From stability to growth in neoclassical multisector models

Mario, Pomini

Department of Economics, University of Padua, Italy

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By Mario Pomini
Department of Economics and Management
University of Padua (Italy)
e-mail: mario.pomini@unipd.it

Abstract

Analysis of the multisector models was an important strand of inquiry within neoclassic growth theory from the early 1960s and at the end of the decade the multisector approach constituted one of the most promising areas of inquiry within growth theory as a whole. Studies in this area dwindled away at the end of the 1970s but the situation abruptly changed with the advent of endogenous growth theory in the second half of the 1980s which with Lucas (1988) and Romer (1990) was from the outset framed in a multisectorial perspective. The multisector approach was resumed in the literature on endogenous growth, but with features different from those that had previously characterized it. The aim of this paper is to analyze the evolution of some particular aspects of the neoclassical multisector approach from the first studies of the 1960s until current theorization.

Keywords: multisector economic growth, neoclassical growth models

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1. Introduction

Analysis of the multisector models was an important strand of inquiry within neoclassic growth theory from the early 1960s onwards, and it had two main purposes. From a theoretical point of view, it sought to determine whether the properties of the Solow model – in particular, the stability of the equilibrium path – were maintained also in an economy with several sectors and many capital goods. Secondly, the neoclassical authors endeavoured to give greater realism to growth theory by abandoning the simplistic hypothesis of a single-sector economy. At the end of the 1960s, the multisector approach constituted one of the most promising areas of inquiry within growth theory as a whole, both in the neoclassical and other strands of research (Wan, 1971). From this perspective it is no coincidence that half of one of the principal handbooks of the topic, *Mathematical Theories of Economic Growth* by Burmeister and Dobell (1970), was devoted to the study of the dynamic multisector models, and an entire chapter (chapter 9) to neoclassical multisector models without joint production. The other frontier of research at that time was the transition from descriptive models of economic growth to optimal ones (Britto 1974). It was on these two themes – the need for a multisector vision of growth and its microeconomic foundations using dynamic optimization models – that neoclassical theory made its most outstanding contributions.

The extension of Solow's model to the case with several capital goods yielded notable results both in analysis and interpretation, clarifying the role of prices also in a dynamic economy. Studies in this area dwindled away at the end of the 1970s, partly because important results had already been achieved, but mainly because there was a general decline of interest in growth theory as such. In those years, macroeconomic inquiry shifted decisively to the study of economic fluctuations and therefore to short-period macroeconomics (Snowdon and Vane, 2006). The situation abruptly changed with the advent of endogenous growth theory in the second half of the 1980s which with Lucas (1988) and Romer (1990) was from the outset framed in a multisectorial perspective. The multisector approach was resumed in the literature on endogenous growth, but with features different from those that had previously characterized it.

The aim of this paper is to analyze the evolution of some particular aspects of the neoclassical multisector approach from the first studies of the 1960s until current theorization. As often happens, also in economics theoretical changes have been determined by the choice of the problems to address. During the 1960s the main concern was to consider the dynamic stability of a market economy with heterogeneous capital goods. During the 1980s the central problem was the entirely different one of identifying the mechanisms which sustain economic growth in the long period as well. This change of perspective will help us understand the different roles performed by the multisectorial approach first in the exogenous theory of economic growth and then in the endogenous theory.

The paper is structured as follows. The second section considers the pioneering work of U. Uzawa, whose studies were of fundamental importance for the development of neoclassical growth theory in its entirety. The third section considers the problem addressed by F. Hahn (1966), namely the instability of neoclassical multisector models in the version proposed by K. Shell and J. Stiglitz (1967). The fourth and fifth sections are devoted to the problem of stability in models with optimal growth, starting from the contribution of M. Kurz (1968). The sixth and seventh sections discuss the multisector perspective in endogenous growth theory mainly by considering so-called neo-Schumpeterian thought (Romer 1990, Grossman and Helpman 1991, Aghion and Howitt 1992, 2009). The eighth section makes some concluding remarks.

2. H. Uzawa and the problem of stability in two-sector models

Neoclassical literature in this field was launched by Uzawa, who produced a notable series of studies (1961, 1963, 1964) which extended the Solow model to the simplest case: that of an economy composed of two sectors. Uzawa considered both the descriptive and optimizing cases, thereby opening the way for the use of optimal control in growth models (Cass, 1965). Aside from its specific results, Uzawa's approach is important because it furnished a general frame of reference for subsequent analyses. Uzawa concentrated on the problem of the stability of the equilibrium solution obtained, and this aspect would become the predominant problem in subsequent neoclassical literature as well.

