A Spot Stochastic Recovery Extension of the Gaussian Copula

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Abstract.[1] The market evolution since the end of 2007 has been characterized by an increase of systemic risk and a high number of defaults. Realized recovery rates have been very dispersed and different from standard assumptions, while 60%-100% super-senior tranches on standard indices have started to trade with significant spread levels.
This has triggered a growing interest for stochastic recovery modelling. This paper presents an extension to the standard Gaussian copula framework that introduces a consistent modelling of stochastic recovery. We choose to model directly the spot recovery, which allows to preserve time consistency, and compare this approach to the standard ones, defined in terms of recovery to maturity. Taking a specific form of the spot recovery function, we show that the model is flexible and tractable, and easy to calibrate to both individual credit spread curves and index tranche markets. Through practical numerical examples, we analyze specific model properties, focusing on default risk.

1. Introduction.

The market evolution since the end of 2007 has been characterized by an increase of systemic risk and a high number of defaults. Default rate has reached record level, with e.g. a rate of around 20% on HY firms in March 2009, a 20 year record.

Realized recovery rates have been in general fairly different from the initial curve marking assumptions, and have covered a very large spectrum of values, from very low for Tribune (1.5%) to very high for FNM (91.51%). The weighted average recovery computed over the last twelve months, on the major CDS auctions, has been around 10%, well below the standard assumptions of 30% to 40% recovery rate.

On the standard index tranche market, the 60%-100% super-senior tranche has started to trade. It has somehow materialized the high level of systemic risk as well as the uncertainty around recovery rate. Quoted prices for this tranche showed a significant switch in value towards the most senior tranches over the period. A direct consequence in terms of modelling has been the failure of standard pricing framework to calibrate the index tranche market. This has been largely due to their embedded assumption of constant recovery.

As an attempt to capture this market evolution, the focus in terms of modelling has been on stochastic recovery. Following early work as in e.g. Andersen and Sidenius [2], different approaches have been proposed and discussed in recent papers, such as Amraoui and Hitier [1], Ech-Chatbi [7], Krekel [10], or Li [11]. They tend to concentrate on the dependency between default and recovery rate, which was previously analyzed empirically in e.g. [8], and on the ability of the model to calibrate to the index tranche market. Tractability and capacity to preserve the pricing and calibration of credit spread curves are also identified as key features.

In this paper, we present an extension of the standard Gaussian copula framework that includes a consistent modelling of stochastic recovery. We start with the definition of the spot recovery, and discuss why this is a suitable underlying for recovery modelling. As most of the recent approaches have been defined in terms of recovery to maturity[2], we analyze the necessary conditions for consistency in this case.

[1] Disclaimer: The views expressed in this paper are the authors' own and may not necessarily reflect those of Barclays Capital.

[2] with some exception, such as Li [11]

Taking a specific representation of the spot recovery, we show that the resulting framework preserves the pricing of credit default swaps and is both flexible and tractable. We focus on time consistency and analyze in details how the introduction of stochastic recovery affects default risk.

The rest of the paper is organized as follows. In section 2, we introduce the main notations and recall briefly the standard Gaussian copula framework. The concepts of spot recovery and recovery to maturity are presented and discussed in details.
Section 3 focuses on the modelling of the spot recovery. Fundamental results are obtained in the general case as well as for the factor model approach. A detailed comparison between modelling spot recovery and recovery to maturity is done with a particular focus on term structure consistency and default risk.
In section 4, we choose a specific form for the spot recovery that enables the overall pricing framework to be tractable and efficient for practical implementation.
Section 5 provides numerical results showing the flexibility of the model for calibration to index tranche markets, and the consistent behaviour of the model for default risk.

2. Notations and Model Setup.

In this section, we introduce the main notations and the standard Gaussian copula set-up used in the rest of the paper.
We will consider a portfolio of N underlying issuers with corresponding market recoveries $\left(R_i^{Mkt}\right)_{i=1,..,N}$, notional weights $(\omega_i)_{i=1,..,N}$ and default time $(\tau_i)_{i=1,..,N}$. The total portfolio notional is set to 1\$, and $\sum_{i=1}^{N} \omega_i = 1$.

2.1. Default Probability, Expected Loss and Spot Recovery Rate.

We start with the definition of the different recovery rates. Then we introduce default probability, cumulated and expected loss.

2.1.1. Spot Recovery Rate.

The spot recovery rate corresponds to the recovery paid at the time default happens.

Definition 2.1. *Spot Recovery Rate*
The spot recovery rate $r(t)$ is defined as the recovery rate paid if default happens at t.

$$r(t) = r(\tau) \,|\tau = t \tag{1}$$

In a factor model, the conditional spot recovery $r(t, X)$ is defined by:

$$r(t, X) = \mathbb{E}\left[r(\tau) \,|\tau = t, X\right] \tag{2}$$

2.1.2. Recovery To Maturity.

The recovery to maturity corresponds to the recovery rate paid if default happens before a specified date.

Definition 2.2. *Recovery To Maturity*
The recovery to maturity $R(t)$ is defined as the recovery rate paid if default happens before t.

$$R(t) = r(\tau) \,|\tau \leq t \tag{3}$$

In a factor model, the conditional recovery to maturity $R(t, X)$ is defined by:

$$R(t, X) = \mathbb{E}\left[r(\tau) \,|\tau \leq t, X\right] \tag{4}$$

Remark 2.3. *It is interesting to note that the spot recovery is directly linked to the default time and corresponds to an instantaneous recovery rate, while the recovery to maturity is a function of the default event, and could be seen as an average recovery rate over the period. The notion of recovery to maturity is commonly used in the literature and is introduced here for comparison purpose only.*

2.1.3. Cumulated and Expected Loss.
Using previous notations, we define the cumulated loss up to time t as

$$L_t = \sum_{i=1}^{N} \omega_i \left(1 - r_i(\tau_i)\right) \mathbf{1}_{\{\tau_i \leq t\}}$$

The default probability for issuer i up to maturity t is noted $p_i(t)$. The corresponding conditional probability is given by:

$$p_i(t, X) = \mathbb{P}\left(\tau_i \leq t \,|X\right) \tag{5}$$

The expected loss $EL_i(t)$ for issuer i up to maturity t is defined as:

$$\begin{aligned} EL_i(t) &= \mathbb{E}\left[\left(1 - r_i(\tau_i)\right) \mathbf{1}_{\{\tau_i \leq t\}}\right] \\ &= \int_0^t \left(1 - \mathbb{E}\left[r_i(s)\right]\right) dp_i(s) \\ &= \left(1 - \mathbb{E}\left[R_i(t)\right]\right) p_i(t) \end{aligned} \tag{6}$$

In a factor model, the conditional expected loss will then be given by:

$$\begin{aligned} EL_i(t, X) &= \mathbb{E}\left[\left(1 - r_i(\tau_i)\right) \mathbf{1}_{\{\tau_i \leq t\}} \,|X\right] \\ &= \int_0^t (1 - r_i(s, X)) dp_i(s, X) \\ &= (1 - R_i(t, X)) p_i(t, X) \end{aligned} \tag{7}$$

From this we can derive directly that:

$$R_i(t, X) = \int_0^t r_i(s, X) \frac{dp_i(s, X)}{\int_0^t dp_i(s, X)}$$

which indicates that the conditional recovery to maturity can be seen as an average of the spot recovery rate weighted by the conditional default probabilities.

