A short step between democracy and dictatorship

Quesada, Antonio

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Antonio Quesada†

Departament d'Economia, Universitat Rovira i Virgili, Avinguda de la Universitat 1, 43204 Reus, Spain

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Abstract
The majority rule and the hierarchically dictatorial rule are both characterized when preferences are defined over two alternatives. The majority rule is characterized in terms of seven axioms. The hierarchically dictatorial rule is characterized in terms of six of these seven axioms and the negation of the seventh, so each rule can be seen as obtained from the other by negating just one of the axioms. The pivotal axiom holds that, for societies with at least three members, the frequency with which indifference is the result of the preference aggregation must be smaller than the frequency with which one of the alternatives is declared preferred to the other.

Keywords: Social welfare function, majority rule, dictatorship, axiomatic characterization, two alternatives.

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1. Introduction

Democracy and dictatorship define two focal procedures to make a collective decision. In a democracy, every member of the collective always has the potential to influence the decision; in a dictatorship, a given member of the collective always determines the decision. Since democracy is usually associated with the adoption of some majority principle, dictatorship will be confronted with the weakest majority concept: the relative majority rule (or, for short, the majority rule).

The majority rule does not create preference cycles, as in the Condorcet paradox, when there are only two alternatives. There are several axiomatizations of the majority rule for the two-alternative case; see, for instance, May (1952, p. 682), Fishburn (1973, p. 58), Aşan and Sanver (2002, p. 411), Woeginger (2003, p. 91; 2005, p. 9), Miroiu (2004, p. 362), and Xu and Zhong (2009). This paper presents another axiomatization of the majority rule, which can be turned into an axiomatization of dictatorship by negating just one the axioms. Specifically, it is shown that majority and dictatorship satisfy the following axioms.

- Unanimity: if all the voters vote for the same candidate *a*, then *a* is the chosen candidate.

- Reducibility: the outcome of an election involving *n* voters can be obtained from a certain election involving $n - 1$ voters.

- Substitutability: in elections involving two voters *i* and *j*, a third voter *k* can replace *i* or *j* without altering the result of the election.

- Exchangeability: in elections involving two voters, if the result of the election differs from both *i*'s and *j*'s vote, then the result remains the same when *i* and *j* exchange their votes.

- Parity: for elections with two given voters, each of the three outcomes is chosen the same number of times.

- Resoluteness: for elections involving three given voters, the proportion of cases in which the outcome "tie" arises is not greater than ⅓.

Given those axioms, majority rule arises, roughly speaking, when resoluteness rather than parity is imposed on elections involving more than two voters, whereas dictatorship emerges when it is parity rather than resoluteness that is imposed. So the emphasis on resoluteness over parity leads to majority, whereas the emphasis on parity over resoluteness leads to dictatorship.

It is worth noticing that the characterization of the majority rule does not resort to such typical axioms as neutrality, anonymity, Pareto efficiency, or monotonicity. In addition, the characterizations of majority and dictatorship seem to be paradoxical in the following sense: dictatorship emerges from a non-discriminatory treatment of all the outcomes, whereas majority derives from a discriminatory treatment of one of the outcomes (the tie).

2. Definitions and assumptions

Members of the set $\mathbb{N}$ of natural numbers are names for individuals. A society is a finite non-empty subset of $\mathbb{N}$. There are two alternatives: A and B. A preference over $\{A, B\}$ is represented by a number from the set $\{-1, 0, 1\}$. If the number is 1, A is preferred to B; if -1, B is preferred to A; if 0, A is indifferent to B. A preference profile for society I is a function $x_I : I \to \{-1, 0, 1\}$ assigning a preference over $\{A, B\}$ to each member of I.

For $n \in \mathbb{N}$, X_n is the set of all preference profiles $x_I : I \to \{-1, 0, 1\}$ such that I has n elements. The set X is the set of all preference profiles $x_I : I \to \{-1, 0, 1\}$ such that I is a society. A member x_I of X can be viewed as an election in which I is the set of voters and, for $i \in I$, x_i is i's vote: if $x_i = 1$, then i votes for candidate A; if $x_i = -1$, then i votes for candidate B; and if $x_i = 0$, then i's vote is a blank vote. For $x_I \in X$, $i \in I$ and non-empty $J \subset I$, x_i abbreviates $x_I(i)$ and x_J is the restriction of x_I to society J.

Definition 2.1. A social welfare function is a mapping $f : X \to \{-1, 0, 1\}$.

A social welfare function takes as input the preferences over $\{A, B\}$ of all the members of any given society I and outputs a collective preference over $\{A, B\}$. For $x_I \in X$: (i) $f(x_I) = 1$ means that, according to f, the collective prefers A to B; (ii) $f(x_I) = -1$, that the collective prefers B to A; and (iii) $f(x_I) = 0$, that the collective is indifferent between A and B. Another interpretation is that f determines the outcome of an election x_I: $f(x_I) = 1$ means that A is the winning candidate; $f(x_I) = -1$ that it is B; and $f(x_I) = 0$ that there is a tie between A and B.

Definition 2.2. The majority rule is the social welfare function $\mu : X \to \{-1, 0, 1\}$ such that, for all $x_I \in X$: (i) if $\sum_{i\in I} x_i > 0$, then $\mu(x_I) = 1$; (ii) if $\sum_{i\in I} x_i < 0$, then $\mu(x_I) = -1$; and (iii) if $\sum_{i\in I} x_i = 0$, then $\mu(x_I) = 0$.

Definition 2.3. A social welfare function $f : X \to \{-1, 0, 1\}$ has a hierarchy of dictators if there a linear order $\Rightarrow$ on $\mathbb{N}$ such that, for all $x_I \in X$, $f(x_I) = x_i$, where i is the member of I such that, for all $j \in I\backslash\{i\}$, $i \Rightarrow j$.

UNA. *Unanimity*. For all $x_I \in X$, if there is $a \in \{-1, 0, 1\}$ such that, for all $i \in I$, $x_i = a$ then, $f(x_I) = a$.