Uzawa's model is commonly defined neoclassical with reference to the structure of the technology, in that both the sector producing the consumption good and the one producing the capital good use neoclassical production functions. However, to make the model analytically tractable, Uzawa introduced a hypothesis, thereafter widely adopted in the subsequent literature, on the behaviour of economic agents which instead pertained to the classical school. From this point of view, his model can also be regarded as classical. Uzawa assumes that the social structure of the model is such that workers spend all their income on purchasing consumption goods, while capitalists spend it on purchasing capital goods. Therefore, as in the classical tradition, the accumulation of new capital goods originates from the savings of capitalists. Formally:

$$Y_1 = wL \qquad\qquad Y_2 = rK \tag{1}$$

where [1] represent the outputs of the two sectors, the former specialized in production of the consumption good Y_1, the latter in production of the investment good, Y_2. With these two hypotheses – the one concerning behaviour, the other technology – Uzawa was able to show that the dynamics of the economy were governed by the pattern of a single magnitude, namely the ratio between the wage and the interest rate. Given the characteristics of the production functions of the two sectors, this ratio becomes:

$$\omega = \frac{f_i(k_i)}{f_i^{'}(k_i)} - k_i \qquad\qquad i = 1,2 \tag{2}$$

This is a monotonic relation where only one value of per-capita capital, k_i, can be associated with each value of ω. In particular, also dependent on ω is the equation that governs accumulation of the capital good. Uzawa shows that, given a certain initial endowment of capital, the system tends monotonically to the stationary state $\bar{k}_i$, provided that the consumption goods sector has greater capital intensity than the sector producing the capital goods. The centrality of this rather peculiar assumption was reiterated in subsequent versions of the model, where saving was no longer constant (Uzawa 1963) or an intertemporal utility function was introduced (Uzawa 1964).

The importance of Uzawa's model, which was reprised and developed by other authors (Takayama 1963, Inada 1963) in the early 1960s, resided in the fact that it opened the way for the multisector dynamic by indicating possible strategies of inquiry, while also pointing out potential problems. Uzawa's findings were ambiguous. On the one hand, Solow's results were confirmed in a two-sector context; on the other, the capital-intensity condition on the model's sectors greatly restricted its interpretative capacity. In turn, this hypothesis was connected to a key feature of the model. In fact, even if there were two sectors, only one capital good was produced. This made the dynamic more tractable because it eliminated any problem of portfolio choice and the savings of capitalists could only be invested in the sole existing capital good. To remedy this limitation it would be necessary to introduce not only two sectors but also two distinctive capital goods with their prices. This would make it possible to overcome the unrealistic hypothesis concerning the capital composition of the two sectors. But at the same time, introducing the prices of the goods would raise issues with a crucial bearing on the problem of stability.

3. F. Hahn and the problem of stability on multisector descriptive models

For a multisector model to be genuine, even in the neoclassical domain, it must comprise a plurality of capital goods. This extension was made by Hahn (1966) in a study which engendered a large body of literature in that it reached the rather problematic conclusion that, in a model with a variety of capital goods, prices usually have a destabilizing effect. The price evolution over time does not lead the economic system to stationary state equilibrium, as in the case of the one-sector model, but away from it. This result had considerable implications because it cast doubt on the capacity of markets to self-regulate themselves on the basis of price movements. Not surprisingly, therefore, subsequent theorists sought to disprove this result or to mitigate its disruptive consequences.

The structure of Hahn's model was entirely similar to that of Uzawa: workers spent all their income on consumption whereas saving only derived from capitalists. The novel feature was that there was now a plurality of sectors, and above all of capital goods, which therefore raised the further problem of the distribution of saving between them. The condition that the economy must be in equilibrium in every period entails that the rate of return on each capital good must be the same. This rate of return can be decomposed into two parts: one derives from its marginal product, the other from the gains or losses on capital asset. Consequently, in equilibrium an arbitrage condition of the following type holds (in regard to two capital goods):

$$\frac{r_1}{p_1} + \frac{\dot{p}_1}{p_1} = \frac{r_2}{p_2} + \frac{\dot{p}_2}{p_2} \qquad [3]$$

which is also a portfolio equilibrium condition where it is assumed that the rate of return is uniform in the two productive sectors.

Hahn provided examples to show that when this equation was added, the system's dynamics was unstable owing to the presence of the second term, i.e. the gain/ loss on capital asset, which does not depend on the quantity of accumulated factor but on the psychology of economic agents. It may happen, for example, that even if a capital good has low marginal productivity, there is high demand for that good owing to prospects for future gain on capital asset. In this case, the price and the quantity are driven in the same direction by speculation, and the economy does not tend towards any equilibrium position. Hahn concludes that, by virtue of [3], the economic dynamic is compatible with a wide range of trajectories among which the stable ones constitute an entirely particular case. On this view, the economic system is characterized by strong instability due to the expectations of economic agents.