2.2. A Standard Gaussian Copula for Default Times.

Here we recall briefly the standard Gaussian copula framework - see e.g. [9] for more details.
To each issuer i and corresponding default time τ_i corresponds a random variable X_i defined by:

$$\mathbb{P}\left(\tau_i \leq t\right) = \mathbb{P}\left(X_i \leq c_i(t)\right)$$

where $c_i(t) = F_{X_i}^{-1}(p_i(t))$.
The latent variable X_i is then constructed as:

$$X_i = \sqrt{\rho} X + \sqrt{1 - \rho} Y_i \tag{8}$$

where X and Y_i are independent, $\mathcal{N}(0, 1)$ random variables. For $i \neq j$, Y_i and Y_j are also independent which implies that the dependency between the different names is driven by the common factor X and the correlation ρ. Note that conditional on X, $(X_i)_{i=1,..,N}$ are independent.

The conditional default probability is given by:

$$p_i(t, X) = \Phi\left(\frac{c_i(t) - \sqrt{\rho} X}{\sqrt{1 - \rho}}\right) \tag{9}$$

2.3. Pricing and Conditional Independence.

2.3.1. Conditional Independence.
The factor model defined for the dependency structure of default times has the very important property of conditional independence. We extend this property to the specification of the recovery rate framework and assume that conditional on the common factor X, default times and recovery rates are independent.

Hypothesis 2.4. *Conditional on the common factor X, default times and recovery rates are independent.*

The pricing framework defined in this paper is essentially an extension of the Gaussian copula to integrate stochastic recovery. As such, it can be applied to the pricing of various portfolio Credit Derivatives, from N^{th}-to-Default to CDO and CDO^2. As we will show in the next sections, this can be achieved using either Monte-Carlo simulation or semi-analytic methods.
As the pricing of CDOs represents one of the main challenges in portfolio Credit Derivatives, the rest of the paper will focus on this product.

2.3.2. Loss Distribution and CDO Pricing.
The expected loss of the $[K-100\%]$ tranche with maturity t is defined by:

$$EL(t,K) = \mathbb{E}\left[(L_t - K)_+\right]$$

From the model specification for the conditional recovery and conditional default probability, this can be computed using standard numerical methods such as recursion or the conditional normal approximation - see [3] and [12] respectively.
The conditional normal approximation is a very efficient and accurate method in the case of stochastic recovery. It consists in assuming that the cumulated loss conditional on the common factor follows a normal distribution, i.e.

$$L_t\,|X \sim \mathcal{N}\left(m(t,X), \sigma(t,X)\right)$$

where $m(t,X)$ and $v(t,X)$ are respectively the conditional mean and variance of the conditional loss with maturity t. Under the conditional independence assumptions these two quantities can be computed directly from

$$\begin{aligned} L_t(X) &= \mathbb{E}\left[L_t\,|X\right] \\ &= \sum_{i=1}^{N} \omega_i EL_i(t,X) \\ L_t^2(X) - L_t(X)^2 &= \mathbb{E}\left[L_t^2\,|X\right] - \mathbb{E}\left[L_t|X\right]^2 \\ &= \sum_i \omega_i^2 \left\{\mathbb{E}\left[(1-r_i(\tau_i))^2 \mathbf{1}_{\{\tau_i \le t\}}|X\right] - EL_i(t,X)^2\right\} \end{aligned}$$

Note that $EL(t,K)$ is then given directly by a closed form:

$$EL(t,K) = \mathbb{E}\left[\sigma(t,X)\varphi\left(\frac{K-m(t,X)}{\sigma(t,X)}\right) - (K-m(t,X))\left(1-\Phi\left(\frac{K-m(t,X)}{\sigma(t,X)}\right)\right)\right]$$

The tranche expected loss can be computed by a direct numerical integration. As soon as the tranche expected loss is known for all maturities, the protection and coupon leg of the corresponding CDO can be derived directly.

3. Spot Recovery Rate Modelling.

3.1. General Results.

In this section, we derive the condition (EL) on the expected loss, which is the only condition for the spot recovery to be valid. We also derive the limiting distribution when the instantaneous default probability p_{0^+} tends to 1. The spot recovery process is supposed to be a general bounded process between 0 and 1.

3.1.1. Expected Loss Condition.

Let l_t^i be the loss at time t of a given issuer i, τ_i its default time and $r_i(\tau)$ its recovery rate. To simplify notations, we will note explicitly use the index i in the rest this section. By definition:

$$l_t = (1 - r_\tau)\mathbf{1}_{\{\tau \leq t\}}$$

with $r \in [0, 1]$. This is the only condition required a priori. Another condition is given by the single name CDS as we want the expected loss to be matched for any maturity:

Definition 3.1. *Expected Loss Condition (EL)*
The general expected loss condition is defined by:

$$\forall t,\ \mathbb{E}[l_t] = (1 - R^{Mkt})p_t \tag{10}$$

If we assume that the default can happen at any point in time with a non-zero density, i.e. that $\mathbb{P}(\tau \in [t, t + dt[) = f_t dt$ with $f > 0$, then:

Proposition 3.2. *The (EL) condition is equivalent to:*

$$\mathbb{E}[r_\tau | \tau = t] = R^{Mkt} \tag{11}$$

Proof.: Considering (10) at t and $t + \epsilon$ for an arbitrary $\epsilon > 0$ we get:

$$\mathbb{E}\left[r_\tau \mathbf{1}_{\{\tau \in [t, t+\epsilon[\}}\right] = R^{\text{Mkt}}(p_{t+\epsilon} - p_t)$$

or equivalently, as $f > 0$:

$$\frac{\int_t^{t+\epsilon} \psi_s f_s ds}{\int_t^{t+\epsilon} f_s ds} = R^{\text{Mkt}}$$

with $\psi(t) = \mathbb{E}[r_\tau | \tau = t]$. Taking the limit when ϵ tends to 0, leads to $\psi_t = R^{\text{Mkt}}$.