UNA states that if all the members of a society have the same preference, then that preference constitutes the collective preference.

For $x_I \in X$, $i \in \mathbb{N}\backslash I$ and $a \in \{-1, 0, 1\}$, (x_I, a^i) designates the member y_J of X such that: (i) $J = I \cup \{i\}$; (ii) for all $j \in I$, $y_j = x_j$; and (iii) $y_i = a$. In words, (x_I, a^i) is the member of X obtained from x_I by adding another individual i with preference a. For the case in which $I = \{i, j\}$, (a^i, b^j) stands for the member x_I of X such that $x_i = a$ and $x_j = b$.

RED. *Reduction*. For all $x_I \in X$, $i \in I$ and $j \in I\backslash\{i\}$, if $x_i \neq x_j$, then, for some $k \in \{i, j\}$, $f(x_I) = f(x_{I\backslash\{i,j\}}, f(x_{\{i,j\}})^k)$.

RED asserts that the result of aggregating n preferences (or of an election involving n voters) can be obtained as the result of aggregation of $n - 1$ preferences (or an election involving $n - 1$ voters). Specifically, RED holds that the preference $f(x_I)$ can be obtained as follows. Choose any two individuals i and j whose preferences x_i and x_j are different. Determine the preference $f(x_{\{i,j\}})$ of society $\{i, j\}$. Select a representative $k \in \{i, j\}$ of society $\{i, j\}$. Replace, in the original aggregation problem x_I, the preferences (x_i, x_j) by the preference $f(x_{\{i,j\}})$ and ascribe $f(x_{\{i,j\}})$ to the representative k. Finally, compute the preference $f(x_{\{i,j\}}, f(x_{\{i,j\}})^k)$ and make $f(x_I)$ equal to that preference.

The condition of weak path independence in Aşan and Sanver (2002, p. 411) and the property of reducibility to subsocieties in Woeginger (2003, p. 90) are similar reduction properties. RED is also related to Chambers' (2008, p. 350) representative consistency, which is a condition of gerrymandering proofness. When combined with UNA, representative consistency implies that, for all $x_I \in X$ and $J \subset I$, $f(x_I) = f(x_{I\backslash J}, (f(x_J)^i)_{i\in J})$. This says that the outcome of election x_I coincides with the outcome of any election obtained from x_I by replacing the vote of each voter in any given strict subset J of I with the vote $f(x_J)$, which can be viewed as the representative vote of the group J. RED differs from Chamber's consistency in having the whole set J of voters replaced by a representative voter casting the representative vote and in requiring J to have just two members.

Definition 2.4. For $i \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{i\}$, k can replace i (abbreviated "$k \equiv i$") if, for every $j \in \mathbb{N}\backslash\{i, k\}$, every preference profile x_I for $I = \{i, j\}$ and every preference profile y_J for $J = \{j, k\}$, if $x_i = y_k$ and $x_j = y_j$, then $f(x_I) = f(y_J)$.

SUB. *Substitutability*. For all $i \in \mathbb{N}$, $j \in \mathbb{N}\backslash\{i\}$ and $k \in \mathbb{N}\backslash\{i, j\}$, $k \equiv i$ or $k \equiv j$ (or both).

SUB claims that, in every aggregation problem (or election) involving just two individuals i and j, any other individual k can replace i or j without causing any change in the final result.

EXC. *Exchangeability*. For all $i \in N$, $j \in N\backslash\{i\}$ and preference profile x_I for $I = \{i, j\}$, if $x_i \neq f(x_I) \neq x_j$, then $f(y_I) = f(x_I)$, where $y_i = x_j$ and $y_j = x_i$.

EXC says that if the collective preference associated with a society with two individuals disagrees with the preference of each member of the society, then the same collective preference results when the individuals exchange their preferences. Both SUB and EXC can be regarded as anonymity conditions.

For society I of $\mathbb{N}$ and $a \in \{1, 0, -1\}$, define π_a^I to be the number of preference profiles x_I for I such that $f(x_I) = a$ divided by the number of preference profiles for I. Hence, π_1^I is the proportion of elections involving the set I of voters in which the chosen candidate is A; π_{-1}^I is the proportion in which the chosen candidate is B; and π_0^I is the proportion in which no candidate is chosen (there is a tie).

PAR_2. *Parity*. For every society I having two members, $\pi_0^I = \pi_1^I = \pi_{-1}^I$.

According to PAR_2, in societies with two members, every possible collective preference should be obtained the same total number of times. In terms of elections, if all elections are equally likely, then all the outcomes are also equally likely.

RES_3. *Resoluteness*. For every society I having three members, $\pi_0^I \leq ⅓$.

RES_3 requires that, in societies with three voters, the proportion of cases in which indifference results cannot be greater than ⅓, which is the proportion associated with the situation in which all the collective preferences are equally likely. As a result, the proportion of cases in which the rule is resolute (an alternative is chosen) is at least ⅔.

IND. *Indifference disliked*. For every $n \geq 3$, there is $I \subset \mathbb{N}$ having n members such that $\pi_0^I < \pi_1^I$.

IND*. *Indifference not necessarily disliked*. There is $n \geq 3$ such that, for all $I \subset \mathbb{N}$ having n members, $\pi_0^I \geq \pi_1^I$.

IND* is the negation of IND and IND can be interpreted in the sense that, in societies with at least members, indifference is less likely than having alternative A be the collectively preferred alternative (both IND and IND* could be defined with π_{-1}^I instead of π_1^I, because the majority rule and the hierarchically dictatorial rule satisfy, for society $I \subset \mathbb{N}$, $\pi_{-1}^I = \pi_1^I$). IND discriminates indifference (the outcome "0"), whereas IND* denies that discrimination, as a general rule. IND is a condition of the sort "∀∃". The results in Section 3 hold if IND is replaced by any condition of the sort "∀∀","∃∀" or "∃∃" (with IND* redefined accordingly).

