Double Impact on CVA for CDS: Wrong-Way Risk with Stochastic Recovery

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31 December 2009

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December 31, 2009

Abstract

Current CVA modeling framework has ignored the impact of stochastic recovery rate. Due to the possible negative correlation between default and recovery rate, stochastic recovery rate could have a doubling effect on wrong-way risk. In the case of a payer CDS, when counterparty defaults, the CDS value could be higher due to default contagion while the recovery rate may also be lower if the economy is in a downturn. Using our recently proposed model of correlated stochastic recovery in the default time Gaussian Copula framework, we demonstrate this double impact on wrong-way risk in the CVA calculation for a payer CDS. We also present a new form of Gaussian copula that correlates both default time and recovery rate.

1. Introduction

Counterparty credit risk has been a hot topic. In the recent document of Basel Committee's reform proposal [3], counterparty credit risk is identified as a key area where capital requirement needs to be strengthened. How to value counterparty credit risk in the form of credit valuation adjustment (CVA) is an active research field as of late, see for example [2, 5, 6, 7, 8, 14]. All these papers have tried to capture the wrong-way risk that counterparty defaults when the market value of an over-the-counter (OTC) payer credit default swap (CDS) contract is high, through assumptions of correlation or contagion between defaults. However, one aspect that is missing is that recovery rate is usually not deterministic, but instead is stochastic and could be negatively correlated with default rate, see Altman [1] and references therein. In an economic downturn, default rates are higher than usual and recovery rates are also lower at the same time. This could lead to a doubling effect on CVA in case of wrong-way risk, where counterparty credit quality is negatively correlated with total exposure to the same counterparty. A CVA calculation without the consideration of stochastic recovery could easily underestimate the counterparty credit risk. An obvious example would be the default of Lehman Brothers in the credit crisis. The recovery rate after the Lehman CDS auction was set at 8.625 cents on the dollar for senior unsecured debt. The recovery rate on OTC contracts might be different due to netting and collateral posting, and also depending on the bankruptcy workout process. It is the purpose of this paper to start quantifying this double impact of wrong-way risk.

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One reason that recovery effect was not seriously considered in the previous work is because a consistent stochastic recovery modeling framework was lacking until the recent work of Bennani-Maetz [4] and Li [13]. Although their work has been focused on modeling of CDO senior tranche risk, there is no reason why the framework can not be used in other credit areas to capture the recovery risk. In this paper, we will focus on how stochastic recovery deepens the wrong-way risk on a payer CDS contract. The other area that the recovery modeling might help is downturn LGD in the Basel capital requirement, which is also an interesting topic.

In a previous paper, we discussed a simple way to calculate CVA for CDS on super senior ABS CDO [11]. It turns out that the method was too simplistic in that it totally ignored wrong-way risk and stochastic recovery effect. Armed with our stochastic recovery model, it seems to be the right time to revisit it. We will illustrate the wrong-way risk and recovery effect through an example of a payer CDS deeply in the money with a stressed counterparty.

The paper is organized as follows. In section 2, we will detail our model of stochastic spot recovery model in a default time copula framework and derive the copula function for both default and recovery in the Gaussian case. In section 3, we will setup our model for the bilateral CVA calculation on an OTC payer CDS contract. In section 4, we first give two simple numeric examples. Then we compare the new method with the simplistic method in our previous paper to show the double impact from wrong-way risk and stochastic recovery in the case of a deeply-in-the-money CDS with a stressed counterparty. Section 5 concludes the paper.

2. Stochastic Recovery in the Default Time Copula Framework

In the default time copula framework of D. X. Li [10], the joint distribution of default times is determined by the marginal default time distributions (given by default probability curve) and the default time copula. In the following, we will build a correlated stochastic recovery model in a one factor Gaussian copula setup, following our recent work [13]. It is straight forward to extend the model to multi-factor or non-Gaussian copula cases.

In the Gaussian Copula setup, a latent variable $V_i = \sqrt{\rho_d^i} Z + \sqrt{1-\rho_d^i}\,\varepsilon_i$ drives the default of obligor i of a credit portfolio, where Z and ε_i are independent normal random variables $\sim N(0,1)$ and Z is the systematic factor. The default event $1_{\tau_i \le t}$ can be characterized by $V_i \le v_i = \Phi^{-1}(p_i(t))$, where τ_i is the default time random variable, $p_i(t)$ is the cumulative default probability of the obligor i and $\Phi(x)$ is the standard cumulative normal distribution function. In other words, we can define the default time random variable as

$$\tau_i = p_i^{-1}(\Phi(V_i)) \tag{1}$$

We can assume that stochastic recovery is driven by another latent variable $W_i = \sqrt{\rho_r^i} Z + \sqrt{1-\rho_r^i}\xi_i$ through a time-independent cumulative distribution function $F_R(r)$, where Z, ε_i, ξ_i are independent normal random variables. In a previous paper [12], we specify that stochastic recovery is defined by $R_i = F_R^{-1}(\Phi(W_i))$ conditional on $\tau_i \le t$ or $V_i \le \Phi^{-1}(p_i(t))$. The recovery defined this way is not the spot recovery at default and may lead to arbitrage conditions. To build a consistent stochastic recovery model, we have to start with the spot recovery upon default at an arbitrary time t.