■

Note that we have the equivalence:

$$\forall t,\ \mathbb{E}[L_t] = (1 - R^{\text{Mkt}})p_t \Leftrightarrow \forall t,\ \mathbb{E}[r_\tau | \tau = t] = R^{\text{Mkt}}$$

The (EL) condition is a direct translation of the intuitive concept that the expected recovery given default is the market recovery.

3.1.2. Limiting Distribution.

The limiting distribution is trivial to obtain as by definition:

$$\lim_{p(0_+) \to 1} \tau = 0_+ \text{ a.s.}$$

Hence, when the instantaneous default probability tends to 1, the loss process is defined by the spot recovery value at 0^+:

$$\lim_{p(0_+) \to 1} l_t = 1 - r(0_+) \text{ a.s.} \tag{12}$$

The previous result is particularly useful as it shows how to decouple what happens when the name is close to default, which is driven essentially by its recovery rate, from what happens when the name is far from default, which is driven by its spread dynamics. In particular, the continuity on default is directly guaranteed if $r(0_+) = R^{\text{Mkt}}$. By doing this, we guarantee the continuity on default of the specific issuer but also of the whole basket as the loss of the issuer is decoupled from the rest of the basket.

3.2. Factor Model for Spot Stochastic Recovery.

We work with a factor representation and make the following assumption.

Hypothesis 3.3. *In a factor model, the spot recovery depends only on τ and X. This implies that $r(t,x)$ is a deterministic function of t and x.*

Under (3.3.), $r_\tau = r(\tau, X)$, the conditional expected loss and squared loss given default are therefore respectively given by their integral over time:

$$\begin{cases} l_t(X) &= \int_0^t (1 - r(s,X)) dp(s,X) \\ l_t^2(X) &= \int_0^t (1 - r(s,X))^2 dp(s,X) \end{cases}$$

Proposition 3.4. *The (EL) condition reads:*

$$\mathbb{E}\left[r(\tau, X) | \tau = t\right] = \mathbb{E}\left[r\left(t, \sqrt{\rho}\Phi^{-1}(p_t) + \sqrt{1-\rho}X\right)\right] = R^{Mkt}$$

Proof.: The result relies on the conditional distribution of X conditional on the default time τ. One can derive $X|\tau = t \sim \mathcal{N}\left(\sqrt{\rho}\Phi^{-1}(p_t), (1-\rho)\right)$.

■

Let us now assume the following form for the recovery $r_\tau = r\left(\Phi^{-1}(p_\tau), X\right)$. Such a transformation is always possible as long as $dp_t > 0$. Replacing the dependency in t by a dependency in $\Phi^{-1}(p_t)$ leads to an integration over the space domain compared to an integration over time.

Proposition 3.5. *The loss process is equivalently given by:*

$$r_\tau \mathbf{1}_{\{\tau \le t\}} \sim r(\sqrt{\rho}X + \sqrt{1-\rho}Y, X)\ \mathbf{1}_{\left\{Y \le \frac{\Phi^{-1}(p_t) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right\}} \tag{13}$$

Proof. By definition $r_\tau \mathbf{1}_{\{\tau \le t\}} \sim r\left(\Phi^{-1}(p_\tau), X\right) \mathbf{1}_{\left\{\sqrt{\rho}X + \sqrt{1-\rho}Y \le \Phi^{-1}(p_t)\right\}}$.

■

As a consequence, the different moments are simply given by:

$$l_t^k(X) = \mathbb{E}\left[\left(1 - r(\sqrt{\rho}X + \sqrt{1-\rho}Y, X)\right)^k \mathbf{1}_{Y \le \frac{\Phi^{-1}(p_t) - \sqrt{\rho}X}{\sqrt{1-\rho}}} | X\right] \tag{14}$$

In particular the integral over time can be avoided as long as r is such that the previous expectation can be computed in closed form.

Proposition 3.6. *The (EL) condition reads:*

$$\forall \mu \in \mathbb{R}, \quad \mathbb{E}\left[r\left(\mu, \sqrt{\rho}\mu + \sqrt{1-\rho}X\right)\right] = R^{Mkt} \tag{15}$$

Proof. $\forall t$, the (EL) condition reads:

$$\mathbb{E}\left[r\left(\Phi^{-1}(p_t), \sqrt{\rho}\Phi^{-1}(p_t) + \sqrt{1-\rho}X\right)\right] = R^{\text{Mkt}}$$

Changing $\Phi^{-1}(p_t) \to \mu$ ends the proof.

■

In particular, if $r(x,y) = r\left(\frac{y - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)$ then the (EL) condition simplifies to:

$$\mathbb{E}\left[r(X)\right] = R^{\text{Mkt}} \tag{16}$$

The change of variable removes the time dependency and all we are left with is a space constraint.

3.3. Monte Carlo Simulation.

One of the main advantages of the spot recovery formulation is its simplicity in a Monte Carlo environment since both the recovery and the default time can be simulated independently. This is particularly useful for basket default swaps and CSO^2 valuation. The procedure is described hereafter:

- Simulate the common factor X
- Simulate each issuer default time given X, $\tau_i|X$
- Compute the recovery value $r(\tau_i, X)$
- Compute the loss $(1 - r(\tau_i, X))\mathbf{1}_{\tau_i \leq t}$

This procedure can easily be implemented in any standard Monte Carlo pricer as it does not require any extra work to simulate the recovery on default.

4. Spot Recovery vs Recovery to Maturity.

In the literature, stochastic recovery modelling has been tackled from different angles. One approach however, which consists in modelling the recovery to maturity, seems to benefit from the favor of the practitioners. In this section, we discuss the link between spot recovery and recovery to maturity and focus on the specific conditions required for a recovery to maturity approach to be arbitrage-free and numerically efficient.

4.1. Linking Spot Recovery and Recovery to Maturity.

Even when specifying the model in terms of spot recovery, it is possible for the actual implementation to use an equivalent representation with a recovery to maturity. This requires some specific approximations, as detailed in the proposition below.