3. Results

Lemma 3.1. Let $f : X \to \{-1, 0, 1\}$ be a social welfare function satisfying UNA, RED, SUB, EXC, and PAR_2. Let $i \in \mathbb{N}$, $j \in \mathbb{N}\backslash\{i\}$ and $I = \{i, j\}$. If $f(1^i, -1^j) = 0$, then, for every preference profile x_I for I, $f(x_I) = \mu(x_I)$.

Proof. Let $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$. Suppose that $f(1^i, -1^j) = 0$. There are 9 preference profiles for $I = \{i, j\}$. It must be shown that for each such preference profile x_I, $f(x_I) = \mu(x_I)$. By UNA, $f(1^i, 1^j) = 1 = \mu(1^i, 1^j)$, $f(0^i, 0^j) = 0 = \mu(0^i, 0^j)$ and $f(-1^i, -1^j) = -1 = \mu(-1^i, -1^j)$. By assumption, $f(1^i, -1^j) = 0 = \mu(1^i, -1^j)$. By EXC, $f(1^i, -1^j) = f(1^j, -1^i)$, so $f(-1^i, 1^j) = 0 = \mu(-1^i, 1^j)$. Since the number of preference profiles for I is 9, it follows from $f(0^i, 0^j) = f(1^i, -1^j) = f(-1^i, 1^j) = 0$ that $\pi_0^I \geq 3/9$. By PAR_2, the remaining four preference profiles for I satisfy $f(1^i, 0^j) \neq 0$, $f(0^i, 1^j) \neq 0$, $f(-1^i, 0^j) \neq 0$ and $f(0^i, -1^j) \neq 0$.

Case 1: $f(1^i, 0^j) = -1$. By EXC, $f(0^i, 1^j) = -1$. Choose $k \in \mathbb{N}\backslash\{i, j\}$. By RED, there are $\alpha \in \{i, j\}$ and $\beta \in \{i, k\}$ such that $f(f(1^i, -1^j)^\alpha, 0^k) = f(1^i, -1^j, 0^k) = f(f(1^i, 0^k)^\beta, -1^j)$. That is, $f(0^\alpha, 0^k) = f(f(1^i, 0^k)^\beta, -1^j)$. Since, by UNA, $f(0^\alpha, 0^k) = 0$, it follows that $f(f(1^i, 0^k)^\beta, -1^j) =$

0. By UNA $f(-1^{\beta}, -1^{j}) = -1$. Accordingly, $f(1^{i}, 0^{k}) \neq -1$. By SUB, $j \equiv i$ or $j \equiv k$. If $j \equiv k$, then $f(1^{i}, 0^{k}) \neq -1$ implies $f(1^{i}, 0^{j}) \neq -1$: contradiction. Therefore, $j \equiv i$. By SUB, $k \equiv i$ or $k \equiv j$. If $k \equiv j$, then $f(1^{i}, 0^{j}) = -1$ implies $f(1^{i}, 0^{k}) = -1$: contradiction. As a result, $k \equiv i$. Since $j \equiv i$, $f(1^{i}, 0^{k}) \neq -1$ implies $f(1^{j}, 0^{k}) \neq -1$; and since $k \equiv i$, $f(0^{i}, 1^{j}) = -1$ implies $f(0^{k}, 1^{j}) = -1$: contradiction. Case 2: $f(1^{i}, 0^{j}) = 1 = \mu(1^{i}, 0^{j})$. Recall that $f(0^{i}, 1^{j}) \neq 0$. If $f(0^{i}, 1^{j}) = -1$, then, by EXC, $f(1^{i}, 0^{j}) = -1$: contradiction. If $f(0^{i}, 1^{j}) = 1 = \mu(0^{i}, 1^{j})$, then, by PAR_2, $f(-1^{i}, 0^{j}) = -1 = \mu(-1^{i}, 0^{j})$ and $f(0^{i}, -1^{j}) = -1 = \mu(0^{i}, -1^{j})$.■

Lemma 3.2. Let $f : X \to \{-1, 0, 1\}$ be a social welfare function satisfying UNA, RED, SUB, EXC, and PAR_2. If there are $i \in \mathbb{N}$ and $j \in \mathbb{N}\setminus\{i\}$ such that $f(1^{i}, -1^{j}) = 0$, then f is the majority rule.

Proof. Let $i \in \mathbb{N}$ and $j \in \mathbb{N}\setminus\{i\}$ satisfy $f(1^{i}, -1^{j}) = 0$. It must be shown that, for all $n \in \mathbb{N}$, $f = \mu$ on X_n. Case 1: $n = 1$. By UNA, $f = \mu$ on X_1. Case 2: $n = 2$. Choose $I \subset \mathbb{N}$ having two members and $x_I \in X_2$. Case 2a: $I = \{i, j\}$. By Lemma 3.1, $f(1^{i}, -1^{j}) = 0$ implies $f(x_I) = \mu(x_I)$. Case 2b: $I \cap \{i, j\} = i$. Let $k \in I\setminus\{i\}$. By SUB, $k \equiv i$ or $k \equiv j$. If $k \equiv j$, then $f(1^{i}, -1^{j}) = 0$ implies $f(1^{i}, -1^{k}) = 0$. Given this, by Lemma 3.1, $f(x_I) = \mu(x_I)$. If $k \equiv i$, then $f(1^{i}, -1^{j}) = 0$ implies $f(1^{k}, -1^{j}) = 0$. By SUB, $j \equiv k$ or $j \equiv i$. If $j \equiv i$, then $f(1^{k}, -1^{i}) = f(1^{k}, -1^{j})$. By Lemma 3.1, $f(1^{k}, -1^{i}) = 0$ yields $f(x_I) = \mu(x_I)$. If $j \equiv k$, then $f(1^{i}, -1^{k}) = f(1^{i}, -1^{j}) = 0$. By Lemma 3.1, $f(1^{i}, -1^{k}) = 0$ yields $f(x_I) = \mu(x_I)$.

