Optimal option pricing and trading: a new theory

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14 December 2009

OPTIMAL OPTION PRICING AND TRADING: A NEW THEORY

ABSTRACT. We introduce a new utility-based approach to pricing European and American options. In so doing, we overcome some of the limitations of the existing models.

1. Introduction

Much of the literature on the European and American options is based on risk neutral pricing (see for example Bensoussan (1984) and Elliott and Kopp (2005), among many others). Other studies incorporated preferences (utility) into the valuation of options, such as the certainty equivalence and indifference pricing. Examples include Musiela and Zariphopolou (2007) and Musiela and Zariphopolou (2004). However, these approaches are somewhat impractical since it is cumbersome to compute the prices of the options.

Moreover, American options impose an additional problem known as the free-boundary problem. Even under risk neutrality, it is very difficult to price American options. In sum, using the existing utility-based methods, it is cumbersome to price both European and American options. Therefore other utility-based models should be investigated.

Consequently, the goal of this paper is to overcome these shortcomings. That is, we introduce a new utility-based approach, which enables us to easily price both European and American options. In so doing, we circumvent the free-boundary problem and provide general explicit solutions to the optimal hedge ratio, optimal stock shares and optimal option shares.

2. European options

As usual, the risky asset price process is defined as

$$dS_u = S_u \left(\mu_u dt + \sigma_u dW_u \right), \tag{1}$$

where r_u is the risk-free rate of return, μ_u and σ_u are the rate of return and the volatility, respectively; W_u is a standard Brownian motion, based on the probability space $(\Omega, \mathcal{F}_u, P)$. The wealth process is given by

$$X_T = x + \int_t^T \left\{ rX_u + \left(\left(\mu - r \right) \pi_u \right) \right\} du + \int_t^T \pi_u \sigma dW_u, \tag{2}$$

where x is the initial wealth, $\{\pi_u, \mathcal{F}_u\}_{t \le t \le T}$ is the portfolio process based on the probability space $(\Omega, \mathcal{F}_u, P)$ with $\int_t^T \pi_u^2 du < \infty$.

The writer's gain/loss from trading options is

$$g\left(u, S_u\right) = \left(K - C\left(u, S_u\right)\right) q_u = \left(K - S_u\right) q_u, \tag{3}$$

where K is the strike price, q_u is the quantity of option contracts with maturity time T, and C is the price of the option. The total wealth is given by

$$
\begin{aligned}
w_T &= X_T + g = x + \int_t^T \{rX_u + ((\mu - r)\,\pi_u) + (K - s\mu)\,q_u\}\,du + \\
&\int_t^T (\pi_u - sq_u)\,\sigma dW_u; \qquad (4)
\end{aligned}
$$

and $\int_t^T q_u^2 du < \infty$. The trading strategy $\mathcal{A}(\pi_u, q_u)$ is admissible.

The firms's objective is to maximize the expected utility of total wealth with respect to the portfolio and the option quantity

$$
V(t,x) = \underset{\pi_t, q_t}{Sup} E\left[u(w_T) \mid \mathcal{F}_t\right], \qquad (5)
$$

where $V(.)$ is the value function, $u(.)$ is a differentiable, bounded and concave utility function.

The value function satisfies the Hamiltonian-Jacobi-Bellman PDE

$$
V_t + rxV_x +
$$

$$
\underset{\pi_t, q_t}{Sup}\left\{\left[\pi_t(\mu - r) + (K - s\mu)\,q_t\right]V_x + \frac{1}{2}\left(\pi_t^2 - s^2 q_t^2\right)\sigma^2 V_{xx}\right\} = 0,
$$

$$
V(T,x) = u(x). \qquad (6)
$$

The solutions are

$$(\mu - r)\, V_x + \pi_t^* \sigma^2 V_{xx} = 0, \tag{7}$$

$$(s\mu - K)\, V_x + q^* \sigma^2 V_{xx} = 0. \tag{8}$$

Thus

$$\pi_t^* = -\frac{(\mu - r)\, V_x}{\sigma^2 V_{xx}}, \tag{9}$$

$$q_t^* = -\frac{(s\mu - K)\, V_x}{s^2 \sigma^2 V_{xx}}. \tag{10}$$

The optimal number of the stock shares $\delta_t^* = \pi_t^*/s$; therefore the optimal hedge ratio is explicitly expressed as

$$\frac{\delta_t^*}{q_t^*} = \frac{(\mu - r)\, s}{s\mu - K}, \tag{11}$$

which is clearly independent of the investor's preferences. Similarly, the optimal number of the stock shares can be explicitly expressed as a function of the optimal option quantity

$$\delta_t^* = \frac{(\mu - r)\, s q_t^*}{s\mu - K};\, q_t^* = \frac{(s\mu - K)\, \delta_t^*}{(\mu - r)\, s}. \tag{12}$$

To determine the option price, we simply divide the initial wealth minus the portfolio and the bank account by the optimal option quantity

$$C_t = \frac{x - \pi_t^* - B}{q_t^*}, \tag{13}$$

where B is the amount of money invested in the bank account.