An answer to the problems raised by Hahn was provided by Shell and Stiglitz already in the following year (1967). The crucial question was the stability of a model with several capital goods. K. Shell and J. Stiglitz anticipated the problem's principal components that M. Kurz (1969) would thereafter clarify through the application of optimum control. They too considered the simplified case of an economy with one consumption good and two capital goods. As in Hahn, and in Uzawa before him, workers consume all their income and accumulation originates from saving by capitalists. The consumption good and the two capital goods are produced with the same production function, $y = k_1^a k_2^b$ with $(1-a-b) > 0$, and given the national income equation, aggregate consumption becomes $c = (1-a-b)y$. Because the two capital goods are produced with the same technology, the maximum profit condition requires the good with the higher price to be produced. In formal terms:

$$\max(p_1, p_2) = p_c = 1 \qquad [4]$$

if p_c is assumed as numeraire.

On these hypotheses, the two accumulation equations of the two capital goods become the following:

$$\begin{aligned} \dot{k}_1 &= \sigma(k_1 f_1 + k_2 f_2) - \delta k_1 \\ \dot{k}_2 &= (1-\sigma)(k_1 f_2 + k_2 f_2) - \delta k_2 \end{aligned} \qquad [5]$$

with σ is a correspondence given by the following expression:

$$\sigma = \begin{cases} = 1 & if \quad p_1 > p_2 \\ \in [0,1] & if \quad p_1 = p_2 \\ = 0 & if \quad p_2 > p_1 \end{cases} \qquad [6]$$

which describes the possible price configurations of the two capital goods.

Let us first determine the stationary state. In the stationary state, $\dot{p}_1 = \dot{p}_2 = 0$, whereby the portfolio equilibrium equation [3] reduces to equality between the marginal productivities,

$$f_1 = f_2 \qquad [7]$$

Taking account of the structure of the production function, [7] defines a path along which the marginal product of each sector is decreasing. In the long period the system tends to the stationary state position with respect the quantities $(\bar{k}_1, \bar{k}_2)$.

The price dynamics is very different because it is regulated by equation [6]. Three dynamic regions can be identified on the basis of this relation. In fact, if $p_1 > p_2$, the economy specializes in the production of the first good; if $p_2 > p_1$ in that of the second good; while in the third case σ is indefinite. On analysing all possible paths of prices and quantities in the three dynamic regions, Shell and Stiglitz again obtain Hahn's results whereby, as a rule, given a generic vector of prices, the system tends to move away from the stationary state position. At the same time, however, the negative conclusions reached by Hahn are radically scaled down. Firstly, even if the equilibrium point identified is unstable, it possesses a particular type of instability because it is a saddle point in the space of prices and quantities. This means that there exists a vector of prices which, once assigned, makes the economic system converge to the stationary state. Secondly, Shell and Stiglitz point out that all the trajectories different from the one leading to the stationary state sooner or later become inefficient from the economic point of view because at a certain point prices or quantities become negative magnitudes. The only remaining problem was how the system might happen to be on the stable path from the outset. To solve it Shell and

Stiglitz resorted to the hypothesis of perfect foresight with regard to all future prices, exactly as in the scheme of static general economic equilibrium.

With Shell and Stiglitz's contribution, the problem of the multisector dynamic began to assume its definitive form. Firstly, they clarified that the stationary point, if it exists, has distinctive characteristics because it is a saddle point in the space of the prices and quantities of capital goods. In economic terms, for some equilibrium position to be achievable, prices and quantities must move in opposite directions. Secondly, on reprising the traditional postulate of perfect foresight, it becomes possible to exclude all erratic trajectories. In this way the economic oscillations provoked by speculation can be related to particular short-period phenomena, while in the long period the critical conclusions reached by Hahn no longer held. Of course, outside the Walrasian world and without perfect forecast anything may happen in relation to the state of the expectations. To be noted is that expectations became an important topic for macroeconomic inquiry during the 1970s.

4. M. Kurz and the problem of stability in optimal growth models

One of the distinctive features of growth theory in the second half of the 1960s was its widespread application of optimal control theory. The initial study by Cass (1965) was followed by the growth of a large body of literature which sought to provide a microeconomic foundation for growth theory through application of dynamic optimization techniques (Burmeister and Dobell 1970). Functional calculus had already made its appearance in economics during the 1930s (Evans, 1930), but it had remained an area of inquiry restricted to a small group of mathematical economists. During the 1960s, also thanks to the simplifications induced by optimal control, functional calculus became the main tool of intertemporal macroeconomic analysis – namely optimal growth theory. In the one-sector case, Cass had shown that the endogenization of consumption choices through the explicit introduction of an intertemporal utility function did not alter the general result of the descriptive model: having assigned the initial stock of capital, it was possible to identify a path that led the system to the stationary state. Extension of the neoclassic model to the multisector case was achieved by M. Kurz (1968), in a study that marked a turning-point in the neoclassic multisector literature, both because of its general formulation and because of its results.