Proposition 4.1. *The conditional mean can be represented in terms of recovery to maturity:*

$$l_t(X) \simeq (1 - R(t, X))p(t, X)$$

with

$$\begin{cases} R(t,X) &= r(\sqrt{\rho}X + \sqrt{1-\rho}Y(t,X), X) + \frac{1}{2}\sigma^2(t,X)\frac{\partial^2}{\partial x^2} r(\sqrt{\rho}X + \sqrt{1-\rho}Y(t,X), X) \\ c(t,X) &= \frac{\Phi^{-1}(p_t) - \sqrt{\rho}X}{\sqrt{1-\rho}} \\ Y(t,X) &= \frac{\varphi(c(t,X))}{p(t,X)} \\ \sigma^2(t,X) &= 1 - c(t,X)Y(t,X) - Y^2(t,X) \end{cases}$$

This provides an equivalent formulation to the one obtained in the previous section.

Proof. Using (14) with $k = 1$:

$$\begin{aligned} R(t,X)p(t,X) &= \mathbb{E}\left[r(\sqrt{\rho}X + \sqrt{1-\rho}Y, X)\mathbf{1}_{\left\{Y \leq \frac{\Phi^{-1}(p_t) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right\}} | X \right] \\ &= \mathbb{E}\left[r(\sqrt{\rho}X + \sqrt{1-\rho}Y, X) \left| X, Y \leq \frac{\Phi^{-1}(p_t) - \sqrt{\rho}X}{\sqrt{1-\rho}} \right. \right] p(t,X) \end{aligned}$$

The Taylor expansion of $r(x, y)$ around $x = x_0$ is given by:

$$r(x,y) = r(x_0,y) + \partial_x r(x_0,y)(x - x_0) + \frac{1}{2}\partial_x^2 r(x_0,y)(x - x_0)^2 + ...$$

Here we take $x_0 = \sqrt{\rho}X + \sqrt{1-\rho}Y(t,X)$ where $Y(t,X)$ is chosen to cancel the first order term or in other words:

$$Y(t,X) = \mathbb{E}\left[Y \left| X, Y \leq \frac{\Phi^{-1}(p_t) - \sqrt{\rho}X}{\sqrt{1-\rho}} \right. \right]$$

Keeping the first three terms of the Taylor expansion leads to the result.

■

4.2. No-Arbitrage Constraints for Recovery To Maturity.

Spot recovery and recovery to maturity behave differently and have different existence conditions. If the conditions for the spot recovery are easy to derive, the ones for the recovery to maturity are more difficult to obtain.

By definition, expressing the conditional loss in terms of spot recovery and recovery to maturity:

$$\int_0^t r(s,X)dp(s,X) = R(t,X)p(t,X)$$

Or equivalently:

$$r(t,X) = \frac{\partial_t \left(R(t,X)p(t,X)\right)}{\partial_t p(t,X)} = R(t,X) + p(t,X)\frac{\partial_t R(t,X)}{\partial_t p(t,X)} \tag{17}$$

For the recovery to maturity to be valid, the following partial derivative equation must be satisfied:

$$0 \leq R(t,X) + p(t,X)\frac{\partial_t R(t,X)}{\partial_t p(t,X)} \leq 1 \tag{18}$$

This condition is generally not satisfied by current representations of $R(t,X)$. In particular, the recovery to maturity defined in [1]:

$$R(t,X) = 1 - \frac{\widetilde{p}(t,X)}{p(t,X)}$$

with $\widetilde{p}(t,X) = \Phi\left(\frac{\Phi^{-1}(\widetilde{p}_t)-\sqrt{\rho}X}{\sqrt{1-\rho}}\right)$ and $\widetilde{p}_t = (1-R^{\text{Mkt}})p_t$, does not satisfy this condition.

Proof. Starting from (18) we have:

$$\begin{array}{rccc}
 & R(t,X) + p(t,X)\frac{\partial_t R(t,X)}{\partial_t p(t,X)} & \in & [0,1] \\
\Leftrightarrow & \frac{\partial_t \widetilde{p}(t,X)}{\partial_t p(t,X)} & \in & [0,1] \\
\Rightarrow & \partial_t \widetilde{p}(t,X) & \leq & \partial_t p(t,X) \\
\Rightarrow & \left(X - \sqrt{\rho}\Phi^{-1}(p_t)\right)^2 & \leq & \left(X - \sqrt{\rho}\Phi^{-1}(\widetilde{p}_t)\right)^2 \\
\Rightarrow & 0 & \leq & \left(\Phi^{-1}(p_t) - \Phi^{-1}((\widetilde{p}_t)\right)\left(X - \sqrt{\rho}\left(\frac{\Phi^{-1}(p_t)+\Phi^{-1}(\widetilde{p}_t)}{2}\right)\right)
\end{array}$$

As $\widetilde{p}_t < p_t$, as soon as $X < \sqrt{\rho}\left(\frac{\Phi^{-1}(p_t)+\Phi^{-1}(\widetilde{p}_t)}{2}\right)$ the previous equation will not be satisfied.

■

4.3. Recovery to Maturity in Practice.

Modelling the recovery to maturity is essentially motivated in practice by its simplicity and tractability in terms of numerical implementation. This is strongly linked to the following assumption:

Hypothesis 4.2. *In a factor model, the recovery to maturity depends only on $\{\tau \leq t\}$ and X. In particular, the recovery to maturity $R(t,x)$ is a deterministic function of t and x.*

Under (4.2.), the loss conditioned on the common factor X is simply given by:

$$l_t^i | X \sim (1 - R_i(t,X))p_i(t,X)$$

It is important to note at this stage that without (4.2.), a recovery to maturity approach would be significantly more difficult to implement. It is also not clear that it would be an efficient approach anymore.
This critical assumption has very important consequences. Combined with the no-arbitrage conditions derived in the previous section, it leads to the following result:

Proposition 4.3. *Under (4.2.), the only valid recovery to maturity is $R(t,X) = R^{Mkt}$.*

Remark 4.4. *This implies that under (4.2.) the spot recovery itself is deterministic.*

Proof. By definition, for $k \geq 0$ we have:

$$\int_0^t \mathbb{E}\left[r_\tau^k \,|\tau = t, X\right] dp(t,X) = R^k(t,X)p(t,X)$$

And hence:

$$\mathbb{E}\left[r_\tau \,|\tau = t, X\right] = R(t,X) + p(t,X)\frac{\partial_t R(t,X)}{\partial_t p(t,X)}$$

$$\mathbb{E}\left[r_\tau^2 \,|\tau = t, X\right] = R^2(t,X) + 2R(t,X)p(t,X)\frac{\partial_t R(t,X)}{\partial_t p(t,X)}$$

