed that goods are exogenously divided into money and credit goods, which makes an intuitive interpretation of the model more difficult.
[27] The original model described in Stockman's article concerned a discrete time setting. In order to preserve the consistency of the way in which models are presented here, Stockman's original model was re-formulated to the continuous-time case.
[28] One of the possible explanations is the assumption that investment goods are purchased with the use of credit, not cash. In the model no such assumption was formulated explicitly and its implications for e.g. the effect of inflation on the availability and cost of credit were not analyzed either.
[29] Similarly to Stockman's model, Abel's original model concerned discrete-time case. In order to preserve the consistency of the way in which models are presented here, the original model was re-formulated to the continuous-time setting in which the objective function corresponds to the problem of a representative household maximizing utility in an infinite time horizon.
[30] A similar model was proposed by Cooley and Hansen (1989) but they presented only a numerical and not the analytic solution of the model. Aschauer and Greenwood (1983) and Carmichael (1989) proposed models with endogenous labour but without capital.
[31] One of the first models in this vein was proposed by Kiyotaki and Wright (1989, 1993) and Trejos and Wright (1995). The present section, in the part presenting the basic assumptions describing the functioning of the market and the exchange process is based on the reviews by Rupert, Schindler, Schevchenko and Wright (2000) and Shi (2006).
[32] Formally, it is assumed that a continuum of agents exists in an infinite time horizon as well as a continuum of various consumption goods.
[33] In the literature one can find many other possibile notations for this constraint. The basic distinction concerns whether barter is possible (just as in the above specification) or not (e.g. the specification proposed by Kiyotaki and Wright 1991).
[34] Agent j is randomly drawn from the population.
[35] According to the definition by Wallace (1980) the additional characteristic of paper money, beside the lack of value of its own, is the lack of government guarantee (e.g. gold backing).
[36] The lack of communication costs would allow agents to eliminate the problem of dual coincidence of wants without the use of money. The lack of costs of monitoring the actions of agents would allow for the use of a system of exchange in which a given agent provided the good it produced always when another agent demanded it. If the agent refused to provide the good it would be permanently excluded from the system of exchange.
[37] Menner (2005) and Molico and Zhang (2006) also analyze search models with capital, but they only present numerical solutions of the model.
[38] Similarly to neoclassical models there is no distinction between consumption and capital goods. However, contrary to neoclassical models there is a difference between output and the consumption good, which is the reason why non-consumption goods are not used by households in the exchange process.
[39] Shi defines this type of agents as *leisure-seekers*.
[40] A non-Walrasian exchange process implies that the exchange market does not clear, that is some agents are unable to conclude a transaction.
[41] This however makes it impossible to eliminate the individual risk faced by agents, which is the result of the random character of the process of concluding transactions. As a result the distribution of money and capital is not degenerate., which makes it necessary to use numerical methods to solve the model.
[42] Such a framework is a simplification allowing to obtain a degenerate distribution of the money and capital stock, which makes it possible to obtain an analytic solution of the model.
[43] In the context of monetary search models this problem is treated more widely in Rocheteau and Waller (2005).
[44] Wang and Yip consider only ST models. The equivalence of different models based on the SC concept has been demonstrated by Zhang (2000).
[45] Qualitative equivalence occurs when the results of a comparative-static analysis of two models are identical with respect to the sign. This is thus a type of equivalence weaker than functional equivalence.