Case 2c: $I \cap \{i, j\} = j$. Analogous to case 2b. Case 2d: $I \cap \{i, j\} = \varnothing$. Let $I = \{k, r\}$. By SUB, $i \equiv k$ or $i \equiv r$. Without loss of generality, suppose $i \equiv r$. With $J = \{i, k\}$, by case 2b, for all $x_J \in X$, $f(x^{J}) = \mu(x_J)$. Since $i \equiv r$ implies $r \equiv i$, for all $x_I \in X$, $f(x_I) = \mu(x_I)$. Case 3: $n \geq 3$. Taking case 2 as the base case of an induction argument, choose $n \geq 3$ and suppose that, for all $t \in \{2, \ldots, n-1\}$, $f = \mu$ on X_t. To prove that $f = \mu$ on X_n, choose $I \subset \mathbb{N}$ having n members and $x_I \in X_n$. If, for some $a \in \{1, 0, -1\}$, all the components of x_I are equal to a, then, by UNA, $f(x_I) = a = \mu(x_I)$. If two components x_k and x_r are different, then, by RED, for some $\alpha \in \{k, r\}$, $f(x_I) = f(x_{I\setminus\{k,r\}}, f(x_{\{k,r\}})^{\alpha})$. By the induction hypothesis, $f(x_{I\setminus\{k,r\}}, f(x_{\{k,r\}})^{\alpha}) = \mu(x_{I\setminus\{k,r\}}, \mu(x_{\{k,r\}})^{\alpha})$. Since μ satisfies RED, $\mu(x_{I\setminus\{k,r\}}, \mu(x_{\{k,r\}})^{\alpha}) = \mu(x_I)$. As a consequence, $f(x_I) = \mu(x_I)$.■

Lemma 3.3. Let $f : X \to \{-1, 0, 1\}$ be a social welfare function satisfying UNA, RED, SUB, EXC, PAR_2, and RES_3. Let $i \in \mathbb{N}$, $j \in \mathbb{N}\setminus\{i\}$ and $I = \{i, j\}$. If $f(1^{i}, -1^{j}) = 1$, then, for every preference profile x_I for I, $f(x_I) = x_i$.

Proof. Suppose that $f(1^{i}, -1^{j}) = 1$. With $I = \{i, j\}$, there are another 8 preference profiles for I. It must be shown that for each such preference profile x_I, $f(x_I) = x_i$. By UNA, $f(1^{i},$

$1^j) = 1$, $f(0^i, 0^j) = 0$ and $f(-1^i, -1^j) = -1$. Therefore, only six profiles have to be considered: $(1^i, -1^j)$, $(-1^i, 1)$, $(1^i, 0^j)$, $(0^i, 1^j)$, $(-1^i, 0^j)$, and $(0^i, -1^j)$. Those profiles are represented by the first two columns in Table 1 by letting $\alpha = i$ and $\beta = j$. The remaining six columns in Table 1 show the possible values of each such profile when UNA, RED, SUB, EXC, and PAR_2 are assumed. The values are obtained as follows.

α	β	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
1	−1	1	1	1	1	1	1
−1	1	1	1	−1	−1	−1	−1
1	0	−1	0	0	0	1	1
0	1	−1	0	1	1	0	0
−1	0	0	−1	−1	0	0	−1
0	−1	0	−1	0	−1	−1	0

Table 1

By assumption, $f(1^i, -1^j) = 1$. By EXC, $f(-1^i, 1^j) = 0$ would imply $f(1^i, -1^j) = 0$: contradiction. Therefore, $f(-1^i, 1^j) \in \{1, -1\}$. If $f(-1^i, 1^j) = 1$, then, by PAR_2, the value of each of the four remaining profiles $(1^i, 0^j)$, $(0^i, 1^j)$, $(-1^i, 0^j)$, and $(0^i, -1^j)$ is different from 1. If $f(1^i, 0^j) = -1$, then, by EXC, $f(0^i, 1^j) = -1$. Given this, by PAR_2, $f(-1^i, 0^j) = f(0^i, -1^j) = 0$. This set of values defines Case 1 in Table 1. If $f(1^i, 0^j) = 0$, then, by EXC, $f(0^i, 1^j) \neq -1$, so $f(0^i, 1^j) = 0$. In view of this, by PAR_2, $f(-1^i, 0^j) = f(0^i, -1^j) = -1$. This is Case 2 in Table 1.

If $f(-1^i, 1^j) = -1$, then $f(1^i, 0^j) \in \{0, 1\}$: if $f(1^i, 0^j) = -1$, by EXC, $f(0^i, 1^j) = -1$ and, as a result, $\pi_{-1}{}^I \geq 4/9$, contradicting PAR_2. If $f(1^i, 0^j) = 0$, by EXC, $f(0^i, 1^j) \neq -1$ and, accordingly, $f(0^i, 1^j) \in \{0, 1\}$. If $f(0^i, 1^j) = 0$, then, by PAR_2, $\{f(-1^i, 0^j), f(0^i, -1^j)\} = \{1, -1\}$, which is not consistent with EXC. Consequently, $f(0^i, 1^j) = 1$. By PAR_2, $f(-1^i, 0^j) = -1$ and $f(0^i, -1^j) = 0$ (Case 3 in Table 1) or $f(-1^i, 0^j) = 0$ and $f(0^i, -1^j) = -1$ (Case 4 in Table 1).