Applying the conditional Jensen inequality for the function: $x \to x^2$ leads to:

$$\begin{array}{rccc} & \mathbb{E}\left[r_\tau^2 \,|\tau = t, X\right] & \geq & \mathbb{E}\left[r_\tau \,|\tau = t, X\right]^2 \\ \Leftrightarrow & R^2(t,X) + 2R(t,X)p(t,X)\frac{\partial_t R(t,X)}{\partial_t p(t,X)} & \geq & \left(R(t,X) + p(t,X)\frac{\partial_t R(t,X)}{\partial_t p(t,X)}\right)^2 \\ \Leftrightarrow & 0 & \geq & p^2(t,X)\left(\frac{\partial_t R(t,X)}{\partial_t p(t,X)}\right)^2 \\ \Leftrightarrow & \partial_t R(t,X) & = & 0 \end{array}$$

Now, the (EL) condition reads:

$$\forall \mu,\ \mathbb{E}\left[R(\sqrt{1-\rho}X + \sqrt{\rho}\mu)\right] = R^{\text{Mkt}}$$

Let $F(u) = \mathbb{E}\left[e^{iu\frac{Y}{\sqrt{\rho}}} R\left(\sqrt{1-\rho}X + \sqrt{\rho}Y\right)\right]$. F can be computed either by conditioning on Y or by noticing that $Z = \sqrt{1-\rho}X + \sqrt{\rho}Y$ is a standard gaussian random variable.

- On one hand:
$$F(u) = \mathbb{E}\left[e^{iu\frac{Y}{\sqrt{\rho}}}\mathbb{E}\left[R\left(\sqrt{1-\rho}X + \sqrt{\rho}Y\right)|Y\right]\right] = R^{\text{Mkt}}e^{-\frac{1}{2}\frac{u^2}{\rho}}$$

- On the other hand:
$$F(u) = e^{-\frac{1}{2}\frac{1-\rho}{\rho}u}\mathbb{E}\left[e^{iuZ}R(Z)\right]$$

Therefore, we have:

$$\mathbb{E}\left[e^{iuZ}R(Z)\right] = R^{\text{Mkt}}e^{-\frac{1}{2}u^2}$$

Applying the inverse Fourier transform leads to $R(x) = R^{\text{Mkt}}$.

■

Working with a recovery to maturity is possible but the simplified assumption (4.2.) required to compute the conditional variance introduces arbitrage in time. One way to get around this issue is to start from a recovery to maturity definition and define the loss process as:

$$l_t = (1 - r(t,X))\mathbf{1}_{\{\tau \leq t\}}$$

with $r(t,X) = R(t,X) + p(t,X)\frac{\partial_t R(t,X)}{\partial_t p(t,X)}$. We preserve the tractability of the conditional expected loss computation as :

$$l_t(X) = (1 - R(t,X))p(t,X)$$

However, the variance term now requires some work and the gain compared to the spot recovery formulation is not guaranteed.

5. A Specific Form for the Spot Recovery Rate.

In this section, we specify the spot recovery as a simple function of the common factor, which provides additional tractability. The calibration and practical implementation of the model is then discussed in details.

5.1. Conditional Normal Spot Recovery.

The spot recovery is defined as

$$r_i(t, X) = \Phi\left(\alpha_\rho X + \beta_i(t)\right) \tag{19}$$

This choice is motivated by at least three fundamental properties:

- $r \in [0, 1]$ and is a continuous function of t and X.
- The dependency in ρ is controlled by α_ρ. In particular, $\alpha_\rho = 0$ corresponds to the constant recovery case, while the spot recovery will convergence to a Dirac function as $\alpha_\rho \to +\infty$.
- The resulting modelling framework is very tractable. In particular, closed forms can be derived for the moments of the cumulated loss, and for the calibration to the individual expected loss.

It is sometime more convenient to adopt the following equivalent formulation:

$$r_i(t, X) = \Phi\left(\gamma_\rho\left(\frac{X - \sqrt{\rho}\Phi^{-1}(p_i(t))}{\sqrt{1-\rho}}\right) + \delta_i\right) \tag{20}$$

or with the notations defined in the previous section:

$$\begin{aligned} r(x, y) &= r\left(\frac{y - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \\ &= \Phi\left(\gamma_\rho\left(\frac{y - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) + \delta\right) \end{aligned}$$

The parameters for the two specifications are directly linked by:

$$\left\{\begin{array}{rcl} \alpha_\rho &=& \gamma_\rho/\sqrt{1-\rho} \\ \beta_i(t) &=& \delta_i - \gamma_\rho\sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}\left(p_i(t)\right) \end{array}\right.$$

We will use either formulation of the spot recovery function in the following sections, depending on which is more convenient. Note that both are strictly equivalent.

5.2. Calibration to Credit Default Swaps.

As part of the modelling approach, the conditional probability arising from the choice of a Gaussian copula are preserved following the introduction of stochastic recovery.
Then, calibrating the initial credit spread curve is strictly equivalent to calibrating the model to the expected loss curve. This ensures that standard CDS pricing is preserved for any par and off par contract. With the specific form of the spot recovery (19), this can be done using a simple closed form.

Proposition 5.1. *For a spot recovery function defined by (19), the (EL) condition is equivalent to*

$$\left\{\begin{array}{rcl} \delta &=& \Phi^{-1}\left(R_i^{Mkt}\right)\sqrt{1+\gamma_\rho^2} \\ \beta_i(t) &=& \Phi^{-1}\left(R_i^{Mkt}\right)\sqrt{1+(1-\rho)\alpha_\rho^2} - \alpha_\rho\sqrt{\rho}\Phi^{-1}\left(p_i(t)\right) \end{array}\right.$$

Proof. As detailed in the previous section, the (EL) condition is given by

$$\mathbb{E}\left[r_i\left(t, \sqrt{\rho}\Phi\left(p_i(t)\right) + \sqrt{1-\rho}X\right)\right] = R_i^{Mkt}$$

Given the equivalent form (20) of the spot recovery, this is equivalent to:

$$\mathbb{E}\left[\Phi\left(\gamma_\rho X + \delta\right)\right] = R_i^{\text{Mkt}}$$

Using the fact that (see e.g. [2])

$$\mathbb{E}\left[\Phi\left(\gamma_\rho X + \delta\right)\right] = \Phi\left(\frac{\delta}{\sqrt{1+\gamma_\rho^2}}\right)$$

leads to the result for δ.