Kurz's strategy was to exploit the significant properties deriving from the application of optimal control theory in economics. His aim was to furnish a more general and possibly definitive solution to Hahn's problem by clearly defining the stability conditions of a dynamic economic system of neoclassical type. Kurz started from the observation that a problem of intertemporal optimum consists in a search for the maximum of a intertemporal utility function $U = U(k, \dot{k})e^{\rho t}$, given the dynamic budget constraint $\dot{k} = f(k) - c$. To solve this problem Kurz introduced an auxiliary function, called Hamiltonian function by analogy with the mechanics in which is often found,

$$H(k,q) = \max\left[(U(k,\dot{k})e^{-\rho t} + q\dot{k}\right] \qquad [8]$$

in which the determinants of the utility function are the stock of capital k and investment $\dot{k}$, while q represents the shadow price of capital valued in terms of utility. Since we are considering a multisector model, all the variables express vector magnitudes. Expression [8] has an economic meaning: it can be interpreted as the current value of the national income valued in terms of utility, on the consumption side through the utility function, and on the investments side through their shadow prices.

On the basis of optimal control theory, the necessary conditions, relatively to the vector of the capital goods k and their shadow price q, to determine the optimum conditions become the following:

$$\begin{aligned} \dot{k} &= \frac{\partial H(k,q)}{\partial q} = H_q(k,q) \\ \dot{q} &= -\frac{\partial H(k,q)}{\partial k} + \rho q = -H_k(k,q) + \rho q \end{aligned} \qquad [9]$$

Immediately evident from system [9] is the symmetrical role performed by the prices and quantities of capital goods; a symmetry which is only disturbed by intertemporal discount factors in the second equation. In his dynamic analysis, Kurz exploited the particular symmetry properties of the system defined by the first-order conditions.

The dynamic analysis first requires determining the system's stationary points [9] and then studying the linearized system around these points. If stationary solutions exist, they must satisfy the following equations:

$$\frac{\partial H(\bar{k},\bar{q})}{\partial q} = 0 \qquad \frac{\partial H(\bar{k},\bar{q})}{\partial k} - \rho\bar{q} = 0 \tag{10}$$

Mathematically, the dynamics of system is determined by the signs of the eigenvalues of the Jacobian matrix of [9], evaluated at the fixed point $(\bar{k},\bar{q})$

(Guckenheimer and Holmes 1983). After the calculation, the matrix is the following:

$$J = \begin{pmatrix} H_{qk} & H_{qq} \\ -H_{kk} & -H^{T}{}_{qk} + \rho I \end{pmatrix} \tag{11}$$

where J is a square matrix and I is the identity matrix.

Written as [11], the Jacobian matrix is difficult to interpret. Kurz first notes that it can be reformulated as follows:

$$F = \begin{pmatrix} H_{qk} - \frac{\rho}{2} I & H_{qq} \\ -H_{kk} & -H^{T}{}_{qk} + \frac{\rho}{2} I \end{pmatrix} \tag{12}$$

When the matrix is written in this form, the marked symmetry of the terms contained in it becomes evident. This has considerable consequences: if the matrix F admits an eigenvalue λ, then it also admits its opposite, $-\lambda$. Taking account of the matrix F, the original matrix J can now be written as follows:

$$J = F + \frac{\rho}{2} I \tag{13}$$

The convenience of the change from the original matrix *J* to matrix *F* is that the eigenvalues of the former, μ, depend on the latter, λ, according to this relation:

$$\mu = \lambda + \frac{\rho}{2} \qquad [14]$$

We may now draw the strands of the argument together. The dynamics of the economic system depends on the sign of the eigenvalues of matrix *J*, μ, which in their turn depend, according to expression [14], on those of the auxiliary matrix *F*, which I symmetric. Inspection of equation [14] shows that, as Kurz first pointed out, only two cases are possible: (a) if $|\lambda| < \rho/2$ both the eigenvalues have positive signs, or (b) they have discordant signs, one positive and the other negative, by virtue of the properties of matrix *F*. This entails that also the dynamic of the economic system has two possible forms. In the first case the system is unstable because the eigenvalues are positive: the system tends to move away from equilibrium. In the second case, the system possesses a saddle-type stability because there are as many positive eigenvalues as negative ones, the former relating to the quantities of capital goods, the latter to their prices. Moreover, this analytical result has a local nature because it shows the dynamic trend only around the stationary point.

Given its analytical depth, this contribution by Kurz can be considered a decisive step towards the construction of neoclassical multisector models, as regards not only growth but also macroeconomic dynamics in general. For now leaving aside the instability corridor determined by the condition $|\lambda| < \rho/2$, the fundamental result is that the equilibrium point, if it exists, can only be a saddle point in the vector space of capital goods and their prices. What Shell and Stiglitz anticipated in their descriptive model, and in some respects Hahn as well, is thus confirmed and extended in natural manner to the case of *n* capital goods. The underlying economic intuition becomes sharply defined: for the system to converge to the equilibrium position, prices and quantities must move in opposite directions, given that as the accumulated factor increases, so the price in terms of utility must diminish. This characterization of the equilibrium point as a saddle point subsequently found wider application not only in growth theory but also in other areas of dynamic macroeconomics, as in the case of rational expectations. Finally to be pointed out is that the problem that the system may assume a non-optimal trajectory does not arise in

Kurz's optimizing model. It can be excluded because in the long period it violates the transversality condition, i.e. of intertemporal consistency in economic behaviour.