We have

$$\tau_i = t \qquad \leftrightarrow \qquad V_i = \Phi^{-1}(p_i(t)) \tag{2}$$

Conditional on default at time t or $V_i = \Phi^{-1}(p_i(t))$, W_i follows a normal distribution with mean $\sqrt{\rho_d^i \rho_r^i}\Phi^{-1}(p_i(t))$ and standard deviation $\sqrt{1-\rho_d^i \rho_r^i}$. To ensure that $F_R(r)$ is indeed the marginal cumulative distribution for the spot recovery upon default at time t, we define

$$R_i = F_R^{-1}\left(\Phi\left(\frac{W_i - \sqrt{\rho_d^i \rho_r^i}\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_d^i \rho_r^i}}\right)\right) \tag{3}$$

Thus

$$P(R_i \le r \mid \tau_i = t) = P\left(F_R^{-1}\left(\Phi\left(\frac{W_i - \sqrt{\rho_d^i \rho_r^i}\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_d^i \rho_r^i}}\right)\right) \le r \mid \tau_i = t\right) = F_R(r) \tag{4}$$

Conditional on $\tau_i = t$ or $V_i = \Phi^{-1}(p_i(t))$, Z follows a normal distribution with mean $\sqrt{\rho_d^i}\Phi^{-1}(p_i(t))$ and standard deviation $\sqrt{1-\rho_d^i}$, while ξ_i still follows the standard normal distribution. If we fix $Z = z$, then the conditional spot recovery distribution will be

$$\begin{aligned} &P(R_i \le r \mid \tau_i = t, Z = z) \\ &= \Phi\left(\frac{-\sqrt{\rho_r^i} z + \sqrt{1-\rho_d^i \rho_r^i}\Phi^{-1}(F_R(r)) + \sqrt{\rho_d^i \rho_r^i}\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_r^i}}\right) \end{aligned} \tag{5}$$

Conditional on the systematic factor Z, obligor defaults are independent and the conditional default probability for obligor i is given by

$$p_i(t,z) = p(\tau_i \le t \mid Z = z) = \Phi\left(\frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d^i}\, z}{\sqrt{1-\rho_d^i}}\right) \tag{6}$$

Now we can derive the distribution for conditional period recovery rate defined as

$$\begin{aligned} &P(R_i \le r \mid \tau_i \le t, Z = z) \\ &= \frac{1}{p_i(t,z)} \cdot \int_0^t \Phi\left(\frac{-\sqrt{\rho_r^i}\, z + \sqrt{1-\rho_d^i \rho_r^i}\,\Phi^{-1}(F_R(r)) + \sqrt{\rho_d^i \rho_r^i}\,\Phi^{-1}(p_i(s))}{\sqrt{1-\rho_l}}\right) \cdot d\,p_i(s,z) \qquad (7) \\ &= \frac{1}{p_i(t,z)} \cdot \Phi_2\left(\frac{-(1-\rho_d^i)\sqrt{\rho_r^i}\, z + \sqrt{1-\rho_d^i \rho_r^i}\,\Phi^{-1}(F_R(r))}{\sqrt{1-\rho_r^i + \rho_d^i \rho_r^i - {\rho_d^i}^2 \rho_r^i}}, c_i(t,z); -\tilde{\rho}_i\right) \end{aligned}$$

where

$$c_i(t,z) = \frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d^i}\, z}{\sqrt{1-\rho_d^i}} \text{ and } \tilde{\rho}_i = \frac{\sqrt{\rho_d^i \rho_r^i (1-\rho_d^i)}}{\sqrt{1-\rho_r^i + \rho_d^i \rho_r^i - {\rho_d^i}^2 \rho_r^i}}$$

We also have

$$\begin{aligned} &P(1_{\{\tau_i \le t\}} \cdot 1_{\{R_i \le r\}} \mid Z = z) \\ &= E\left[1_{\left\{\varepsilon_i \le \frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d^i} z}{\sqrt{1-\rho_d^i}}\right\}} \cdot \Phi\left(\frac{-\sqrt{\rho_r^i}\, z + \sqrt{1-\rho_d^i \rho_r^i}\,\Phi^{-1}(F_R(r)) + \sqrt{\rho_d^i \rho_r^i}\,\Phi^{-1}(p_i(\tau_i))}{\sqrt{1-\rho_r^i}}\right) \Big| Z = z\right] \\ &= E\left[1_{\left\{\varepsilon_i \le \frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d^i} z}{\sqrt{1-\rho_d^i}}\right\}} \cdot \Phi\left(\frac{-\sqrt{\rho_r^i}\, z + \sqrt{1-\rho_d^i \rho_r^i}\,\Phi^{-1}(F_R(r)) + \sqrt{\rho_d^i \rho_r^i}\,(\sqrt{\rho_d^i}\, z + \sqrt{1-\rho_d^i}\,\varepsilon_i)}{\sqrt{1-\rho_r^i}}\right) \Big| Z = z\right] \\ &= \Phi_2\left(\frac{-(1-\rho_d^i)\sqrt{\rho_r^i}\, z + \sqrt{1-\rho_d^i \rho_r^i}\,\Phi^{-1}(F_R(r))}{\sqrt{1-\rho_r^i + \rho_d^i \rho_r^i - {\rho_d^i}^2 \rho_r^i}}, c_i(t,z); -\tilde{\rho}_i\right) \end{aligned} \tag{8}$$