Finally, if $f(1^i, 0^j) = 1$, by EXC, $f(0^i, 1^j) \neq -1$ and, hence, $f(0^i, 1^j) \in \{0, 1\}$. If $f(0^i, 1^j) = 1$, then $\pi_1{}^I \geq 4/9$, contradicting PAR_2. As a result, $f(0^i, 1^j) = 0$. By PAR_2, $f(-1^i, 0^j) = 0$ and $f(0^i, -1^j) = -1$ (Case 5 in Table 1) or $f(-1^i, 0^j) = -1$ and $f(0^i, -1^j) = 0$ (Case 6 in Table 1). The proof amounts to reaching a contradiction from each case different from Case 6. Choose $k \in \mathbb{N}\backslash\{i, j\}$. By SUB, in both Case 1 and Case 2, f is symmetric on the domain of preference profiles for societies having two members: for all $a \in \{-1, 0, 1\}$, $b \in \{-1, 0, 1\}$, $\alpha \in \mathbb{N}$, and $\beta \in \mathbb{N}\backslash\{\alpha\}$, $f(a^\alpha, b^\beta) = f(a^\beta, b^\alpha)$.

• Case 1. There are 27 preference profiles for $J = \{i, j, k\}$. By RED and the symmetry between i, j and k, f assigns the value 0 to the following nine profiles: (i) the three profiles in which two components are 0 and the third one is −1; (ii) the three profiles in which two components are −1 and the third one is 0; and (iii) the three profiles in which two components are 0 and the third one is 1. By UNA, $f(0^i, 0^j, 0^k) = 0$. The conclusion is then that $\pi_0^J \geq 10/27$, which contradicts RES_3.

• Case 2. By RED, there are $\delta \in \{i, k\}$ and $\gamma \in \{j, k\}$ such that $f(f(-1^i, 0^k)^\delta, 1^j) = f(-1^i, 1^j, 0^k) = f(f(1^j, 0^k)^\gamma, -1^i)$. Since $f(-1^i, 0^k) = -1$ and $f(1^j, 0^k) = 0$, $f(-1^\delta, 1^j) = f(0^\gamma, -1^i)$. By symmetry between i, j and k, for all $\delta \in \{i, k\}$, $f(-1^\delta, 1^j) = 1$; and, for all $\gamma \in \{j, k\}$, $f(0^\gamma, -1^i) = -1$: contradiction.

By SUB, $k \equiv i$ or $k \equiv j$. If $k \equiv i$, then, by SUB, $j \equiv k$ or $j \equiv i$. If $k \equiv j$, then, by SUB, $i \equiv k$ or $i \equiv j$. Since the substitutability relation is, by definition, symmetric, having $k \equiv i$ and $j \equiv k$ represents the same case as having $k \equiv j$ and $i \equiv k$. Summarizing, by SUB: (i) $j \equiv k$ and $k \equiv i$; (ii) $j \equiv i$ and $i \equiv k$; or (iii) $k \equiv j$ and $j \equiv i$. If (i) holds, then, for all $a \in \{-1, 0, 1\}$ and $b \in \{-1, 0, 1\}/\{a\}$, $f(a^i, b^j) = f(a^i, b^k) = f(a^k, b^j)$. This means that, for any $c \in \{3, 4, 5\}$, Case c in Table 1 yields the value of the corresponding profiles not only when $(\alpha, \beta) = (i, j)$ but also when

$$(\alpha, \beta) = (i, k) \text{ and } (\alpha, \beta) = (k, j). \tag{1}$$

Similarly, when (ii) holds, Table 1 provides the values when $(\alpha, \beta) = (i, j)$,

$$(\alpha, \beta) = (k, j) \text{ and } (\alpha, \beta) = (k, i). \tag{2}$$

Lastly, when (iii) holds, Table 1 provides the values when $(\alpha, \beta) = (i, j)$,

$$(\alpha, \beta) = (i, k) \text{ and } (\alpha, \beta) = (j, k). \tag{3}$$

• Case 3. Case 3a: $j \equiv k$ and $k \equiv i$, so (1) holds. By RED, there are $\delta \in \{i, k\}$ and $\gamma \in \{j, k\}$ such that $f(f(1^i, -1^k)^\delta, 0^j) = f(1^i, 0^j, -1^k) = f(f(0^j, -1^k)^\gamma, 1^i)$. Since $j \equiv k$, $1 = f(1^i, -1^j) = f(1^i, -1^k)$. If $\delta = i$, then $f(f(1^i, -1^k)^\delta, 0^j) = f(1^i, 0^j) = 0$. If $\delta = k$, then $k \equiv i$ and $f(1^i, 0^j) = 0$ imply $f(1^k, 0^j) = 0$, so $f(f(1^i, -1^k)^\delta, 0^j) = f(1^k, 0^j) = 0$. Therefore, $f(f(0^j, -1^k)^\gamma, 1^i) = 0$. As $k \equiv i$, $-1 = f(0^j, -1^i) = f(0^j, -1^k)$. Hence, $f(-1^\gamma, 1^i) = 0$. If $\gamma = j$, then $f(-1^\gamma, 1^i) = 1$: contradiction. If $\gamma = k$, then, since $j \equiv k$, $1 = f(-1^j, 1^i) = f(-1^k, 1^i)$: contradiction.

Case 3b: $j \equiv i$ and $i \equiv k$, so (2) holds. By RED, there are $\delta \in \{i, j\}$ and $\gamma \in \{j, k\}$ such that $f(f(-1^i, 0^j)^\delta, 1^k) = f(-1^i, 0^j, 1^k) = f(f(0^j, 1^k)^\gamma, -1^i)$. By (2), $f(0^j, 1^k) = 0$. As $f(-1^i, 0^j) = -1$, $f(-1^\delta, 1^k) = f(0^\gamma, -1^i)$. By (2), $f(-1^j, 1^k) = 1 = f(-1^i, 1^k)$. But $f(0^j, -1^i) = -1$ and, by (2), $f(0^k, -1^i) = 0$: contradiction.

Case 3c: $k \equiv j$ and $j \equiv i$, so (3) holds. By RED, there are $\delta \in \{i, j\}$ and $\gamma \in \{j, k\}$ such that $f(f(1^i, -1^j)^\delta, 0^k) = f(1^i, -1^j, 0^k) = f(f(-1^j, 0^k)^\gamma, 1^i)$. By (3), $f(-1^j, 0^k) = -1$. As $f(1^i, -1^j) = 1$, $f(1^\delta, 0^k) = f(-1^\gamma, 1^i)$. By (3), $f(1^j, 0^k) = 0 = f(1^i, 0^k)$. But, by (3), $f(-1^k, 1^i) = 1 = f(-1^j, 1^i)$: contradiction.