3. American options

It is well-known that the price of the American option is defined as $A_t = \max\limits_{t \leq \tau \leq T} E_t \left[g\left(\tau \right) \right]$. In this paper, we redefine the price of the American option based on the price of its European counterpart

$$A_t = \max_{t \leq \tau \leq T} E_t \left[g\left(\tau \right) \right] = E_t \left[g\left(T \right) \right] + \varpi_u, \tag{14}$$

where ϖ_u is a random variable such that

$$\begin{cases} \varpi_\tau = 0 \text{ if } \tau = T \\ \varpi_\tau > 0 \text{ if } \tau < T \end{cases} \tag{15}$$

The dynamics of ϖ_u are given by

$$d\varpi_u = a_u dt + \sigma_{2u} dW_{u2}; \varpi_t = \omega \tag{16}$$

where σ_{2u} is the volatility, a_u is the mean, W_{u2} is a standard Brownian motion; $\bar{p}_u, \sigma_u$, σ_{2u} and $a_u \in C_b$. The gain/loss process from trading options is given by

$$g_u = q_u \left(K - S_u - \varpi_u\right) \tag{17}$$

Therefore the total wealth process is given by

$$\begin{aligned} w_T &= X_T + g = x + \int\limits_t^T \left\{rX_u + \left(\left(\mu - r\right)\pi_u\right) + \left(K - s\mu - a_u\right)q_u\right\}du + \\ & \int\limits_t^T \left(\pi_u - sq_u\right)\sigma_1 dW_{u1} - \int\limits_t^T q_u\sigma_2 dW_2. \end{aligned} \tag{18}$$

Since the random process ϖ_u is included in the wealth process and it accounts for the possibility of earlier exercise of the option, the trading horizon can be set at $[t, T]$.

The objective function is given by

$$V\left(t, x, \omega\right) = \underset{\pi_t, q_t}{Sup} E\left[u\left(w_T\right) \mid \mathcal{F}_t\right]. \tag{19}$$

The value function satisfies the HJP PDE

$$V_t + rxV_x + -a_t V_\omega \frac{1}{2}\sigma_2^2 V_{\omega\omega} +$$

$$\underset{\pi_t, q_t}{Sup} \left\{ \begin{array}{c} \left[\pi_t \left(\mu - r\right) + s\left(K - \mu\right) q_t\right] V_x + \\ \frac{1}{2}\left(\pi_t^2 + s^2 q_t^2\right) \sigma_1^2 V_{xx} - \rho\sigma_1\sigma_2 \left(s q_t + \pi_t\right) V_{x\omega} \end{array} \right\} = 0.$$

$$V\left(T, x, \omega\right) = u\left(x, \omega\right). \tag{20}$$

The above equation holds with equality and thus the usual free boundary problem is avoided. The solutions yield

$$\left(\mu - r\right) V_x + \pi_t^* \sigma_1^2 V_{xx} + \rho\sigma_1\sigma_2 V_{x\omega} = 0, \tag{21}$$

$$\left(s\mu - K\right) V_x + q_t^* s^2 \sigma_1^2 V_{xx} + \rho\sigma_1\sigma_2 s V_{x\omega} = 0, \tag{22}$$

where ρ is the factor of correlation between the two Brownian motions. Since $V_{x\omega} = -q_t^* V_{xx}$, we obtain

$$\pi_t^* = -\frac{\left(\mu - r\right) V_x - \rho\sigma_1\sigma_2 q_t^* V_{xx}}{\sigma_1^2 V_{xx}}, \tag{23}$$

$$q_t^* = -\frac{\left(s\mu - K\right) V_x - \rho\sigma_1\sigma_2 q_t^* V_{xx}}{s^2 \sigma_1^2 V_{xx}}. \tag{24}$$

Let $c_1 \equiv \rho\sigma_1\sigma_2/s\sigma_1^2$ and $c_2 \equiv \frac{(\mu-r)s^2}{(s\mu-K)}$; from (23)-(24)

$$\frac{\pi_t^* - c_1 q_t^*}{q_t^* - c_1 \pi_t^*} = c_2, \tag{25}$$

and thus the optimal portfolio can be explicitly expressed as a function of the optimal option quantity

$$\pi_t^* = \frac{(c_1 + c_2)\, q_t^*}{1 - c_1 c_2}. \tag{26}$$

In addition, the optimal hedge ratio has an explicit solution independent of preferences

$$\frac{\delta_t^*}{q_t^*} = \frac{(c_1 + c_2)}{(1 - c_1 c_2)\, s}. \tag{27}$$

As before the price of the American option is calculated as

$$A_t = \frac{x - \pi_t^* - B}{q_t^*}. \tag{28}$$

References

[1] Bensoussan, A. (1984). "On the theory of option pricing." *Acta Appl. Math.*, **2**, pp 139-158.

[2] Elliott, R. and P. Kopp (2005). Mathematics of financial markets. Springer, NY, USA.

[3] Musiela, M. and T. Zariphopoulou (2007). "Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model." in Advances in Mathematical Finance, Birkhauser, Boston, pp 303-334.

[4] Musiela, M. and T. Zariphopoulou (2004). "Indifference prices of early exercise claims." *Contemporary Mathematics*, **351,** pp 259-272.