The unconditional period recovery distribution can be calculated as follows

$$P(R_i \le r \mid \tau_i \le t) = \frac{1}{p_i(t)} \int_{-\infty}^{\infty} P(1_{\tau_i \le t} \cdot 1_{\{R_i \le r\}} \mid Z = z) \cdot \phi(z) dz = F_R(r) \tag{9}$$

So the marginal distribution of period recovery rate is the same as the marginal distribution of spot recovery rate and is time-independent. If $F_R(r)$ has expected recovery of R^{MKT} implied by the single name CDS market, then the model is automatically consistent with the single name CDS market. Note that, in a dynamic model, the spot recovery distribution $F_R(r)$ could be time dependent, although the integration in equation (7) would be more complicated.

Consider two obligors with correlated default and recovery rate, here we derive the copula of default time and recovery rate. Conditional on Z, the default and recovery process are independent for the two obligors, and we have

$$\begin{aligned} & P(1_{\{\tau_1 \le t_1\}} \cdot 1_{\{R_1 \le r_1\}} \cdot 1_{\{\tau_2 \le t_2\}} \cdot 1_{\{R_2 \le r_2\}} \mid Z = z) \\ & = \Phi_2\left(\frac{-(1-\rho_d^1)\sqrt{\rho_r^1}\, z + \sqrt{1-\rho_d^1 \rho_r^1}\, \Phi^{-1}(F_R^1(r_1))}{\sqrt{1-\rho_r^1 + \rho_d^1 \rho_r^1 - {\rho_d^1}^2 \rho_r^1}}, c_1(t,z); -\tilde{\rho}_1 \right) \\ & \quad \cdot \Phi_2\left(\frac{-(1-\rho_d^2)\sqrt{\rho_r^2}\, z + \sqrt{1-\rho_d^2 \rho_r^2}\, \Phi^{-1}(F_R^2(r_2))}{\sqrt{1-\rho_r^2 + \rho_d^2 \rho_r^2 - {\rho_d^2}^2 \rho_r^2}}, c_2(t,z); -\tilde{\rho}_2 \right) \end{aligned} \tag{10}$$

Integrating over z, we will have the copula as

$$\begin{aligned} C(p_1(t_1), F_R^1(r_1); p_2(t_2), F_R^2(r_2)) &= P(\tau_1 \le t_1, R_1 \le r_1; \tau_2 \le t_2, R_2 \le r_2) \\ &= \int_{-\infty}^{+\infty} P(1_{\{\tau_1 \le t_1\}} \cdot 1_{\{R_1 \le r_1\}} \cdot 1_{\{\tau_2 \le t_2\}} \cdot 1_{\{R_2 \le r_2\}} \mid Z = z) \cdot \phi(z) dz \\ &= \Phi_4(\Phi^{-1}(p_1(t_1)), \Phi^{-1}(F_R^1(r_1), \Phi^{-1}(p_2(t_2)), \Phi^{-1}(F_R^2(r_2)); \Sigma_\rho) \end{aligned} \tag{11}$$

where Φ_4 is the 4-variable cumulative normal distribution and the correlation matrix is defined as

$$
\Sigma_\rho = \begin{pmatrix}
1 & 0 & \sqrt{\rho_d^1 \rho_d^2} & \frac{(1-\rho_d^2)\sqrt{\rho_r^2 \rho_d^1}}{\sqrt{1-\rho_d^2 \rho_r^2}} \\
0 & 1 & \frac{(1-\rho_d^1)\sqrt{\rho_r^1 \rho_d^2}}{\sqrt{1-\rho_d^1 \rho_r^1}} & \frac{(1-\rho_d^1)(1-\rho_d^2)\sqrt{\rho_r^1 \rho_r^2}}{\sqrt{(1-\rho_d^1 \rho_r^1)(1-\rho_d^2 \rho_r^2)}} \\
\sqrt{\rho_d^1 \rho_d^2} & \frac{(1-\rho_d^1)\sqrt{\rho_r^1 \rho_d^2}}{\sqrt{1-\rho_d^1 \rho_r^1}} & 1 & 0 \\
\frac{(1-\rho_d^2)\sqrt{\rho_r^2 \rho_d^1}}{\sqrt{1-\rho_d^2 \rho_r^2}} & \frac{(1-\rho_d^1)(1-\rho_d^2)\sqrt{\rho_r^1 \rho_r^2}}{\sqrt{(1-\rho_d^1 \rho_r^1)(1-\rho_d^2 \rho_r^2)}} & 0 & 1
\end{pmatrix}
$$

