Allocation of objects with conditional property rights

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Abstract
This paper considers the allocation of indivisible goods among members of a collective assuming that individuals are given the right to retain certain goods when some individuals alter the allocation problem. The assignment of rights is exhaustive in that, for every good *x*, either individual *i* can exercise a right over *x* against *j* or *j* against *i*. It is shown that the only Pareto efficient allocation rules satisfying these requirements are those having a hierarchy of diarchies.

Keywords: Allocation rule, hierarchy of diarchies, indivisible good, property right.

JEL Classification: D61, D70

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1. Introduction

Shapley and Scarf (1974) consider an allocation model, the Shapley-Scarf housing market, consisting of traders owning heterogeneous indivisible goods (houses) to offer in trade. This models has nice properties when preferences over the goods are strict. For instance, there is always a unique competitive equilibrium allocation, which can be identified using the top trading cycle algorithm. In addition, the strict core coincides with the set of competitive equilibrium allocations. See Kamijo and Kawasaki (2009, pp. 3-5) for a summary of basic results of this model and Pápai (2007) for a generalization of the model that allows individuals to be endowed with a set of goods.

The same problem of allocating heterogeneous indivisible goods among individuals has been studied under the assumption that the individuals do not initially own the goods. The recurrent result in this literature is that the allocation can be understood as dictated by some hierarchical structure that establishes priorities to obtain the goods. Pápai (2000) characterizes a general class of hierarchical structures, of which the hierarchy of diarchies obtained by Ehlers (2002) and Ehlers and Klaus (2003), and the hierarchy of dictators obtained by Svensson (1999) and Ergin (2000), are particular cases.

This paper follows a mixed approach: though individuals are assumed not to possess full property rights over the goods, they are attributed some sort of property rights over them. These rights are not absolute, as in the Shapley-Scarf housing market, because they can only be exercised against some individuals and only in certain circumstances. Specifically, for individual i and good x, a set I_i^x defines the set of individuals before whom i can exercise the right to retain x when one of those individuals changes his preference over the objects. In the Shapley-Scarf housing model, owning x means that I_i^x consists of the rest of individuals. In the present case, I_i^x may be any subset of the rest of individuals, even the empty set.

In an attempt to make the concession of those rights general, it is assumed as well that the diffusion of such rights is maximized: for every good and two individuals i and j, one can tell whether it is i who has the right over j or vice versa. This assumption is also consistent with the view that, in a society, any type of right must be clearly defined: there should not be uncertainty as to who is entitled to exercise a right. In this respect, the definition of the sets I_i^x could be an expression of the collective will of constraining, by virtue of equity considerations, the way an unrestrained economic mechanism would dictate the allocation of objects. The sets I_i^x would capture the idea that the welfare of certain individuals has to be preserved in certain circumstances.

As an illustration, consider the case in which the custody of a child has to be assigned to someone. Imagine that the mother is granted the custody in any situation in which the father does not ask for it. Then, in general, no matter the opinion of third parties, it seems reasonable for the mother to retain the custody of the child. This concedes her a right over the custody with respect to any individual different from the father. But it is conceivable that, if the father asks for the custody, it may no longer be granted to the mother. In that case, the mother does not have a right over the custody with respect to the father.

The third assumption made is a standard Pareto efficiency requirement: the resulting allocation cannot make it possible to increase the welfare of some individual without reducing the welfare of any other individual. It turns out that the only allocation rules satisfying Pareto efficiency and the complete specification of conditional property rights are those having a hierarchy of diarchies, a hierarchy in which, at each level, there are either one or two individuals having priority over the individuals below in the hierarchy.

2. Definitions

Two finite sets will be assumed given throughout: a set I whose $n \geq 2$ members represent names for individuals and a set X whose $m \geq n$ members designate objects or anything that can be assigned to the individuals.

Definition 2.1. A (strict) preference on X is sequence $(x_1, \dots, x_m)$ of members of X such that $X = \{x_1, \dots, x_m\}$. The set of preferences on X is denoted by L and the set of preference profiles (the set of ways of assigning preferences to individuals) by L^n.

The sequence $(x_1, \dots, x_m)$ representing a preference is interpreted in the sense that x_s is preferred to x_t if and only if $s < t$. In this interpretation, no object is indifferent to another object. For preference p on X and $r \in \{1, \dots, m\}$, ${}^{r}p$ designates the rth element in the sequence p, so ${}^{r}p$ is the rth most preferred object according to p.