■

5.3. Model Properties and Default Risk.

5.3.1. Basic Properties.

Given (19), it is clear that any value of recovery rate in $[0,1]$ is attainable. The main properties of the recovery function are driven by the coefficient α_ρ, which depends on the actual default correlation. We require α_ρ to have the following properties:

$$\begin{cases} \forall \rho \in [0,1],\ \alpha_\rho \geq 0 \\ \lim_{\rho \to 0} \alpha_\rho = 0 \\ \lim_{\rho \to 1} \alpha_\rho = +\infty \end{cases}$$

This ensures that for high levels of systemic risk, i.e. very negative values of the common factor X, the recovery will be low. It implies as well that the model will converge to constant recovery as default times become independent, and that conversely, the recovery function converges to a Dirac function as they become perfectly correlated.

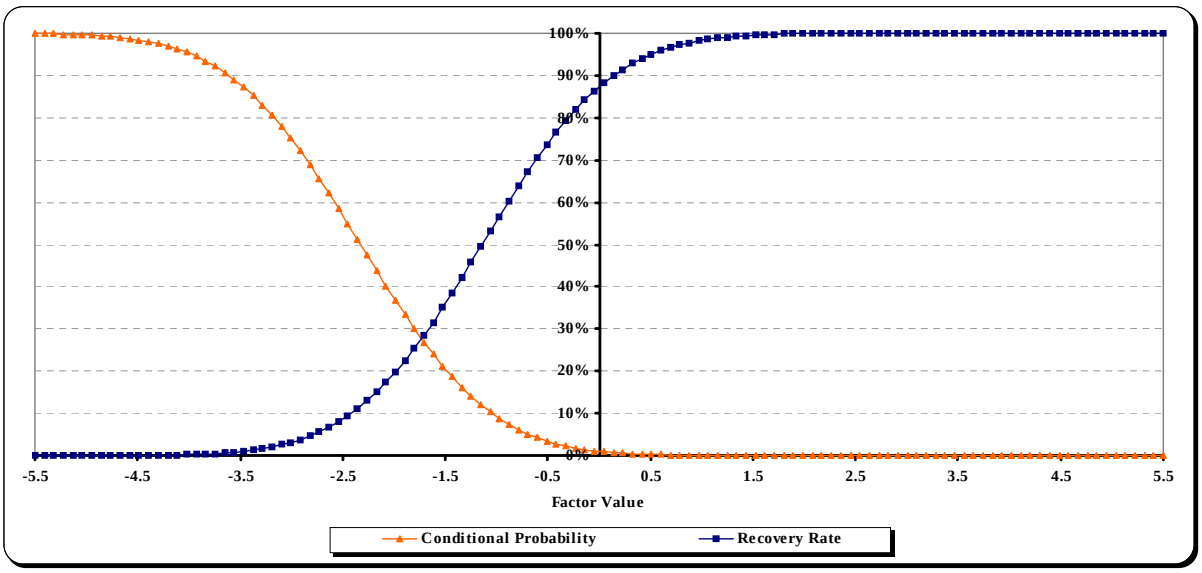

Figure 1: Spot Recovery and Conditional Probability Functions

Figure (1) gives an idea of the typical shape of the recovery rate as a function of the common factor. The conditional probability function is provided as well for comparison.

5.3.2. Time Consistency and Default Risk.
A critical element of risk management for portfolio Credit Derivatives in the current market conditions is the monitoring and hedging of default risk. Default risk is also one of the main measures of portfolio risk used for reserves, regulatory risk reporting and capital allocation. It is therefore a fundamental requirement for a pricing model to be continuous if an issuer goes smoothly to default. This notion of continuity on default can be defined more precisely as follows.

Definition 5.2. *Continuity on Default*
Introducing the following notations:

- $P(T, K_1, K_2)$ *is the price of a given CDO of maturity* T*, initial strikes* K_1 *and* K_2 *on the initial portfolio.*
- $P^{-i_0}(T, K_1^{-i_0}, K_2^{-i_0})$ *is the price of the same CDO after the realised default of issuer* i_0 *- at market recovery rate* $R_{i_0}^{Mkt}$ *- for which the strikes and portfolio have been adjusted.*

Continuity on default is defined as:

$$\lim_{p_{i_0}(0_+)\to 1} P(T, K_1, K_2) = P^{-i_0}(T, K_1^{-i_0}, K_2^{-i_0})$$

In practice, there is still uncertainty on the recovery until the auction process is completed. Continuity in default does not suggest that there is no uncertainty on recovery once a given issuer has defaulted. It simply outlines that this uncertainty cannot be modelled adequately using the dependency of the recovery with the common factor - which drives the dependency structure of default times.

As shown in Section 3, the spot recovery model has the specific property that

$$\forall x, \lim_{p(0_+)\to 1} r(t, x) = r(0_+, x)$$

which implies that continuity on default will be guaranteed as soon as

$$\forall x, \lim_{p(0_+)\to 1} r(0_+, x) = R^{Mkt} \tag{21}$$

It is interesting to note that continuity on default can only be achieved because of the flexibility of the spot recovery model w.r.t the time dimension. Indeed, there is no condition on the evolution of $r(t, x)$ w.r.t to t, which allows to specify a consistent behaviour at $t = 0_+$ for the model to satisfy default continuity.

5.4. Practical Implementation.

As seen in section 3, there are two equivalent ways to compute the cumulated loss conditional moments. They correspond to two different approaches of spot recovery modelling, which we can categorized as representations in space or in time.
As discussed in section 2.3.2, the loss distribution is fully defined by the conditional expected loss and squared expected loss when using the conditional normal approximation. We will therefore focus on the numerical computation of these two quantities in the following sections.