This can be proven through the following result

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \Phi_2(a_1 z + b_1, c_1 z + d_1; \rho_1) \cdot \Phi_2(a_2 z + b_2, c_2 z + d_2; \rho_2) \cdot \phi(z) dz \\
&= \Phi_4\left(\frac{b_1}{\sqrt{1+a_1{}^2}}, \frac{d_1}{\sqrt{1+c_1{}^2}}, \frac{b_2}{\sqrt{1+a_2{}^2}}, \frac{d_2}{\sqrt{1+c_2{}^2}}; \Sigma \right)
\end{aligned}
\tag{12}
$$

where

$$
\Sigma = \begin{pmatrix}
1 & \frac{\rho_1 + a_1 c_1}{\sqrt{(1+a_1^2)(1+c_1^2)}} & \frac{a_1 a_2}{\sqrt{(1+a_1^2)(1+a_2^2)}} & \frac{a_1 c_2}{\sqrt{(1+a_1^2)(1+c_2^2)}} \\
\frac{\rho_1 + a_1 c_1}{\sqrt{(1+a_1^2)(1+c_1^2)}} & 1 & \frac{c_1 a_2}{\sqrt{(1+c_1^2)(1+a_2^2)}} & \frac{c_1 c_2}{\sqrt{(1+c_1^2)(1+c_2^2)}} \\
\frac{a_1 a_2}{\sqrt{(1+a_1^2)(1+a_2^2)}} & \frac{c_1 a_2}{\sqrt{(1+c_1^2)(1+a_2^2)}} & 1 & \frac{\rho_2 + a_2 c_2}{\sqrt{(1+a_2^2)(1+c_2^2)}} \\
\frac{a_1 c_2}{\sqrt{(1+a_1^2)(1+c_2^2)}} & \frac{c_1 c_2}{\sqrt{(1+c_1^2)(1+c_2^2)}} & \frac{\rho_2 + a_2 c_2}{\sqrt{(1+a_2^2)(1+c_2^2)}} & 1
\end{pmatrix}
$$

If we define

$$
\begin{cases}
X_1 = \sqrt{\rho_1}\varepsilon + \sqrt{1-\rho_1}\varepsilon_1 - a_1 Z \\
Y_1 = \sqrt{\rho_1}\varepsilon + \sqrt{1-\rho_1}\varepsilon_2 - c_1 Z \\
X_2 = \sqrt{\rho_2}\xi + \sqrt{1-\rho_2}\xi_1 - a_2 Z \\
X_1 = \sqrt{\rho_2}\xi + \sqrt{1-\rho_2}\xi_2 - c_2 Z
\end{cases}
$$

where ε, ε_1, ε_2, ξ, ξ_1, ξ_2, Z are independent standard normal random variables, then

$$\int_{-\infty}^{+\infty} \Phi_2(a_1 z + b_1, c_1 z + d_1; \rho_1) \cdot \Phi_2(a_2 z + b_2, c_2 z + d_2; \rho_2) \cdot \phi(z) dz$$
$$= E(X_1 \le b_1, Y_1 \le d_1, X_2 \le b_2, Y_2 \le d_2)$$

which leads to the equation (12).

Equation (11) can be compared with the standard Gaussian copula of default times with fixed recovery

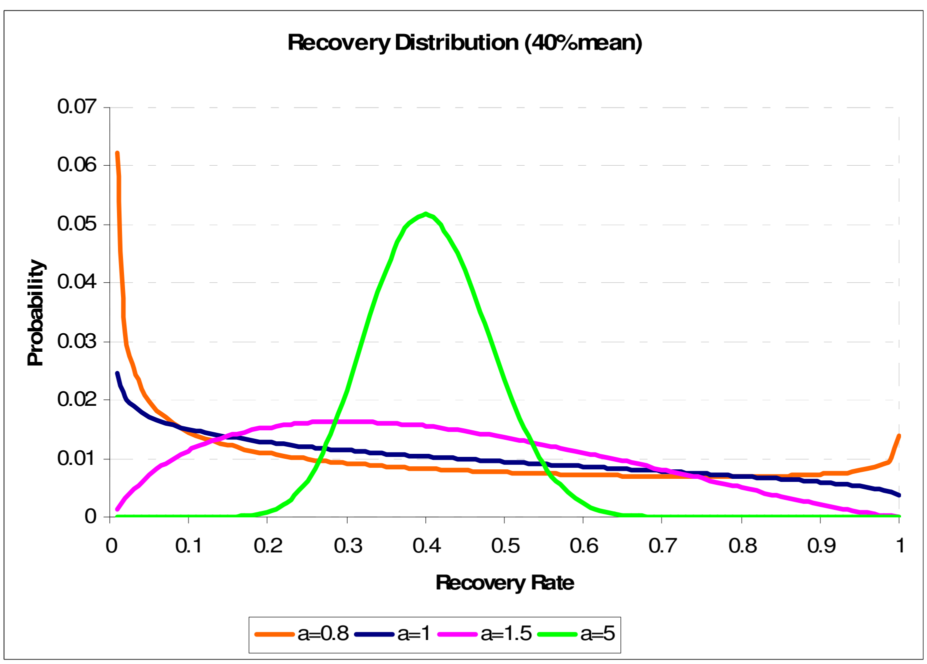
$$C(p_1(t_1), p_2(t_2)) = P(\tau_1 \le t_1, \tau_2 \le t_2) = \Phi_2(\Phi^{-1}(p_1(t_1)), \Phi^{-1}(p_2(t_2)); \sqrt{\rho_d^1 \rho_d^2}) \qquad (13)$$