• Case 4. Case 4a: $j \equiv k$ and $k \equiv i$, so (1) holds. By RED, there are $\delta \in \{i, k\}$ and $\gamma \in \{j, k\}$ such that $f(f(1^i, 0^k)^\delta, -1^j) = f(1^i, -1^j, 0^k) = f(f(-1^j, 0^k)^\gamma, 1^i)$. By (1), $f(-1^j, 0^k) = -1$. Since $f(1^i, 0^k) = 0$, $f(0^\delta, -1^j) = f(-1^\gamma, 1^i)$. By (1), $f(0^k, -1^j) = -1 = f(0^i, -1^k)$. But, by (1), $f(-1^k, 1^i) = 1 = f(-1^j, 1^i)$: contradiction.

Case 4b: $j \equiv i$ and $i \equiv k$, so (2) holds. By RED, there are $\delta \in \{i, k\}$ and $\gamma \in \{i, j\}$ such that $f(f(0^i, -1^k)^\delta, 1^j) = f(0^i, 1^j, -1^k) = f(f(0^i, 1^j)^\gamma, -1^k)$. By (2), $f(0^i, -1^k) = 0$. Given that $f(0^i, 1^j) = 1$, it must be that $f(0^\delta, 1^j) = f(1^\gamma, -1^k)$. By (2), $f(0^k, 1^j) = 1 = f(0^i, 1^j)$. But, by (2), $f(1^i, -1^k) = -1$ and $f(1^j, -1^k) = -1$: contradiction.

Case 4c: $k \equiv j$ and $j \equiv i$, so (3) holds. By RED, there are $\delta \in \{i, j\}$ and $\gamma \in \{j, k\}$ such that $f(f(-1^i, 0^j)^\delta, 1^k) = f(-1^i, 0^j, 1^k) = f(f(0^j, 1^k)^\gamma, -1^i)$. By (3), $f(0^j, 1^k) = 1$. In addition, $f(-1^i, 0^j) = 0$, so $f(0^\delta, 1^k) = f(1^\gamma, -1^i)$. By (3), $f(0^i, 1^k) = 1$ and $f(0^j, 1^k) = 1$. But, by (3), $f(1^k, -1^i) = -1 = f(1^j, -1^i)$: contradiction.

• Case 5. Case 5a: $j \equiv k$ and $k \equiv i$, so (1) holds. By RED, there are $\delta \in \{i, j\}$ and $\gamma \in \{j, k\}$ such that $f(f(-1^i, 0^j)^\delta, 1^k) = f(-1^i, 0^j, 1^k) = f(f(0^j, 1^k)^\gamma, -1^i)$. By (1), $f(0^j, 1^k) = 1$. Since $f(-1^i, 0^j) = 0$, $f(0^\delta, 1^k) = f(1^\gamma, -1^i)$. By (1), $f(0^j, 1^k) = 1$ and $f(0^i, 1^k) = 0$. But, by (1), $f(1^k, -1^i) = -1 = f(1^j, -1^i)$: contradiction.

Case 5b: $j \equiv i$ and $i \equiv k$, so (2) holds. By RED, there are $\delta \in \{i, k\}$ and $\gamma \in \{i, j\}$ such that $f(f(1^i, -1^k)^\delta, 0^j) = f(1^i, 0^j, -1^k) = f(f(1^i, 0^j)^\gamma, -1^k)$. By (2), $f(1^i, -1^k) = -1$. Moreover, $f(1^i, 0^j) = 1$. Hence, $f(-1^\delta, 0^j) = f(1^\gamma, -1^k)$. By (2), $f(-1^k, 0^j) = 0 = f(-1^i, 0^j)$. But, by (2), $f(1^i, -1^k) = -1$ and $f(1^j, -1^k) = -1$: contradiction.

Case 5c: $k \equiv j$ and $j \equiv i$, so (3) holds. By RED, there are $\delta \in \{i, j\}$ and $\gamma \in \{j, k\}$ such that $f(f(0^i, 1^j)^\delta, -1^k) = f(0^i, 1^j, -1^k) = f(f(1^j, -1^k)^\gamma, 0^i)$. By (3), $f(1^j, -1^k) = 1$. As $f(0^i, 1^j) =$

0, $f(0^\delta, -1^k) = f(1^\gamma, 0^i)$. By (3), $f(0^i, -1^k) = -1$ and $f(0^j, -1^k) = -1$. But, by (3), $f(1^k, 0^i) = 0 = f(1^j, 0^i)$: contradiction.■

Lemma 3.4. Let $f : X \to \{-1, 0, 1\}$ be a social welfare function satisfying UNA, RED, SUB, EXC, PAR_2, and RES_3. Let $i \in \mathbb{N}$, $j \in \mathbb{N}\backslash\{i\}$ and $I = \{i, j\}$. If $f(1^i, -1^j) = -1$, then, for every preference profile x_I for I, $f(x_I) = x_j$.

Proof. The proof is analogous to that of Lemma 3.3. Suppose that $f(1^i, -1^j) = -1$. By UNA, $f(1^i, 1^j) = 1$, $f(0^i, 0^j) = 0$, and $f(-1^i, -1^j) = -1$. By EXC, $f(-1^i, 1^j) = 0$ would imply $f(1^i, -1^j) = 0$: contradiction. In consequence, $f(-1^i, 1^j) \in \{1, -1\}$. If $f(-1^i, 1^j) = -1$, then, by PAR_2, the value of each of the four remaining profiles $(1^i, 0^j)$, $(0^i, 1^j)$, $(-1^i, 0^j)$, and $(0^i, -1^j)$ is different from -1. If $f(-1^i, 0^j) = 0$, then, by EXC, $f(0^i, -1^j) \neq -1$, so $f(0^i, -1^j) = 0$. In view of this, by PAR_2, $f(1^i, 0^j) = f(0^i, 1^j) = 1$. This set of values defines Case 1 in Table 2. If $f(-1^i, 0^j) = 1$, then, by EXC, $f(0^i, -1^j) = 1$. Given this, by PAR_2, $f(1^i, 0^j) = f(0^i, 1^j) = 0$. This is Case 2 in Table 2.