Kurz's article deserves special mention for another reason, one of minor nature but equally interesting for comparison between old and new growth theory. When analysing the stability problem, Kurz observed that there may be a case in which the system does not reach a stationary state because the growth rate is constant over time. This position would be the one adopted twenty years later by endogenous growth theory. Kurz was able to specify the features of this constant growth over the long period by introducing the following production function, $f(k) = k^a + bk$, together with the usual intertemporal utility function. Kurz pointed out that no stationary state exists in this case, but rather an optimal growth in which the growth rate of consumption is the following: $\dot{c}/c = (b-\rho)/\gamma$, provided that the system is sufficiently productive, that is, $(b-\rho) > 0$, and provided that the convergence condition of the integral of the utility function is respected. This comment by Kurz highlights that some recently raised problems in reality have roots in the literature of the past. At that time, however, the case of such constant growth was considered of little interest, because it was too closely bound up with particular values of the parameters of technology and preferences.

5. From local to global stability

In his paper Kurz clarified the workings of a dynamic market system in which prices perform a fundamental allocative role. However, this was only a local result: that is, one restricted to the equilibrium solution. Subsequent inquiry shifted to investigation of the conditions that guarantee stability of a global nature and therefore eliminate the instability corridor associated with the discount preference rate. To this end, sophisticated techniques based on some form of the Liapunov function were introduced into economics (Brock and Malliaris, 1989, ch. 4). A major contribution to this strand of analysis was made by Cass and Shell (1976), who were the first to address the problem of global stability by emphasising the analogies between economic and mechanical dynamic systems. They also proposed the definition of Hamiltonian economic system in order to exploit the similarities between economics and physics in dynamic analysis. Although their work was not unique in this area, and probably not the most refined in mathematical terms, it nevertheless offers a clear indication of the potential results and problems.

Cass and Shell's (1976) starting-point was the notion of a conservative dynamical system. This type of dynamic system is often found in mechanical phenomena. For instance, a pendulum which swings without friction is a classic example of a conservative dynamic system. The term 'conservative' derives from the fact that in models of this type, despite the variation of the variables over time, some initial magnitude, in general some kind of energy, is constant: whence derives the expression 'conservative systems'. For this to happen, the dynamic system, expressed by the differential equations of motion, must have an entirely particular structure. Suppose that we are considering a physical system of the type: $H = H(x(t), y(t)) = C$, where C can be conceived as representing a level curve. On executing the total derivative of function H with respect to time, the following expression is obtained:

$$\frac{\partial H}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} \qquad [15]$$

For the system to be conservative it must be the case that $\frac{\partial H}{\partial t} = 0$. It is evident from [15] that this condition is only fulfilled if there is marked symmetry among the variables, in the sense that the following conditions must hold:

$$\frac{\partial H}{\partial x} = -\dot{y} \qquad \frac{\partial H}{\partial y} = \dot{x} \qquad [16]$$

Inserting [16] into [15] yields the result sought. The importance of conservative systems derives from the fact that their dynamics are completely characterized: a conservative system moves around a centre if the eigenvectors are imaginary numbers or the equilibrium points are saddle points, if they are real (Tu 1994). A notable example of a Hamiltonian system in economics is Lotka-Volterra's prey-predator model in which competition has a cyclical trend.

According to Cass and Shell something very similar happens in multisector economic systems owing to the symmetrical roles performed by the quantities of capital goods and by their prices – an aspect first emphasised by Kurz. In effect, if we ignore for a moment the perturbing role of the intertemporal discount rate, the analogy is complete. Reprising equation [8], this expresses the initial magnitude that the system conserves: the value of the national income in terms of utility, $H(k,q) = U(k,\dot{k}) + q\dot{k}$. The total derivative with respect to time is:

$$\frac{\partial H}{dt} = H_k \dot{k} + H_q \dot{q} \qquad [17]$$

Since the first-order conditions are $\dot{k} = H_q$ and $\dot{q} = -H_k$, [17] reduces to:

$$\frac{\partial H}{dt} = H_k H_q - H_q H_k = 0 \qquad [18]$$

That is, the value in terms of utility of the initial national income has not changed over time. The economic system functions as a conservative system which redistributes the initial utility over time through variations in consumption and saving, which is transformed into capital, maintaining the level of initial utility constant. If [18] holds, then also the economy's stationary state solutions are saddle points in the space of capital goods and their prices – as happens in every conservative dynamic system.