Note that, in equation (11), default and recovery of an obligor are not correlated, this is because recovery is always conditional on default. The copula for default and recovery is still Gaussian. However, the correlation matrix can not be generated by a simple one-factor model. Equation (11) can be easily extended to more than two obligors, multi-factors and other types of copulas.

For CVA calculation, we need the conditional expected loss for obligor i before time t

$$L_t^i(z) = \int_0^1 (1-r) \cdot d_r P(1_{\{\tau_i \le t\}} \cdot 1_{\{R_i \le r\}} \mid Z = z) = \int_0^1 P(1_{\{\tau_i \le t\}} \cdot 1_{\{R_i \le r\}} \mid Z = z) \cdot dr \qquad (14)$$

For numeric purpose, we consider the recovery distribution discussed in [13], which is similar to the beta distribution as shown in the Figure below.



It has the following form

$$F_R(r) = P(R \le r) = \Phi(a \cdot \Phi^{-1}(r) - \sqrt{1+a^2}\,\Phi^{-1}(r_0)) \tag{15}$$

or, for the density function,

$$f_R(r) = a \cdot \frac{\phi(a \cdot \Phi^{-1}(r) - \sqrt{1+a^2}\,\Phi^{-1}(r_0))}{\phi(\Phi^{-1}(r))} \tag{16}$$

where $a \ge 0$ and $0 \le r_0 \le 1$. This distribution will simplify calculation for Gaussian Copula model. The expected recovery rate is r_0 and the variance of recovery rate is

$$Var(R) = \Phi_2\left(\Phi^{-1}(r_0), \Phi^{-1}(r_0); \frac{1}{1+a^2}\right) - r_0^2 \tag{17}$$

Assume $r_0 = R^{MKT}$. When a goes to zero, the variance goes to the maximum value $R^{MKT}(1-R^{MKT})$, which corresponds to the case where R takes the values 0 or 1. When a goes to infinity, the variance goes to zero and the distribution reduces to a constant recovery R^{MKT}.

The original spot recovery equation (3) can be written as

$$R = \Phi\left(\frac{\sqrt{\rho_r^i}Z + \sqrt{1-\rho_r^i}\,\xi_i - \sqrt{\rho_d^i\rho_r^i}\,\Phi^{-1}(p_i(t))}{a\sqrt{1-\rho_d^i\rho_r^i}} + \sqrt{1+\frac{1}{a^2}}\,\Phi^{-1}(R^{MKT})\right) \tag{18}$$

Then we have

$$\begin{aligned} &P(R_i \le r \mid \tau_i = t, Z = z) \\ &= \Phi\left(\frac{-\sqrt{\rho_r^i}z + \sqrt{1-\rho_d^i\rho_r^i} \cdot (a\Phi^{-1}(r) - \sqrt{1+a^2}\,\Phi^{-1}(R^{MKT})) + \sqrt{\rho_d^i\rho_r^i}\,\Phi^{-1}(p_i(t))}{\sqrt{1-\rho_r^i}}\right) \end{aligned} \tag{19}$$

The expected conditional spot recovery is

$$
\begin{aligned}
r_i(t,z) &= \int_0^1 r \cdot d_r P(R_i \le r \mid \tau_i = t, Z = z) \\
&= \Phi\left(\frac{\sqrt{\rho_r^i}\, z + \sqrt{1-\rho_d^i \rho_r^i} \cdot \sqrt{1+a^2}\, \Phi^{-1}(R^{MKT}) - \sqrt{\rho_d^i \rho_r^i}\, \Phi^{-1}(p_i(t))}{\sqrt{1-\rho_r^i + a^2(1-\rho_d^i \rho_r^i)}} \right)
\end{aligned} \tag{20}
$$

The expected conditional loss up to time t is

$$
L_t^i(z) = \int_0^t (1 - r_i(s,z)) \cdot dp_i(s,z) = \Phi_2(c_i(t,z), b_i(z); -\hat{\rho}_i) \tag{21}
$$

where $c_i(t,z)$ is defined in equation (7) and

$$
b_i(z) = -\frac{(1-\rho_d^i)\sqrt{\rho_r^i}\, z + \sqrt{1-\rho_d^i \rho_r^i}\, \sqrt{1+a^2}\, \Phi^{-1}(R^{MKT})}{\sqrt{1-\rho_r^i + a^2(1-\rho_d^i \rho_r^i) + \rho_d^i \rho_r^i - {\rho_d^i}^2 \rho_r^i}}
$$

$$
\hat{\rho}_i = \frac{\sqrt{\rho_d^i \rho_r^i (1-\rho_d^i)}}{\sqrt{1-\rho_r^i + a^2(1-\rho_d^i \rho_r^i) + \rho_d^i \rho_r^i - {\rho_d^i}^2 \rho_r^i}}
$$

Conditional on Z, the expected recovery rate will be time-dependent through $p_i(t)$.