5.4.1. Representation in the Space Dimension.
We recall from (13):

$$r_\tau \mathbf{1}_{\{\tau \le t\}} \sim r(\sqrt{\rho}X + \sqrt{1-\rho}Y, X)\ \mathbf{1}_{\left\{Y \le \frac{\Phi^{-1}(p_t) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right\}}$$

Proposition 5.3. *The conditional moments of the cumulated loss are respectively given by:*

$$\begin{aligned} l_t(X) &= \Phi_2\left(c(t, X), d(X), \Theta_\rho\right) \\ l_t^2(X) &= \Phi_3\left(c(t, X), d(X), d(X), \Sigma_\rho\right) \end{aligned}$$

with

$$\left\{\begin{array}{rcl} c(t,X) & = & \frac{\Phi^{-1}(p_t)-\sqrt{\rho}X}{\sqrt{1-\rho}} \\ d(X) & = & -\frac{\gamma_\rho\sqrt{1-\rho}X+\sqrt{1+\gamma_\rho^2}\Phi^{-1}\left(R^{Mkt}\right)}{\sqrt{1+\gamma_\rho{}^2\rho}} \\ \Theta_\rho & = & -\frac{\gamma_\rho\sqrt{\rho}}{\sqrt{1+\gamma_\rho^2\rho}} \\ \Sigma_\rho & = & \begin{pmatrix} 1 & \Theta_\rho & \Theta_\rho \\ & 1 & \Theta_\rho^2 \\ & & 1 \end{pmatrix} \end{array}\right.$$

Proof. Given the previous definition of the conditional recovery, the conditional expected loss is given by

$$\begin{aligned} l_t(X) &= \mathbb{E}_X\left[\left(1-\Phi(\gamma_\rho\sqrt{1-\rho}X+\Phi^{-1}\left(R_i^{Mkt}\right)\sqrt{1+\gamma_\rho^2}-\gamma_\rho\sqrt{\rho}Y)\right)\mathbf{1}_{\left\{Y\leq\frac{\Phi^{-1}(p_t)-\sqrt{\rho}X}{\sqrt{1-\rho}}\right\}}\right] \\ &= \mathbb{E}_X\left[\Phi\left(\gamma_\rho\sqrt{\rho}Y+d(X)\sqrt{1+\gamma_\rho^2\rho}\right)\mathbf{1}_{\{Y\leq c(t,X)\}}\right] \\ &= \Phi_2\left(c(t,X),d(X);\Theta_\rho\right) \end{aligned}$$

using the fact that (see e.g. [2])

$$\int_{-\infty}^{c}\Phi(ax+b)\varphi(x)dx=\Phi_2\left(\frac{b}{\sqrt{1+a^2}},c;\frac{-a}{\sqrt{1+a^2}}\right)$$

In a similar way, the conditional squared expected loss is computed as

$$\begin{aligned} l_t^2(X) &= \mathbb{E}_X\left[\left(1-\Phi(\gamma_\rho\sqrt{1-\rho}X+\Phi^{-1}\left(R_i^{Mkt}\right)\sqrt{1+\gamma_\rho^2}-\gamma_\rho\sqrt{\rho}Y)\right)^2\mathbf{1}_{\left\{Y\leq\frac{\Phi^{-1}(p_t)-\sqrt{\rho}X}{\sqrt{1-\rho}}\right\}}\right] \\ &= \mathbb{E}_X\left[\Phi\left(\gamma_\rho\sqrt{\rho}Y+d(X)\sqrt{1+\gamma_\rho^2\rho}\right)^2\mathbf{1}_{\{Y\leq c(t,X)\}}\right] \end{aligned}$$

Defining Y_1 and Y_2 as

$$\left\{\begin{array}{rcl} Y_1 & = & \Theta_\rho Y+\frac{1}{\sqrt{1+\gamma_\rho^2\rho}}\varepsilon_1 \\ Y_2 & = & \Theta_\rho Y+\frac{1}{\sqrt{1+\gamma_\rho^2\rho}}\varepsilon_2 \end{array}\right.$$

where ε_1 and ε_2 are independent, $\mathcal{N}(0,1)$ random variables.

$$\begin{aligned} \mathbb{E}_X\left[\mathbf{1}_{\{Y_1\leq d(X)\}}\mathbf{1}_{\{Y_2\leq d(X)\}}\mathbf{1}_{\{Y\leq c(t,X)\}}\right] &= \mathbb{E}_X\left[\mathbb{E}\left[\mathbf{1}_{\{Y_1\leq d(X)\}}\mathbf{1}_{\{Y_2\leq d(X)\}}\mathbf{1}_{\{Y\leq c(t,X)\}}\,|Y\right]\right] \\ &= \mathbb{E}_X\left[\Phi\left(\gamma_\rho\sqrt{\rho}Y+d(X)\sqrt{1+\gamma_\rho^2\rho}\right)^2\mathbf{1}_{\{Y\leq c(t,X)\}}\right] \\ &= l_t^2(X) \\ &= \Phi_3\left(c(t,X),d(X),d(X);\Sigma_\rho\right) \end{aligned}$$

■

This approach relies heavily on the bivariate and trivariate Gaussian distributions which can be computationally intensive. Approximations exist for both (see e.g. [6]), but one might as well consider the alternative representation in the time dimension as it might be a more tractable solution.

5.4.2. Representation in the Time Dimension.
The representation in the time dimension uses the fact that the cumulated loss moments can be written as integrals over time that are easy to discretise. In fact,

$$\begin{aligned} l_i^k(t,X) &= \int_0^t\left(1-r_i(s,X)\right)^k dp_i(s,X) \\ &\simeq \sum_{j=0}^{J-1}\left(1-r_i(t_{j+1},X)\right)^k\left(p_i(t_{j+1},X)-p_i(t_j,X)\right) \end{aligned} \tag{22}$$

where $(t_j)_{j=0,..,J}$ is a discretisation of $[0,t]$ with $t_0 = 0$ and $t_J = t$.

For each time step $[t_j, t_{j+1}]$, the value of $\beta_i(t_{j+1})$ is computed to ensure the absence of bias for the expected loss. This can be done easily using the following result.

Proposition 5.4. *For the interval $[t_j, t_{j+1}]$, the value of $\beta_i(t_{j+1})$ is taken as solution of the following equation:*

$$\begin{aligned} R_i^{Mkt}\left(p_i(t_{j+1}) - p_i(t_j)\right) &= \mathbb{E}\left[\Phi\left(\alpha_\rho X + \beta_i(t_{j+1})\right)\left(p_i(t_{j+1}, X) - p_i(t_j, X)\right)\right] \\ &= \Phi_2\left(\frac{-\alpha_\rho\sqrt{\rho}}{\sqrt{1+\alpha_\rho^2}}; \Phi^{-1}(p_i(t_{j+1})), \frac{\beta_i(t_{j+1})}{\sqrt{1+\alpha_\rho^2}}\right) \\ &- \Phi_2\left(\frac{-\alpha_\rho\sqrt{\rho}}{\sqrt{1+\alpha_\rho^2}}; \Phi^{-1}(p_i(t_j)), \frac{\beta_i(t_{j+1})}{\sqrt{1+\alpha_\rho^2}}\right) \end{aligned} \tag{23}$$