However, there is a further complication in economics. This consists in the fact that, along the optimal path, the growth rate of consumption must not be so high that it does not fulfill the transversality condition. From this derives the instability corridor dependent on the intertemporal preference rate. If this rate is nil, the system becomes of conservative type. In order to determine the analytical conditions that make it possible to obtain global stability conditions, Cass and Shell (1976b) used very advanced mathematical methods: in particular, they introduced a specific Liapunov function. Although this enabled them to obtain some analytical results, it had the effect of emphasizing the importance of mathematical formalism even further. Their treatment is now briefly considered.

The Liapunov function (*V*) is, in general, a quadratic form that describes orbits around a stationary state point whose dynamic properties are to be studied. Aside from mathematical details, of interest here is that the function is constructed in such a way that if $\dot{V} > 0$, the trajectory points towards the exterior and the equilibrium point is unstable. If, instead $\dot{V} < 0$, the trajectory points towards the interior and the point is stable. The Liapunov (second) method consists in showing, without explicit solution that the distance between $k(t)$ and its equilibrium $\bar{k}$ shrinks overtime. Cass and Shell introduced a Liapunov function of the following type:

$$V(k) = -(k-\bar{k})(q-\bar{q}) \qquad V(k) > 0 \quad for \quad k \neq \bar{k} \tag{19}$$

where the vector $(\bar{k}, \bar{q})$ represents the stationary state position. Taking account of the intertemporal discount rate, Cass and Shell modifies [19] as follows:

$$V_1 = Ve^{-\rho t} \tag{20}$$

The first derivative of [20] is:

$$\dot{V}_1 = (\dot{V} - \rho V)e^{-\rho t} \tag{21}$$

This equations expressed in the variables of the Hamiltonian function, becomes:

$$\dot{V}_1 = [(q-\bar{q})\frac{\partial H}{\partial q} - (k-\bar{k})\frac{\partial H}{\partial k} + \rho\bar{q}(k-\bar{k}) - \rho(q-\bar{q})(k-\bar{k})]e^{-\rho t} \tag{22}$$

[

From the concavity-convexity of Hamiltonian equation $H(k,q)$ expression [22] is negative and the capital tends to decrease along orbits**.** Cass and Shell demonstrate that in general the system have a global steady state solution if for every $\varepsilon > 0$, there is a δ such that $\left|k-\bar{k}\right| > \varepsilon$ implies

$$(q-\bar{q})\frac{\partial H}{\partial q} - (k-\bar{k})\frac{\partial H}{\partial k} + \rho\bar{q}(k-\bar{k}) > -\rho(q-\bar{q})(k-\bar{k}) + \delta \tag{23}$$

This last expression is the fundamental analytical result of Cass and Shell's approach to the Hamiltonian dynamic system (Cass and Shell, 1976a). It is, however, an expression difficult to interpret and which does not have immediate economic relevance. It only states that the global stability condition depends on the intertemporal discount rate. Apart from this, not a great deal can be said without considering the specific structure of the Hamiltonian function.

Expression [23] warrants two further comments. Firstly, it is automatically satisfied in the standard case in which the Hamiltonian function is convex in prices and concave in the quantities of capital goods. In particular, it implies some form of decreasing returns on the production function, which appears to be a key condition for obtaining both local and global stability. Secondly, as Cass and Shell observe, the fact that the trajectory is stable in the long period tells us nothing about its form, which may be very complicated. This strand of inquiry based on the construction of a Liapunov function was developed by other authors as well, in particular by Brock and Sheinkman (1976, 1977). Aside from analytical formalism, which tends to increase substantially in these cases, also for these authors global stability depends essentially on the structure of the production function, which must be the traditional one with decreasing returns to scale in the individual factors (Varian, 1981)

The introduction of the Liapunov function concluded the period of research on stability in the neoclassical multisector model of growth. Thereafter, only sporadic studies appeared on specific issues, and with strongly mathematical content (Tu, 1994). We may say that, apart from a general explanation of how a dynamic market economy works, this literature on global stability did not produce any real theoretical advances in the understanding of economic phenomena; indeed, the economic dimension was entirely subordinate to the mathematical one. At the beginning of the 1980s, the multisector theory of growth appeared to be an abstract and fragile mathematical construct which contributed little to the understanding of economic reality. Some years later, Shell himself acknowledged that the Hamiltonian approach had not yielded the hoped-for results as a general scheme in which to frame the multisector dynamics (1987).

6. Multisector models in endogenous growth theory

Growth theory resumed in the second half of the 1980s by virtue of the pioneering works of Romer (1986) and Lucas (1988), and it was accompanied by a revival of interest in all multisector aspects of the economic dynamic. Indeed, as we shall seek to show, multisectoriality acquired a role even more important than in the traditional approach. The new literature comprised several models with two or more sectors and, after a first phase, the multisector approach in its various forms became the rule rather than the exception (Jones and Manuelli, 1997).