Definition 2.2. An allocation over X is a mapping $\alpha : I \to X$ such that, for all $i \in I$ and $j \in I \setminus \{i\}$, $\alpha(i) \neq \alpha(j)$. The set of allocations over X is denoted by A.

An allocation is a way of assigning an object to each individual with no object being assigned to two different individuals. If $m > n$, an allocation leaves $m - n$ objects unassigned.

Definition 2.3. An allocation rule is a mapping $f : L^n \to A$ associating an allocation $f(P)$ over X with every profile P of preferences on X.

For preference profile $P \in L^n$ and individual $i \in I$, P_i designates the preference corresponding to i in P and, for allocation rule f, $f_i(P) \in X$ designates the object that is assigned to individual i in allocation $f(P)$.

Definition 2.4. A diarchy$_1$ in allocation rule f consists of a subset $D_1 \subseteq I$ of the set of individuals, having either one or two members, such that: (i) if $D_1 = \{i\}$, then, for all $P \in L^n$, $f_i(P) = {}^1P_i$; and (ii) if $D_1 = \{i, j\}$, then there is a partition $\{X_i, X_j\}$ of the set of objects X such that $X_i \neq \emptyset \neq X_j$ and, for all $P \in L^n$:

(a) if ${}^1P_i \neq {}^1P_j$, then $f_i(P) = {}^1P_i$ and $f_j(P) = {}^1P_j$;
(b) if ${}^1P_i = {}^1P_j \in X_i$, then $f_i(P) = {}^1P_i$ and $f_j(P) = {}^2P_j$; and
(c) if ${}^1P_i = {}^1P_j \in X_j$, then $f_i(P) = {}^2P_i$ and $f_j(P) = {}^1P_j$.

A diarchy$_1$ is a set of individuals, having either one or two members, satisfying the following. If the diarchy$_1$ has one member i, then i is always assigned his most preferred object. If the diarchy$_1$ has two members i and j, then both are assigned their most preferred objects when those objects are different and, in case of conflict, a partition $\{X_i, X_j\}$ of the set X determines who has priority: when the object that both like most is in X_i, it is i who receives the object, with j getting his second most preferred object; when it is in X_j, the object is assigned to j and i obtains his second most preferred object.

Given an allocation rule f, subset of individuals $D \subseteq I$ and preference profile $P \in L^n$, define the set $F(D, P) = \{x \in X\text{: for all } i \in D, f_i(P) \neq x\}$ to be the set of objects not assigned to members of D under P and f. Hence, $F(D, P)$ is the set of objects free to be allocated among the members of $I\backslash D$, given preference profile P and allocation rule f. For preference p on X and $Y \subseteq X$, $p|_Y$ is the restriction of preference p to the subset Y of objects.

Definition 2.5. For $r \geq 2$, a diarchy$_r$ in allocation rule f consists of a sequence $(D_1, \ldots , D_r)$ of subsets of I such that, letting $D = D_1 \cup \ldots \cup D_{r-1}$: (i) $D_1 \cup \ldots \cup D_r$ has at most n members; (ii) $(D_1, \ldots , D_{r-1})$ is a diarchy$_{r-1}$; (iii) D_r has either one or two members; (iv) if $D_r = \{i\}$, then, for all $P \in L^n$, $f_i(P) = {}^1P_i|_{F(D,P)}$; and (v) if $D_1 = \{i, j\}$, then there is a partition $\{X_i, X_j\}$ of the set of objects X such that $X_i \neq \emptyset \neq X_j$ and, for all $P \in L^n$:

(a) if ${}^1P_i|_{F(D,P)} \neq {}^1P_j|_{F(D,P)}$, then $f_i(P) = {}^1P_i|_{F(D,P)}$ and $f_j(P) = {}^1P_j|_{F(D,P)}$;

(b) if ${}^1P_i|_{F(D,P)} = {}^1P_j|_{F(D,P)} \in X_i$, then $f_i(P) = {}^1P_i|_{F(D,P)}$ and $f_j(P) = {}^2P_j|_{F(D,P)}$; and
(c) if ${}^1P_i|_{F(D,P)} = {}^1P_j|_{F(D,P)} \in X_j$, then $f_i(P) = {}^2P_i|_{F(D,P)}$ and $f_j(P) = {}^1P_j|_{F(D,P)}$.