If $f(-1^i, 1^j) = 1$, then $f(-1^i, 0^j) \in \{0, -1\}$: if $f(-1^i, 0^j) = 1$, by EXC, $f(0^i, -1^j) = 1$ and, as a result, $\pi_1^I \geq 4/9$, contradicting PAR_2. If $f(-1^i, 0^j) = 0$, by EXC, $f(0^i, -1^j) \neq 1$ and, accordingly, $f(0^i, -1^j) \in \{0, -1\}$. If $f(0^i, -1^j) = 0$, then, by PAR_2, $\{f(1^i, 0^j), f(0^i, 1^j)\} = \{1, -1\}$, which is not consistent with EXC. Consequently, $f(0^i, 1^j) = -1$. By PAR_2, $f(1^i, 0^j) = 1$ and $f(0^i, 1^j) = 0$ (Case 3 in Table 2) or $f(1^i, 0^j) = 0$ and $f(0^i, 1^j) = 1$ (Case 6 in Table 2).

α	β	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
1	−1	−1	−1	−1	−1	−1	−1
−1	1	−1	−1	1	1	1	1
1	0	1	0	1	1	0	0
0	1	1	0	0	0	1	1
−1	0	0	1	0	−1	−1	0
0	−1	0	1	−1	0	0	−1

Table 2

Finally, if $f(-1^i, 0^j) = -1$, by EXC, $f(0^i, -1^j) \neq 1$ and, hence, $f(0^i, -1^j) \in \{0, -1\}$. If $f(0^i, -1^j) = -1$, then $\pi_{-1}^I \geq 4/9$, contradicting PAR_2. Thus, $f(0^i, -1^j) = 0$. By PAR_2, $f(1^i, 0^j) = 1$ and $f(0^i, 1^j) = 0$ (Case 4 in Table 2) or $f(1^i, 0^j) = 0$ and $f(0^i, 1^j) = 1$ (Case 5 in Table 2). The proof amounts to deriving a contradiction from each case different from Case 6. To this end, notice that, by renaming i as j and j as i, for $c \in \{3, 4, 5\}$, Case c in Table 2 is the same as Case c in Table 1. Therefore, the contradictions reached from those cases in the proof of Lemma 3.3 show that Case 3, 4 and 5 cannot hold. With respect to Case 1 and Case 2, by SUB, the columns "Case 1" and "Case 2" in Table 2 are valid for all $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}\backslash\{\alpha\}$. Choose $k \in \mathbb{N}\backslash\{i, j\}$.

• Case 1. By RED, there are $\delta \in \{i, j\}$ and $\gamma \in \{j, k\}$ such that $f(f(1^i, -1^j)^\delta, 0^k) = f(1^i, -1^j, 0^k) = f(f(-1^j, 0^k)^\gamma, 1^i)$. On the one hand, $f(1^i, -1^j) = -1$ implies $f(f(1^i, -1^j)^\delta, 0^k) = f(-1^\delta, 0^k)$. By symmetry, $f(-1^j, 0^k) = f(-1^i, 0^k) = f(-1^i, 0^j) = 0$. On the other hand, by symmetry, $f(-1^j, 0^k) = f(-1^j, 0^i) = 0$, so $f(f(-1^j, 0^k)^\gamma, 1^i) = f(0^\gamma, 1^i)$. By symmetry, $1 = f(0^j, 1^i) = f(0^k, 1^i)$: contradiction.

• Case 2. There are 27 preference profiles for $J = \{i, j, k\}$. By RED and the symmetry between i, j and k, f assigns the value 0 to the following nine profiles: (i) the three profiles in which two components are 0 and the third one is −1; (ii) the three profiles in which two components are 1 and the third one is 0; and (iii) the three profiles in which two components are 0 and the third one is 1. By UNA, $f(0^i, 0^j, 0^k) = 0$. As a consequence, $\pi_0^J \geq 10/27$, which contradicts RES_3.■

Lemma 3.5. Let $f : X \to \{-1, 0, 1\}$ be a social welfare function satisfying UNA, RED, SUB, EXC, PAR_2, and RES_3. If there are $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$ such that $f(1^i, -1^j) \neq 0$, then f has a hierarchy of dictators.

Proof. Let f satisfy UNA, RED, SUB, EXC, PAR_2, and RES_3. For $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$, define $i \to j$ if and only if for every preference profile x_I for $I = \{i, j\}$, $f(x_I) = x_i$. Loosely speaking, $i \to j$ means that i is a dictator in society $\{i, j\}$. Assume that, for some $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$, $f(1^i, -1^j) \neq 0$.

• Step 1: for all $k \in \mathbb{N}$ and $r \in \mathbb{N}\backslash\{k\}$, either $k \to r$ or $k \to r$. Choose $k \in \mathbb{N}$ and $r \in \mathbb{N}\backslash\{k\}$. If $f(1^k, -1^r) = 0$, then, by Lemma 3.2, f is the majority rule, contradicting $f(1^i, -1^j) \neq 0$. Thus, $f(1^k, -1^r) \neq 0$ and, by Lemmas 3.3 and 3.4, either $k \to r$ or $k \to r$.

• Step 2: for all $k \in \mathbb{N}$, $r \in \mathbb{N}\backslash\{k\}$ and $t \in \mathbb{N}\backslash\{r, k\}$, if $k \to r$ and $r \to t$, then $k \to t$. Suppose $k \to r$ and $r \to t$. By SUB, $k \equiv r$ or $k \equiv t$. If $k \equiv t$, then $r \to t$ implies $r \to k$. Therefore, $k \to r$ and $r \to k$, which contradicts step 1. If $k \equiv r$, then $r \to t$ implies $k \to t$.