We have seen in previous sections that the predominant aim of neoclassical analysis from Huzawa onwards was to demonstrate the stability of the model with several capital goods, while long-period growth was explained by exogenous forces such as technical progress or the growth of the working population. Because the technology structure entailed that sooner or later the system would reach a stationary state position, it was important to establish whether this position was also stable. With the new models of endogenous growth, the perspective was reversed, and the principal concern became that of determining the factors able to explain economic growth in the long period. As Kurz had noted in passing in 1969, long-period growth is only possible if in the Euler equation the return on the accumulated factor is always greater than the intertemporal discount rate. On this hypothesis, which entails the absence of decreasing returns, the system's growth has no tendency to diminish. Numerous ways to obtain this outcome were put forward in the literature on endogenous growth, and one of them – which underwent significant development – was that of multisector approach (Barro and Sala-i-Martin, 2004).

To understand the changed role of multisectoriality in the endogenous approach, it is useful to consider the dynamics of a simplified model with only one sector and only two capital goods (k, z). The constraint of the economy's resources is represented by the usual neoclassical production function $y = k^a z^{1-a} = c + \dot{z} + \dot{k}$. The two sectors, one producing the physical good and the other the non-material good, have the same linear production function. On introducing a standard intertemporal utility function $U = c^{1-\gamma}/1-\gamma$, we obtain the following Euler equation from the first-order conditions,

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}[a(\frac{k}{z})^{a-1} - \rho] \qquad [25]$$

on the basis of which the growth rate of consumption depends on the difference between the marginal product of the accumulated factor and the intertemporal preference rate. On the other hand, this productivity is essentially conditioned also by the available quantity of the other factor, so that we may say that the true factor accumulated is (k/z). There thus emerges a new fact which was previously absent, namely the importance of the interaction between the two factors and therefore between the two sectors.

Determining the trend of the growth rate of consumption over time requires that the quantity of factor (k/z) be constant, and that the value of its marginal productivity be greater than the intertemporal discount rate. The study of the pattern of (k/z) requires the use of the arbitrage condition in the use of the two capital goods, whose productivities at the margin must be equal – as previously in Shell and Stiglitz's model (1967). In this case the equality becomes the following:

$$ak^{a-1}z^{1-a} = (1-a)k^{a}z^{-a} \qquad [26]$$

where the right-hand side indicates the marginal product of non-material capital, while the left-hand side indicates the marginal product of physical capital. As a matter of fact, [26] is the same equation [7] considered in the third section. Unsurprisingly, also the result is the same: the two factors must be employed in fixed proportions according to the parameters of the production function,

$$\frac{k}{z} = \frac{a}{1-a} \qquad [27]$$

On inserting [27] into [25], the growth rate of consumption becomes:

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}[a(\frac{a}{1-a})^{a-1} - \rho] \qquad [28]$$

In [28] the growth rate of consumption entirely depends on the model's parameters, and is therefore also constant in the long period. The economy will always grow at this rate unless some variation occurs in the technological or behavioural parameters. To be noted is that profit is a pure technological element in equation [28].

This model, despite its analytical simplicity, clearly shows the role of the heterogeneity of capital goods, and therefore of multisectoriality in endogenous growth theory. The central point is not so much the presence of several capital goods as their interaction – a factor that allows annulment of the law of decreasing returns. To be noted in this regard is that the production function has decreasing returns with respect to the two factors (k, z), but because of [27] the returns become constant with each factor taken individually. Owing to the complementarity of the two capital goods, it is as if the production function comprises only one compound production factor.

In this model, as generally in endogenous growth models, there is no transition dynamic, so that the problem of the growth path's stability – which was so thoroughly analysed in the previous literature – is eliminated by hypothesis. The economic system is on the balanced growth path at every instant, and it moves instantaneously from one equilibrium path to another because of changes in the technological parameters or the utility function. In a certain sense, we may say that endogenous growth has been obtained at the expense of stability analysis. The absence of a transition dynamic provoked severe criticism (first by Solow 1994) because it was considered unlikely and unrealistic that the parameters would assume precisely the configuration necessary to sustain endogenous growth in the long period.

7. General aspects of multisector approach in Schumpeterian endogenous growth theory

Within the broad area of endogenous growth research, the multisectorial approach was developed mainly by the Schumpeterian strand. Models of this type abandoned the hypothesis of perfect competition and directly introduced some form of market power as envisaged by J. Schumpeter. The model which initiated this approach was that of Romer (1990), which was then followed by a large body of studies, outstanding among which were those by Grossman and Helpman (1991) and Aghion and Howitt (1998, 2009). For our purposes here, we may restrict the discussion to Romer's model.