A diarchy$_r$ is a sequence of r subsets of individuals such that: (i) each subset has one or two members; (ii) members of a subset coming before members of other subsets have priority over objects not yet assigned; and (iii) in subsets with two members, each member has priority over the other member for some subset of objects.

Definition 2.6. An allocation rule f has a hierarchy of diarchies if there is a diarchy$_r$ $(D_1, \ldots, D_r)$ in f such that $D_1 \cup \ldots \cup D_r = I$.

3. Assumptions

PAR. *Pareto efficiency*. For all $P \in L^n$, $i \in I$ and $\alpha \in A$, if i prefers $\alpha(i)$ to $f_i(P)$ according to P_i, then there is $j \in I$ who prefers $f_j(P)$ to $\alpha(j)$ according to P_j.

PAR requires that, for every preference profile P, the corresponding allocation $f(P)$ must be such that, to make some individual better off, some other must be worse off.

CPR. *Conditional property rights over objects*. For every $i \in I$ and $x \in X$ there is $I_i^x \subseteq I\backslash\{i\}$ such that, for all $P \in L^n$, $j \in I_i^x$ and $Q_j \in L$, if $f_i(P) = x$, then it is not the case that i prefers x to $f_i(P_{-j}, Q_j)$ according to P_i.

CPR is a condition attributing individuals a non-absolute property right over objects. Specifically, for every individual i and object x, a subset $I_i^x \subseteq I\backslash\{i\}$ is defined that satisfies the following: if i is assigned x when preferences are P, then, when some member j of I_i^x changes his preference in P, i is entitled to receive an object at least as preferred as x. This suggests that i is allowed to retain x when any member of I_i^x modifies his preference. The set I_i^x identifies those individuals over which i can exercise the right to retain x. This right is not absolute: CPR does not assert that i could secure x for himself in case that i is not assigned x. Instead, it asserts that, once i receives x, he can protect himself against a possible welfare loss caused by a preference change of some individuals: those in I_i^x.

NLL. *No legal loophole*. If f satisfies CPR, then, for all $i \in I$, $j \in I\backslash\{i\}$ and $x \in X$, either $i \in I_j^x$ or $j \in I_i^x$.

NLL complements CPR by making the attribution of property rights exhaustive: for every object x and for all individuals i and j, either i has the right to retain x over j or j has the right over i. NLL seems reasonable when individuals want property rights to be clearly defined in order to avoid uncertainty as to who has a right over a given object. In this sense, NLL expresses a condition of legal security: all conditional property rights have been clearly defined and assigned.

4. Result

For $i \in I$, $j \in I\backslash\{i\}$ and $x \in X$, define the binary relation $\rightarrow_x$ on I as follows: $i \rightarrow_x j$ if and only if $i \in I_j^x$. Hence, $i \rightarrow_x j$ can be interpreted as stating that, when j receives x, he has the right to retain x despite a change in i's preference. Informally, if $i \rightarrow_x j$, then i has "less power" than j when trying to keep object x.

For $P \in L^n$, $Q \in L^n$ and non-empty $J \subset I$, both (P_{-J}, Q_J) and (Q_J, P_{-J}) designate the member R of L^n such that: (i) for all $i \in J$, $R_i = Q_i$; and (ii) for all $i \in I\backslash J$, $R_i = P_i$. Similarly, $(P_{-(J\cup\{i\})}, Q_J, S_i)$ designates the member R of L^n such that: (i) $R_i = S_i$; (ii) for all $j \in J$, $R_j = Q_j$; and (iii) for all $j \in I\backslash(J \cup \{i\})$, $R_j = P_j$. In both cases, when $J = \{j\}$, "j" is written instead of "$\{j\}$".

Lemma 4.1. PAR and CPR imply that, for all $x \in X$, $\rightarrow_x$ is transitive.