• Step 3: f has a hierarchy of dictators. By steps 1 and 2, the binary relation $\to$ defines the linear order $(i_1, i_2, \ldots, i_n, \ldots)$ on $\mathbb{N}$ such that $i_1 \to i_2 \to \ldots \to i_n \to \ldots$ It must be shown that $(i_1, i_2, \ldots, i_n, \ldots)$ is a hierarchy of dictators in f; that is, for all $x_I \in X$, $f(x_I) = x_{i_t}$, where $t = min\{r \in \mathbb{N}: i_r \in I\}$. Case 1: I has one member. The result follows from

UNA. Case 2: I has two members. The result follows from Lemmas 3.3 and 3.4 and the definition of $\to$. Case 3: I has at least three members. Choose $n \geq 3$ and, by cases 1 and 2, suppose the result true for every society with at most $n - 1$ members. Choose society I with n members and $x^I \in X$. Let $t = min\{r \in \mathbb{N}: i_r \in I\}$. The proof amounts to showing that $f(x_I) = x_{i_t}$. If all the components of x_I are the same, then, by UNA, $f(x_I) = x_{i_t}$. If two components x_k and x_r are different, then, by RED, for some $\alpha \in \{k, r\}$, $f(x_I) = f(x_{I\setminus\{k,r\}}, f(x_{\{k,r\}})^\alpha)$. If $t \notin \{k, r\}$, then, by the induction hypothesis, $f(x_{\{k,r\}})^\alpha = x_{i_t}$. If $t = \alpha$, then, by the induction hypothesis, $f(x_{I\setminus\{k,r\}}, f(x_{\{k,r\}})^\alpha) = f(x_{\{k,r\}}) = x_{i_t}$. If $t \neq \alpha$, then let $s = min\{r \in \mathbb{N}: i_r \in I\setminus\{i_t\}\}$. If $\alpha = i_s$, then, by the induction hypothesis, $f(x_{I\setminus\{k,r\}}, f(x_{\{k,r\}})^\alpha) = f(x_{\{k,r\}}) = x_{i_t}$. If $\alpha \neq i_s$, then, by the induction hypothesis, $f(x_{I\setminus\{k,r\}}, f(x_{\{k,r\}})^\alpha) = x_{i_s}$. The proof is complete if $x_{i_s} = x_{i_t}$. If $x_{i_s} \neq x_{i_t}$, then by RED, for some $i \in \{i_s, i_t\}$, $f(x_I) = f(x_{I\setminus\{i_t,i_s\}}, f(x_{\{i_t,i_s\}})^i)$. By the induction hypothesis, $f(x_{I\setminus\{k,r\}}, f(x_{\{i_t,i_s\}})^i) = f(x_{\{i_t,i_s\}})$. And by the induction hypothesis as well, $f(x_{\{i_t,i_s\}}) = x_{i_t}$.■

To summarize, let $f : X \to \{-1, 0, 1\}$ satisfy UNA, RED, SUB, EXC, PAR_2, and RES_3. Choose $i \in \mathbb{N}$ and $j \in \mathbb{N}\setminus\{i\}$. If $f(1^i, -1^j) = 0$, then, by Lemma 3.2, f is the majority rule. If $f(1^i, -1^j) \neq 0$, then, by Lemma 3.5, f has a hierarchy of dictators. Now, suppose that f is required to satisfy UNA, RED, SUB, EXC, PAR_2, and RES_3. Then: (i) if f is also required to satisfy IND, then the majority rule is obtained (Proposition 3.6); and (ii) if f is required not to satisfy IND, then a hierarchy of dictators emerges (Proposition 3.7). Hence, in the context defined by UNA, RED, SUB, EXC, PAR_2, and RES_3, the difference between the majority rule and the hierarchically dictatorial rule can be reduced to choosing to concede to the indifference the same status given to the strict preference.

Proposition 3.6. A social welfare function $f : X \to \{-1, 0, 1\}$ satisfies UNA, RED, SUB, EXC, PAR_2, RES_3, and IND if and only if f is the majority rule.

Proof. "$\Leftarrow$" It should not be difficult to verify that f satisfies UNA, RED, SUB, EXC, PAR_2, RES_3, and IND when f is the majority rule. "$\Rightarrow$" Let f satisfy UNA, RED, SUB, EXC, PAR_2, RES_3, and IND. Choose $i \in \mathbb{N}$ and $j \in \mathbb{N}\setminus\{i\}$. Case 1: $f(1^i, -1^j) \neq 0$. By Lemma 3.5, f has a hierarchy of dictators. Therefore, for each society I, $\pi_0{}^I = \pi_1{}^I = \pi_{-1}{}^I$, which contradicts IND. Case 2: $f(1^i, -1^j) = 0$. By Lemma 3.2, f is the majority rule.■

Proposition 3.7. A social welfare function $f : X \to \{-1, 0, 1\}$ satisfies UNA, RED, SUB, EXC, PAR_2, RES_3, and IND* if and only if f has a hierarchy of dictators.

Proof. "⇐" It should not be difficult to verify that f satisfies UNA, RED, SUB, EXC, PAR_2, RES_3, and IND* when f has a hierarchy of dictators. "⇒" Let f satisfy UNA, RED, SUB, EXC, PAR_2, RES_3, and IND*. Choose $i \in \mathbb{N}$ and $j \in \mathbb{N}\backslash\{i\}$. Case 1: $f(1^i, -1^j) = 0$. By Lemma 3.2, f is the majority rule. Choose any society I having at least 3 members. It is not difficult to verify that $\pi_1^I = \pi_{-1}^I > ⅓$, so $\pi_0^I < ⅓$. This contradicts IND*. Case 2: $f(1^i, -1^j) \neq 0$. By Lemma 3.5, f has a hierarchy of dictators.■

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