Romer's model is expressly multisector. The economic system is divided into three sectors: the first produces the final good, the second produces the capital goods necessary for the first one, and a third sector is that of research, which produces inventions and patents. The first sector produces the final good using skilled labour, unskilled labour, and a certain number of intermediate capital goods. Because the number of workers does not change over time, the economy can only grow if it increases the number of intermediate capital goods. In its turn, each intermediate capital good is produced by the second sector using a patent made available by the third sector, that of research. Since there exists a fixed cost represented by the price paid to purchase the patent, the structure of this second sector is monopolistic. Each of the firms present in the intermediate goods sector specializes in the production of a single capital good. Using the interest rate as aggregator, and exploiting the properties of the neoclassic production function, Romer is able to show that every firm uses the same quantity of the intermediate capital good $\bar{x}_i$, so that the stock of final capital is given by $K = A_t\bar{x}$, where the term A_t represents the number of patents available to the economy. At this point we can draw some preliminary considerations on how the multisector dimension is introduced by Romer, and in similar models. This in fact is a multisectoriallity constructed *ad hoc* and with some very distinctive characteristics. Firstly, in equilibrium each firm that produces its capital must absorb the same quantity of the intermediate capital good because only in this case is easy aggregation among all firms possible. Secondly, these capital goods all have the same price that arises in relation to the interest rate and the parameters of the production function.

If we stop at this phase, this Schumpeterian model is solely an elegant elaboration on the single sector model, given that all it does is separate the intermediate capital good into its two components: the material one, $\bar{x}$, and the non-material one, A, the number of patents. From this we may conclude that multisectoriality *per se* is unable to sustain endogenous growth. To obtain long-period growth one must look elsewhere, i.e. to the dynamics of the sector producing new technologies. In Romer's model, new patents are obtained using human capital and the existing technology. The stock of technology evolves according to the following law:

$$\dot{A} = \delta H_A A \qquad [29]$$

This last equation has an important characteristic: the accumulation of new patents does not depend on those already produced, but only on the productivity of the research sector and the number of researchers employed in it. In economic terms, the research sector exhibits constant returns to scale, so that the rate of growth of new ideas does not decrease over time. Romer shows that, if [29] holds, the system reaches a state of equilibrium in which all the magnitudes, C, K, Y, grow at the same rate even if the population is constant. In Aghion and Howitt's model, [29] assumes a more complicated form because it comes to depend on spending on research and on the probability of discovering new ideas. But, also in their model what matters is that the expression $\dot{A}/A$, i.e. the creation of patents, is constant in the long period.

We are now able to understand the role of multisectoriality in Schumpeterian models more thoroughly. There is no doubt that the fact of considering a plurality of sectors and capital goods makes models of this type more interesting in interpretative terms. Nevertheless what is important is not the multisectorial aspect, but rather the fact that among the various sectors there is one with constant returns to scale. It is this sector that drives the economic dynamic, so that what we have is a two-sector model to which other sectors have been added to give greater realism to the model. And it could not be otherwise because the central purpose is not to describe the overall dynamic of the economic system, but rather to determine the conditions for endogenous growth indicated by the Euler equation.

8. Concluding remarks

This paper has sought to reconstruct the evolution of the idea of multisectoriality in the neoclassical approach. The multisectorial aspect is of great importance in growth theory, and not surprisingly it has also attracted considerable attention within the neoclassical school. We have seen that this literature can be distinguished into two distinct phases: that of the 1970s, and that of the 1990s. During the first phase, the dominant concern was to determine the stability conditions of the multisector model. In the wake of the problems raised by Solow's one-sector model, also the model with several sectors focused on this issue. Besides mathematical aspects, the fundamental finding was that there must be decreasing returns to scale on the factors if the system is to achieve long-period equilibrium. Only in this case are price signals coherent and go in the direction of stability.

In the new wave of studies on economic growth of the 1990s, the traditional neoclassical perspective was abandoned as incompatible with endogenous growth. The paper has shown that, if long-period growth is to be achieved, the model must comprise some form of constant returns, either on a single factor or on several factors used jointly. In this case, the problem of the transition dynamic from one equilibrium position to another, and therefore of stability, no longer arises because the system is always on the long-period equilibrium trajectory. Whilst it was not the intention of stability theory to furnish well-specified theoretical models, endogenous growth theory suffered from the opposite shortcoming, given that its conclusions depended on very particular values of the parameters of the functions involved.

In some sense, we may say that the endogeneity of growth was obtained in multisector neoclassical theory at the expense of stability. It was precisely the mechanism enabling convergence – the decrease in marginal product – that hindered long-period growth. This change of perspective had a significant effect in that it brought the neoclassical tradition significantly closer to other traditions of inquiry, resolving differences that previously had been substantial. The models proposed by endogenous growth theory appeared more akin to those of Harrodian type (Aghion and Howitt, 2009) or those of the classical tradition (Kurz and Salvadori, 1998), that to those of the traditional neoclassical approach.

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