Proof. Choose $x \in X$. There is nothing to prove if I has two members, because NLL implies that $\rightarrow_x$ is asymmetric. Hence, assume that I at least three members, so $m \geq 3$. Suppose $k \rightarrow_x j \rightarrow_x i \rightarrow_x k$. The proof amounts to reaching a contradiction. Let $y \in X\backslash\{x\}$, $z \in X\backslash\{x, y\}$ and $X\backslash\{x, y, z\} = \{x_4, \dots , x_m\}$. With $I\backslash\{i, j, k\} = \{i_4, \dots , i_n\}$, choose any $P \in L^n$ such that: (i) ${}^1P_i = z$; (ii) ${}^1P_j = y$; (iii) ${}^1P_k = x$; (iv) for all $r \in \{i, j, k\}$ and $s \in \{1, 2, 3\}$, ${}^sP_r \in \{x, y, z\}$; and (v) for all $r \in \{4, \dots , n\}$, ${}^1P_{i_r} = x_r$. By PAR, $f_k(P) = x$. Let $Q_i \in L$ be obtained from P_i by moving x to the first position, so ${}^1Q_i = x$. As $i \rightarrow_x k$, by CPR, $f_k(P_{-i}, Q_i) = x$.

Let $Q_j \in L$ be obtained from P_j by moving x to the first position, so ${}^1Q_j = x$. With $R = (P_{-\{i, j\}}, Q_{\{i, j\}})$, by PAR, $x \in \{f_i(R), f_j(R), f_k(R)\}$. Case 1: $f_i(R) = x$. Since $j \rightarrow_x i$, by CPR, $f_i(P_{-\{i, j\}}, Q_{\{i, j\}}) = x$ implies $f_i(P_{-i}, Q_i) = x$, contradicting $f_k(P_{-i}, Q_i) = x$. Case 2: $f_j(R) = x$. With $S_k = P_i$, by PAR, $f_i(P_{-\{i, k\}}, Q_i, S_k) = x$. As $j \rightarrow_x i$, by CPR, $f_i(P_{-\{i, k\}}, Q_i, S_k) = x$ implies $f_i(P_{-\{i, j, k\}}, Q_{\{i, j\}}, S_k) = x$. Given $k \rightarrow_x j$, by CPR, $f_j(R) = x$ implies $f_j(R_{-k}, S_k) = x$. But $(P_{-\{i, j, k\}}, Q_{\{i, j\}}, S_k) = (R_{-k}, S_k)$: contradiction. Case 3: $f_k(R) = x$. Let $T_k = P_j$. By PAR,

$f_j(P_{-\{j,k\}}, Q_j, T_k) = x$. Since $k \to_x j$, by CPR, $f_j(P_{-\{j,k\}}, Q_j, T_k) = x$ yields $f_j(P_{-j}, Q_j) = x$. As $i \to_x k$, by CPR, $f_k(R) = x$ implies $f_k(R_{-i}, P_i) = x$. But $(P_{-j}, Q_j) = (R_{-i}, P_i)$: contradiction.■

Remark 4.2. PAR, CPR and NLL imply that, for all $x \in X$, $\to_x$ is a linear order: by definition, $\to_x$ is irreflexive; NLL implies that $\to_x$ is asymmetric; and, by Lemma 4.1, $\to_x$ is transitive.

Lemma 4.3. PAR, CPR and NLL imply that, for all $x \in X$, $i \in I$, $j \in I\backslash\{i\}$ and $k \in I\backslash\{i, j\}$, if $i \to_x j \to_x k$, then, for all $y \in X\backslash\{x\}$, $i \to_y k$.