3. Bilateral Counterparty Risk with Stochastic Recovery

The general bilateral counterparty risk pricing formula without netting or collateralization has been derived in Brigo and Capponi [7] (see equations (2.6) and (2.7) in their paper). We write down the formula for the bilateral CVA at valuation time t here

$$
\begin{aligned}
BR - CVA(t, T, LGD_{0,1,2}) &= E_t\left\{ LGD_2 \cdot 1_{\tau_2 \le \tau_0 \wedge T} \cdot D(t,\tau_2) \cdot [NPV(\tau_2)]^+ \right\} \\
&\quad - E_t\left\{ LGD_0 \cdot 1_{\tau_0 \le \tau_2 \wedge T} \cdot D(t,\tau_0) \cdot [-NPV(\tau_0)]^+ \right\}
\end{aligned} \tag{22}
$$

where the subscripts 0, 1, 2 are for investor, reference credit and counterparty, τ is the default time variable, $D(t,\tau)$ is the deterministic discount factor, $NPV(\tau)$ is the net present value of future (after τ) cashflows of the OTC contract valued at τ not subjected to counterparty risk, the loss given default LGD is one minus the recovery upon default, T is the maturity of the OTC contract. In the case that the OTC contract is a payer credit default swap, if the counterparty defaults first before both the contract expires and the reference name or investor defaults, the value of the contract could be higher to the investor due to the correlation between defaults. Meanwhile, if the economy is in a downturn, the recovery rate could also be lower, which leads to a double impact to the investor. This is the wrong-way risk that is the most interesting part of counterparty risk management.

Brigo and Capponi [7] combine Gaussian copula of default times with a stochastic intensity model to study the impact of default correlation and credit spread volatility on the bilateral CVA. They assume constant LGDs, which ignores the double impact from negatively correlated recovery rates. The present paper will use a simple model to demonstrate the double impact from recovery. We will use the one-factor Gaussian copula model with stochastic recovery described in the previous section where the uncertainty in the systematic factor contains all the randomness in the default probability curve and defaults are independent conditional on the systematic factor. The problem with this model as a dynamic model has been discussed in the literature, see for example the recent paper of Hitier and Huber [9]. In this simple model, we will be able to demonstrate the double impact from correlated defaults and recovery rates. However, we will not be able to study the impact of credit spread volatility, since this is not a true dynamic model. We notice that it is possible to apply our Gaussian copula model of default and recovery to the Brigo-Capponi framework to add stochastic recovery effect in addition to default correlation and spread volatility.

We assume, conditional on the systematic factor Z, the default probability curve is deterministic and takes the form

$$p_i(t,z) = p(\tau_i \le t \mid Z = z) = \Phi\left(\frac{\Phi^{-1}(p_i(t)) - \sqrt{\rho_d^i}\, z}{\sqrt{1-\rho_d^i}} \right)$$

The bilateral CVA can be written as

$$\begin{aligned} BR - CVA(t,T,LGD_{0,1,2}) = {} & E_t\left\{ E\left[LGD_2 \cdot 1_{\tau_2 \le \tau_0 \wedge T} \cdot D(t,\tau_2) \cdot [NPV(\tau_2)]^+ \middle| Z \right] \right\} \\ & - E_t\left\{ E\left[LGD_0 \cdot 1_{\tau_0 \le \tau_2 \wedge T} \cdot D(t,\tau_0) \cdot [-NPV(\tau_0)]^+ \middle| Z \right] \right\} \end{aligned} \qquad (23)$$

Take $t = 0$ and assume the cashflows and default losses happen on discrete time steps $T_0 = t, T_1, \ldots, T_N = T$. We also assume that, if counterparty and the reference credit default in the same time period, CVA loss will be $LGD_2 \cdot LGD_1$. We arrive at the following approximation