Proof. Suppose $i \to_x j \to_x k$. By NLL, either $k \to_y i$ or $i \to_y k$. It suffices to reach a contradiction from $k \to_y i$. Let $z \in X\backslash\{x, y\}$, $X\backslash\{x, y, z\} = \{x_4, \ldots, x_m\}$ and $I\backslash\{i, j, k\} = \{i_4, \ldots, i_n\}$. Choose any $P \in L^n$ such that: (i) ${}^1P_i = y$, ${}^2P_i = x$ and ${}^3P_i = z$; (ii) ${}^1P_j = x$ and ${}^2P_j = z$; (iii) $P_k = P_i$; and (iv) for all $r \in \{4, \ldots, n\}$, ${}^1P_{i_r} = x_r$. Let $R_k \in L$ satisfy ${}^1R_k = z$. By PAR, $f_i(P_{-k}, R_k) = y$. Since $k \to_y i$, by CPR, $f_i(P_{-k}, R_k) = y$ implies $f_i(P) = y$. Let $R_i \in L$ satisfy ${}^1R_i = z$. By PAR, $f_j(P_{-i}, R_i) = x$. Since $i \to_x j$, by CPR, $f_i(P_{-i}, R_i) = x$ implies $f_j(P) = x$. By PAR, it follows from $f_i(P) = y$ and $f_j(P) = x$ that $f_k(P) = z$. To contradict $f_k(P) = z$, let $R_j \in L$ satisfy ${}^1R_j = z$ and ${}^2R_j = x$. Define $Q_k = P_j$. By PAR, $f_i(P_{-\{j,k\}}, R_j, Q_k) = y$. As $k \to_y i$, by CPR, $f_i(P_{-\{j,k\}}, R_j, Q_k) = y$ implies $f_i(P_{-j}, R_j) = y$. Given this, by PAR, $f_k(P_{-j}, R_j) = x$. As $j \to_x k$, by CPR, $f_k(P_{-j}, R_j) = x$ implies $f_k(P) \in \{x, y\}$: contradiction.■

Lemma 4.4. PAR, CPR and NLL imply that, for all $x \in X$, $y \in X\backslash\{x\}$, $i \in I$ and $j \in I\backslash\{i\}$, if $i \in I_j^y$ and $j \in I_i^x$, then $I_i^x\backslash\{j\} = I_j^x\backslash\{i\} = I_i^y\backslash\{j\} = I_j^y\backslash\{i\}$.

Proof. Step 1: if $i \to_y j$, then, for all $k \in I\backslash\{i, j\}$, $k \to_x i$ implies $k \to_x j$. Suppose not: $i \to_y j$ and $k \to_x i$ but not $k \to_x j$. By NLL, $j \to_x k$. Therefore, $j \to_x k \to_x i$. By Lemma 4.3, $j \to_y i$. As a result, $j \to_y i$ and $i \to_y j$, contradicting NLL. Step 2: if $j \to_x i \to_y j$, then $I_i^x\backslash\{j\} = I_j^x\backslash\{i\} = I_i^y\backslash\{j\} = I_j^y\backslash\{i\}$. By step 1, $i \to_y j$ implies $I_i^x\backslash\{j\} \subseteq I_j^x\backslash\{i\}$ and $j \to_x i$ implies $I_j^y\backslash\{i\} \subseteq I_i^y\backslash\{j\}$. By Lemma 4.1, $\to_x$ is transitive and, consequently, $j \to_x i$ implies $I_j^x \subseteq I_i^x$. Similarly, as $\to_y$ is transitive, $i \to_y j$ implies $I_i^y \subseteq I_j^y$.■

For $i \in I$ and $x \in X$, define i to be a dictator for x if, for all $j \in I\backslash\{i\}$, $j \to_x i$.

Lemma 4.5. If allocation rule f satisfies PAR, CPR and NLL, then f has a diarchy$_1$.

Proof. Choose $x \in X$. By Remark 4.2, $\to_x$ is a linear order. Therefore, some $i \in I$ is a dictator for x. Case 1: there are $y \in X$ and $j \in I\backslash\{i\}$ such that j is a dictator for y. By Lemma 4.3, there cannot be a third dictator. Accordingly, there is a partition $\{X_i, X_j\}$ of

X such that: (i) $X_i \neq \emptyset$ and, for all $z \in X_i$, i is a dictator for z; and (ii) $X_j \neq \emptyset$ and, for all $z \in X_j$, j is a dictator for z. Choose $P \in L^n$. Case 1a: ${}^1P_i \neq {}^1P_j$. It must be shown that $f_i(P) = {}^1P_i$ and $f_j(P) = {}^1P_j$. Let $v = {}^1P_i$ and $w = {}^1P_j$. Since both i and j are dictators for some member of X, by Lemma 4.4,

$$\text{for all } k \in I\backslash\{i, j\},\ k \in I_i^v \text{ and } k \in I_j^w. \tag{1}$$

With $X\backslash\{v, w\} = \{x_3, \ldots, x_m\}$ and $I\backslash\{i, j\} = \{i_3, \ldots, i_n\}$, choose any $Q \in L^n$ such that, for all $r \in \{3, \ldots, n\}$, ${}^1Q_{i_r} = x_r$. By PAR, $f_i(P_{\{i, j\}}, Q_{-\{i, j\}}) = v$ and $f_j(P_{\{i, j\}}, Q_{-\{i, j\}}) = w$. Given (1), it follows from $f_i(P_{\{i, j\}}, Q_{-\{i, j\}}) = v$ and CPR that $v = f_i(P_{\{i, j, i_1\}}, Q_{-\{i, j, i_1\}}) = f_i(P_{\{i, j, i_1, i_2\}}, Q_{-\{i, j, i_1, i_2\}}) = \ldots = f_i(P_{-i_n}, Q_{i_n}) = f_i(P)$. Similarly, given (1), $f_j(P_{\{i, j\}}, Q_{-\{i, j\}}) = w$ and CPR imply $w = f_j(P_{\{i, j, i_1\}}, Q_{-\{i, j, i_1\}}) = f_j(P_{\{i, j, i_1, i_2\}}, Q_{-\{i, j, i_1, i_2\}}) = \ldots = f_j(P_{-i_n}, Q_{i_n}) = f_j(P)$.

Case 1b: ${}^1P_i = {}^1P_j$. Without loss of generality, suppose that ${}^1P_i \in X_i$ (the proof of the case ${}^1P_j \in X_j$ is obtained by permuting i and j). Let $v = {}^1P_i$ and $w = {}^2P_j$. With $X\backslash\{v, w\} = \{x_3, \ldots, x_m\}$ and $I\backslash\{i, j\} = \{i_3, \ldots, i_n\}$, choose any $Q \in L^n$ such that ${}^1Q_j = w$ and, for all $r \in \{3, \ldots, n\}$, ${}^1Q_{i_r} = x_r$. By PAR, $f_i(P_i, Q_{-i}) = v$ and $f_j(P_i, Q_{-i}) = w$. As both i and j are dictators for some member of X, by Lemma 4.4, (1) holds. By (1) and CPR, $f_i(P_i, Q_{-i}) = v$ implies $f_i(P_{\{i, j\}}, Q_{-\{i, j\}}) = v$. Given this, by PAR, $f_j(P_{\{i, j\}}, Q_{-\{i, j\}}) = w$. It then follows from $f_i(P_{\{i, j\}}, Q_{-\{i, j\}}) = v$, (1) and CPR that $v = f_i(P_{\{i, j, i_1\}}, Q_{-\{i, j, i_1\}}) = f_i(P_{\{i, j, i_1, i_2\}}, Q_{-\{i, j, i_1, i_2\}}) = \ldots = f_i(P_{-i_n}, Q_{i_n}) = f_i(P)$. Similarly, it follows from $f_j(P_{\{i, j\}}, Q_{-\{i, j\}}) = w$, (1) and CPR that $w = f_j(P_{\{i, j, i_1\}}, Q_{-\{i, j, i_1\}}) = f_j(P_{\{i, j, i_1, i_2\}}, Q_{-\{i, j, i_1, i_2\}}) = \ldots = f_j(P_{-i_n}, Q_{i_n}) = f_j(P)$.

Case 2: there is no $y \in X$ and no $j \in I\backslash\{i\}$ such that j is a dictator for y. As a consequence, for all $y \in X$, i is a dictator for y. It must be shown that $f_i(P) = {}^1P_i$. Letting $v = {}^1P_i$, $X\backslash\{v\} = \{x_2, \ldots, x_m\}$ and $I\backslash\{i\} = \{i_2, \ldots, i_n\}$, choose any $Q \in L^n$ such that, for all $r \in \{2, \ldots, n\}$, ${}^1Q_{i_r} = x_r$. By PAR, $f_i(P_i, Q_{-i}) = v$. As i is a dictator for all the members of X, it follows that, for all $k \in I\backslash\{i\}$, $k \in I_i^v$. Given this and CPR, $f_i(P_i, Q_{-i}) = v$ implies $v = f_i(P_{\{i, i_1\}}, Q_{-\{i, i_1\}}) = f_i(P_{\{i, i_1, i_2\}}, Q_{-\{i, i_1, i_2\}}) = \ldots = f_i(P_{-i_n}, Q_{i_n}) = f_i(P)$.■

Proposition 4.6. An allocation rule f satisfies PAR, CPR and NLL if and only if f has a hierarchy of diarchies.