$$
\begin{aligned}
&BR-CVA(t,T,LGD_{0,1,2})\\
&=E_t\left\{E\left[\sum_{i=1}^{N} LGD_2\cdot 1_{T_{i-1}<\tau_2\le T_i}1_{\tau_0\ge T_i}\cdot D(T_0,T_i)\cdot[NPV(T_i)]^+\big|Z\right]\right\}\\
&\quad+E_t\left\{E\left[\sum_{i=1}^{N} LGD_2\cdot 1_{T_{i-1}<\tau_2\le T_i}1_{\tau_0\ge T_i}\cdot D(T_0,T_i)\cdot LGD_1\cdot 1_{T_{i-1}<\tau_1\le T_i}\big|Z\right]\right\}\\
&\quad-E_t\left\{E\left[\sum_{i=1}^{N} LGD_0\cdot 1_{T_{i-1}<\tau_0\le T_i}1_{\tau_2\ge T_i}\cdot D(T_0,T_i)\cdot[-NPV(T_i)]^+\big|Z\right]\right\}\\
&=E_t\left\{E\left[\sum_{i=1}^{N}(L^2_{T_i}(Z)-L^2_{T_{i-1}}(Z))\cdot P(\tau_0\ge T_i\mid Z)\cdot D(T_0,T_i)\cdot[NPV(T_i)]^+\big|Z\right]\right\}\\
&\quad+E_t\left\{E\left[\sum_{i=1}^{N}(L^2_{T_i}(Z)-L^2_{T_{i-1}}(Z))\cdot P(\tau_0\ge T_i\mid Z)\cdot D(T_0,T_i)\cdot(L^1_{T_i}(Z)-L^1_{T_{i-1}}(Z))\right]\right\}\\
&\quad-E_t\left\{E\left[\sum_{i=1}^{N}(L^0_{T_i}(Z)-L^0_{T_{i-1}}(Z))\cdot P(\tau_2\ge T_i\mid Z)\cdot D(T_0,T_i)\cdot[-NPV(T_i)]^+\big|Z\right]\right\}
\end{aligned}
\tag{24}
$$

where, for a payer CDS contract,

$$
\begin{aligned}
NPV(T_i)\big|Z &= 1_{\tau_1\ge T_i}E_{T_i}[-S_1\cdot\sum_{j=i+1}^{N} DCF_j\cdot D(T_i,T_j)\cdot P(\tau_1\ge T_j\mid T_i)\\
&\quad+\sum_{j=i+1}^{N} LGD_1\cdot D(T_i,T_j)\cdot(P(\tau_1\ge T_{j-1}\mid T_i)-P(\tau_1\ge T_j\mid T_i))]\big|Z \qquad (25)\\
&=-S_1\cdot\sum_{j=i+1}^{N} DCF_j\cdot D(T_i,T_j)\cdot P(\tau_1\ge T_j\mid Z)+\sum_{j=i+1}^{N} D(T_i,T_j)\cdot(L^1_{T_j}(Z)-L^1_{T_{j-1}}(Z))
\end{aligned}
$$

where S_1 is the fixed premium of the CDS contract and DCF is the day count fraction. For simplicity, we have ignored the accrued premium. Greater accuracy can be achieved through Monte Carlo simulation, using schemes similar to those discussed in [6, 7].

In the numeric calculation, all we need are the conditional survival probability $1-p(T_i,z)$ and conditional expected loss $L_{T_i}(z)$ up to each time point T_i conditional on $Z=z$, which have explicit formula in section 2. The final BR-CVA will be calculated as integration over the Gaussian variable Z.

4. Numerical Results

We consider a five-year payer CDS on a reference name. Since we are more interested in wrong-way risk with stochastic recovery effect, we will assume the protection buyer is almost default-free. Interest rate is assumed to be constant at 4%. We experiment with two spread levels 120 bps and 250 bps applied to either counterparty or reference name. The effects of correlation between defaults and recovery rates and the volatility of recovery rates are considered. The results are presented in Table 1 and Table 2.

In general, when default correlation increases, CVA also increases. Adding correlated stochastic recovery, CVA will increase with the volatility of recovery rate. But the stochastic recovery effect is not as strong as default correlation for wrong-way risk. We notice that the same phenomenon appears here as first discussed in Brigo and Chourdakis [6]. In table 1, when the default correlation is extremely high, CVA drops significantly. This is because reference name almost always default before the counterparty so that the counterparty risk is much smaller.

Next we consider a 5 year CDS contract that is deeply in the money. The deal premium is 5 bps while the current market spread is either 15% or 25% for the reference name or the counterparty. The results are presented in Table 3 and Table 4.

In a previous paper [11], we discussed two simple methods to calculate CVA for a deeply-in-the-money CDS contract on a super senior ABS CDO tranche with a distressed counterparty. The first method is to add the counterparty CDS spread to Libor curve to discount the cashflows, which lacks modeling justification. The second method uses an approximation when the exposure is almost always positive and there is no correlation between counterparty and the underlying credit. We compare the results from these two methods (called method 1 and method 2) with the new method discussed in this paper. The first method always gives a higher CVA than the second method, while the second method matches closely with the new method when there is no correlation. However, with correlation and recovery volatility getting higher, CVA based on the new method could be much higher than both method 1 and method 2. This reflects how wrong-way risk and stochastic recovery affects CVA value. However, when correlation is extremely high and the reference name has worse credit quality than that of counterparty, CVA would drop much lower since default time of the reference name is usually earlier than that of counterparty.