Proof. "⇒" Taking Lemma 4.5 as the base case of an induction argument, suppose that $(D_1, \ldots, D_r)$ is a diarchy$_r$ in f, with $D = D_1 \cup \ldots \cup D_r$ having $d < n$ members. It must be shown that, for some D_{r+1}, $(D_1, \ldots, D_r, D_{r+1})$ is a diarchy$_{r+1}$ in f. If $d = n - 1$, then D_{r+1} consists of the remaining individual. If $d < n - 1$, then the proof is analogous to that of Lemma 4.5. For $i \in I\backslash D$ and $x \in X$, define i to be a dictator$_D$ for x (a dictator for x

conditional on D) if, for all $j \in I\backslash(D \cup \{i\})$, $j \rightarrow_x i$. Choose $x \in X$. By Remark 4.2, $\rightarrow_x$ is a linear order. Therefore, some $i \in I$ is a dictator$_D$ for x.

Case 1: there are $y \in X$ and $j \in I\backslash\{i\}$ such that j is a dictator$_D$ for y. By Lemma 4.3, there cannot be a third dictator$_D$. Accordingly, there is a partition $\{X_i, X_j\}$ of X such that: (i) $X_i \neq \varnothing$ and, for all $z \in X_i$, i is a dictator$_D$ for z; and (ii) $X_j \neq \varnothing$ and, for all $z \in X_j$, j is a dictator$_D$ for z. Choose $P \in L^n$. Let $Y \subset X$ be the set of $m - d$ objects not received by the members of D in $f(P)$; that is, $Y = \{z \in X$: for all $k \in D$, $z \neq f_k(P)\}$.

Case 1a: ${}^1P_i|_Y \neq {}^1P_j|_Y$. It must be shown that $f_i(P) = {}^1P_i|_Y$ and $f_j(P) = {}^1P_j|_Y$. Let $v = {}^1P_i|_Y$ and $w = {}^1P_j|_Y$. Since each individual in $\{i, j\}$ is a dictator$_D$ for some member of X, by Lemma 4.4,

$$\text{for all } k \in I\backslash(D \cup \{i, j\}),\ k \in I_i^v \text{ and } k \in I_j^w. \qquad (2)$$

With $Y\backslash\{v, w\} = \{x_{d+3}, \ldots, x_m\}$ and $I\backslash(D \cup \{i, j\}) = \{i_{d+3}, \ldots, i_n\}$, choose, for all $r \in \{d + 3, \ldots, n\}$, any $Q_{i_r} \in L$ such that ${}^1Q_{i_r} = x_r$. Define $J = D \cup \{i, j\}$. By the induction hypothesis, for all $k \in D$, $f_k(P_J, Q_{-J}) = f_k(P)$. Given this, by PAR, $f_i(P_J, Q_{-J}) = v$ and $f_j(P_J, Q_{-J}) = w$. By (2), it follows from $f_i(P_J, Q_{-J}) = v$, CPR and the induction hypothesis that $v = f_i(P_{J\cup\{i_{d+3}\}}, Q_{-(J\cup\{i_{d+3}\})}) = f_i(P_{J\cup\{i_{d+3},\ d+4\}}, Q_{-(J\cup\{i_{d+3},\ d+4\})}) = \ldots = f_i(P_{-i_n}, Q_{i_n}) = f_i(P)$. Similarly, by (2), it follows from $f_j(P_{\{i, j\}}, Q_{-\{i, j\}}) = w$, CPR and the induction hypothesis that $w = f_j(P_{J\cup\{i_{d+3}\}}, Q_{-(J\cup\{i_{d+3}\})}) = f_j(P_{J\cup\{i_{d+3}, d+4\}}, Q_{-(J\cup\{i_{d+3}, d+4\})}) = \ldots = f_j(P_{-i_n}, Q_{i_n}) = f_j(P)$.

Case 1b: ${}^1P_i|_Y = {}^1P_j|_Y$. Without loss of generality, suppose that ${}^1P_i|_Y \in X_i$ (the proof of the case ${}^1P_j|_Y \in X_j$ is obtained by permuting i and j). Let $v = {}^1P_i|_Y$ and $w = {}^2P_j|_Y$.

With $Y\backslash\{v, w\} = \{x_{d+3}, \ldots, x_m\}$ and $I\backslash(D \cup \{i, j\}) = \{i_{d+3}, \ldots, i_n\}$, choose any $Q \in L^n$ such that ${}^1Q_j|_Y = w$ and, for all $r \in \{d + 3, \ldots, n\}$, ${}^1Q_{i_r} = x_r$. Define $J = D \cup \{i, j\}$ and $H = D \cup \{i\}$. By the induction hypothesis, for all $k \in D$, $f_k(P_H, Q_{-H}) = f_k(P)$. Given this, by PAR, $f_i(P_H, Q_{-H}) = v$ and $f_j(P_H, Q_{-H}) = w$. Since each individual in $\{i, j\}$ is a dictator$_D$ for some member of X, by Lemma 4.4, (2) holds. By (2), CPR and the induction hypothesis, $f_i(P_H, Q_{-H}) = v$ implies $f_i(P_J, Q_{-J}) = v$. In view of this, by PAR, $f_j(P_H, Q_{-H}) = w$. Given that $f_i(P_J, Q_{-J}) = v$, by (2), CPR and the induction hypothesis, $v = f_i(P_{J\cup\{i_{d+3}\}}, Q_{-(J\cup\{i_{d+3}\})}) = f_i(P_{J\cup\{i_{d+3},\ i_{d+4}\}}, Q_{-(J\cup\{i_{d+3},\ i_{d+4}\})}) = \ldots = f_i(P_{-i_n}, Q_{i_n}) = f_i(P)$. In a similar vein, it follows from $f_j(P_J, Q_{-J}) = w$, (2), CPR and the induction hypothesis that $w = f_j(P_{J\cup\{i_{d+3}\}}, Q_{-(J\cup\{i_{d+3}\})}) = f_j(P_{J\cup\{i_{d+3}, i_{d+4}\}}, Q_{-(J\cup\{i_{d+3}, i_{d+4}\})}) = \ldots = f_j(P_{-i_n}, Q_{i_n}) = f_j(P)$.

Case 2: for no $y \in X$ and no $j \in I\backslash\{i\}$, j is a dictator$_D$ for y. As a result, for all $y \in X$, i is a dictator$_D$ for y. It must be shown that $f_i(P) = {}^1P_i|_Y$. Letting $v = {}^1P_i|_Y$, $Y\backslash\{v\} = \{x_{d+2}, \dots, x_m\}$ and $I\backslash(D \cup \{i\}) = \{i_{d+2}, \dots, i_n\}$, choose any $Q \in L^n$ such that, for all $r \in \{d + 2, \dots, n\}$, ${}^1Q_{i_r} = x_r$. Let $H = D \cup \{i\}$. By the induction hypothesis, for all $k \in D$, $f_k(P_H, Q_{-H}) = f_k(P)$. Given this, by PAR, $f_i(P_H, Q_{-H}) = v$. Being i a dictator$_D$ for each member of X, for all $k \in I\backslash H$, $k \in I_i^v$. Given this, CPR and the induction hypothesis, $f_i(P_H, Q_{-H}) = v$ yields $v = f_i(P_{H\cup\{i_{d+2}\}}, Q_{-(H\cup\{i_{d+2}\})}) = f_i(P_{H\cup\{i_{d+2}, i_{d+3}\}}, Q_{-(H\cup\{i_{d+2}, i_{d+3}\})}) = \dots = f_i(P_{-i_n}, Q_{i_n}) = f_i(P)$.

"$\Leftarrow$" Let f have a hierarchy of diarchies $(D_1, \dots, D_r)$. PAR holds because, interpreting that the allocation proceeds sequentially according to the hierarchy of diarchies, some individual always receives his most preferred object among those not yet allocated. With respect to CPR and NLL, both hold with sets I_i^x defined as follows. First, for all $x \in X$ and $i \in D_r$, $I_i^x = \varnothing$. Second, for all $s \in \{1, \dots, r - 1\}$, if D_s has only one member i, then, for all $x \in X$, $I_i^x = D_{s+1} \cup \dots \cup D_r$. And third, for all $x \in X$ and $s \in \{1, \dots, r - 1\}$, if D_s has two members, i and j, and i has priority over j when x is allocated, then $I_j^x = D_{s+1} \cup \dots \cup D_r$ and $I_i^x = \{j\} \cup I_j^x$.■

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