5. Conclusion

In this paper, we apply our model of stochastic spot recovery for Gaussian copula to quantify the wrong-way risk due to correlated default and recovery rate in CVA calculation. We follow the general framework for calculating bilateral CVA discussed in Brigo and Capponi [7], but use our one-factor default time Gaussian copula model with stochastic recovery to describe the future uncertainty in default probabilities and recovery rates. We find that, for a payer CDS contract, CVA normally increases in magnitude with default correlation and volatility of correlated recovery rates. However, in the special case when the reference name has worse credit quality than the counterparty and default correlation is extremely high at the same time, CVA could be much smaller even with high recovery volatility, which confirms the results discussed in Brigo and Chourdakis [6]. The effect of the negative correlation between default and recovery rate does increase the CVA noticeably but is not as strong as the default correlation between counterparty and the reference name. We also revisit a simple method for CVA on a deeply-in-the-money CDS with a stressed counterparty proposed in a previous paper [11]. We find that the simple method does not capture the wrong-way risk due to correlated defaults and lower recovery rates in economic downturn.

Tables

Default correlation ρ_d	no correlation with recovery $\rho_r = 0$	same correlation for recovery $\rho_r = \rho_d$ with **a=200** or **vol=0.07%**	same correlation for recovery $\rho_r = \rho_d$ with **a=1** or **vol=2%**	same correlation for recovery $\rho_r = \rho_d$ with **a=0.01** or **vol=48.77%**
20%	23.17	23.19	34.42	39.75
60%	68.88	69.06	99.32	113.31
90%	92.27	92.34	108.13	111.45
99%	28.23	28.24	28.38	27.31

Table 1. The counterparty CVA in basis points for the case when counterparty break-even 5 year CDS spread is 120 bps and reference break-even 5 year CDS spread is 250 bps, which is also the contract spread. Both have the same recovery distribution with mean at 40%. The parameter **a** determines the volatility of the recovery distribution. Compare with the base case CVA = 3 bps where there is no default correlation.

default correlation ρ_d	no correlation with recovery $\rho_r = 0$	same correlation for recovery $\rho_r = \rho_d$ with **a=200** or **vol=0.07%**	same correlation for recovery $\rho_r = \rho_d$ with **a=1** or **vol=2%**	same correlation for recovery $\rho_r = \rho_d$ with **a=0.01** or **vol=48.77%**
20%	26.13	26.12	38.35	44.21
60%	88.93	89.18	130.69	152.23
90%	200.27	200.55	251.43	280.41
99%	296.33	296.47	316.03	324.68

Table 2. The counterparty CVA in basis points for the case when counterparty break-even 5 year CDS spread is 250 bps and reference break-even 5 year CDS spread is 120 bps, which is also the contract spread. Both have the same recovery distribution with mean at 40%. The parameter **a** determines the volatility of the recovery distribution. Compare with the base case CVA = 3 bps where there is no default correlation.

default correlation ρ_d	no correlation with recovery $\rho_r = 0$	same correlation for recovery $\rho_r = \rho_d$ with **a=200** or **vol=0.07%**	same correlation for recovery $\rho_r = \rho_d$ with **a=1** or **vol=2%**	same correlation for recovery $\rho_r = \rho_d$ with **a=0.01** or **vol=48.77%**
0%	925.36			
20%	936.32	936.70	1015.12	1062.69
60%	933.60	934.27	1050.34	1109.64
90%	755.35	755.80	796.70	793.59
99%	346.88	347.01	353.09	348.90

Table 3. The counterparty CVA in basis points for the case when counterparty break-even 5 year CDS spread is 1500 bps, reference break-even 5 year CDS spread is 2500 bps but contract spread is set at 5 bps. Both have the same recovery distribution with mean at 40%. The parameter **a** determines the volatility of the recovery distribution. The contract value without counterparty default risk is -4800.89 bps for the counterparty. CVA from method 1 is 1060.69 bps, while CVA from method 2 is 925.12 bps.

default correlation ρ_d	no correlation with recovery $\rho_r = 0$	same correlation for recovery $\rho_r = \rho_d$ with **a=200** or **vol=0.07%**	same correlation for recovery $\rho_r = \rho_d$ with **a=1** or **vol=2%**	same correlation for recovery $\rho_r = \rho_d$ with **a=0.01** or **vol=48.77%**
0%	1146.86			
20%	1226.23	1226.74	1336.07	1403.53
60%	1468.02	1468.98	1675.22	1807.90
90%	1940.89	1941.62	2107.35	2220.89
99%	2320.54	2320.82	2364.15	2384.13

Table 4. The counterparty CVA in basis points for the case when counterparty break-even 5 year CDS spread is 2500 bps, reference break-even 5 year CDS spread is 1500 bps but contract spread is set at 5 bps. Both have the same recovery distribution with mean at 40%. The parameter **a** determines the volatility of the recovery distribution. The contract value without counterparty default risk is -3863.56 bps for the counterparty. CVA from method 1 is 1421.91 bps, while CVA from method 2 is 1146.45 bps.

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