Proof. First, we note that

$$\mathbb{E}\left[\Phi\left(\alpha_\rho X + \beta_i(t_{j+1})\right)\left(p_i(t_{j+1}, X) - p_i(t_j, X)\right)\right] = A\left(\Phi^{-1}(p_i(t_{j+1}))\right) - A\left(\Phi^{-1}(p_i(t_j))\right)$$

with

$$A(y) = \mathbb{E}\left[\Phi\left(\alpha_\rho X + \beta_i(t_{j+1})\right)\Phi\left(\frac{y - \sqrt{\rho}X}{\sqrt{1-\rho}}\right)\right]$$

Now we introduce the following auxiliary variables:

$$\begin{cases} Y_1 &= \sqrt{\frac{\alpha_\rho^2}{1+\alpha_\rho^2}}X + \sqrt{\frac{1}{1+\alpha_\rho^2}}\varepsilon_1 \\ Y_2 &= \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_2 \end{cases}$$

where ε_1 and ε_2 are independent, $\mathcal{N}(0,1)$ random variables.
Then, conditioning on X leads to

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{\left\{Y_1 \le \frac{\beta_i(t_{i+1})}{\sqrt{1+\alpha_\rho^2}}\right\}}\mathbf{1}_{\{Y_2 \le y\}}\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\left\{Y_1 \le \frac{\beta_i(t_{i+1})}{\sqrt{1+\alpha_\rho^2}}\right\}}\mathbf{1}_{\{Y_2 \le y\}}\,|X\right]\right] \\ &= A(y) \end{aligned}$$

and as Y_1 and Y_2 are also $\mathcal{N}(0,1)$ random variables, with correlation

$$< Y_1, Y_2 >= \frac{-\alpha_\rho\sqrt{\rho}}{\sqrt{1+\alpha_\rho^2}}$$

this completes the proof.

■

From a practical implementation perspective, equation (22) can be computed efficiently by recursion provided that the same time grid in used for all maturities at which the expected loss is required.

Equation (23) needs to be solved numerically, at each time step and for each underlying issuer, which could potentially be time consuming. However, the (EL) condition given in section 5.2 provides a closed form for β that could be used either directly or as an efficient first guess. This has a significant impact in terms of performance.

The implementation of the model used for practical analysis has proved to be reasonably fast. The overhead of using a spot recovery model implies a computation time about 2x higher than for the equivalent constant recovery approach. On a standard PC[3], the computation time e.g. for the full capital structure of iTraxx S9 5Y is ~ 0.5s.

6. Model Properties and Numerical Results.

The spot recovery model presented in details in the previous section is essentially an extension of the standard Gaussian copula approach to allow for consistent stochastic recovery. This implies that it can be combined with a standard base correlation framework - see e.g. [5] for more details - which is still widely used in the industry. In this section, we provide numerical examples of the flexibility of the model for calibration, with a particular focus on the super senior part of the capital structure. Then we analyze some of the model properties, such as the implied term structure of recovery and the continuity of the model w.r.t default events.

We compare different model specifications, which we identify and define as:

- **Constant Recovery**

  This corresponds to the standard case in which each names defaults with its market recovery:

$$r_i(t,x) = R_i^{Mkt},\ \forall t \geq 0,\ \forall x$$

- **Spot Recovery (1)**

  This first version of the model uses the same functional form for the recovery function for all maturities. In particular, the variance of the recovery does not go to zero as the default probability goes to one. This model specification is not continuous w.r.t default, as described in a previous section.

$$\begin{cases} r_i(t,x) = \Phi\left(\alpha_\rho x + \beta_i(t)\right),\ \forall t \geq 0 \\ \alpha_\rho = \frac{\rho}{1-\rho} \end{cases}$$

- **Spot Recovery (2)**

  This second version of the model defines explicitly the recovery function at time $t = 0^+$ to preserve continuity in case of default; the simplest solution being to revert to constant recovery for this case. For $t > 0$, the specification is identical to version (1).

$$\begin{cases} r_i(0^+,x) = R_i^{Mkt} \\ r_i(t,x) = \Phi\left(\alpha_\rho x + \beta_i(t)\right),\ \forall t > 0 \\ \alpha_\rho = \frac{\rho}{1-\rho} \end{cases}$$

6.1. Calibration to the Index Tranche Market.

Using the different model specifications, we calibrate iTraxx and CDX as of 11^{th} March 2009.

iTraxx S9	Upfront	Running	Upfront	Running	Upfront	Running
Index Maturity/Spread	**5Y**	**200.0**	**7Y**	**184.0**	**10Y**	**179.0**
0%-3%	6650	500.0	7000	500.0	7332	500.0
3%-6%	3475	500.0	4050	500.0	4439	500.0
6%-9%	0	890.0	0	900.0	0	901.7
9%-12%	0	442.5	0	467.5	0	529.4
12%-22%	0	151.0	0	160.0	0	174.8
60%-100%	0	29.4	0	31.1	0	39.3

Figure 2: iTraxx S9 Index Tranche Market - 11^{th} March 2009

[3]Intel Core 2 Quad Q6600 @ 2.40Ghz

CDX S9	Upfront	Running	Upfront	Running	Upfront	Running
Index Maturity/Spread	5Y	285.0	7Y	257.0	10Y	217.0
0%-3%	8014	500.0	8212	500.0	8289	500.0
3%-7%	5597	500.0	6089	500.0	6316	500.0
7%-10%	2284	500.0	3059	500.0	3547	500.0
10%-15%	0	672.5	0	681.3	0	663.0
15%-30%	0	149.1	0	168.1	0	167.0
60%-100%	0	18.0	0	19.0	0	19.0

Figure 3: CDX S9 Index Tranche Market - 11^{th} March 2009

Tables (2) and (3) summarize the market quotes for both indices and the corresponding market tranches. We have included quotes for 60%-100% tranches but we will treat these separately as they are not as standard and as liquid as the rest of the index tranches. However, they provide valuable market information and a good benchmark to assess the calibration flexibility of the model.

The context is still one of a distressed market with the price for the 60%-100% tranches remaining high. In particular, it is not possible to calibrate the standard index tranches using a constant recovery model, even when excluding the super-senior tranches. The standard base correlation approach fails to calibrate the senior tranches - 12%-22% and 15%-30% respectively for iTraxx and CDX.

iTraxx S9	Constant Recovery			Spot Recovery (1)			Spot Recovery (2)		
Index Strike/Maturity	5Y	7Y	10Y	5Y	7Y	10Y	5Y	7Y	10Y
3%	45.9%	48.9%	52.6%	32.5%	34.4%	36.3%	33.0%	34.7%	36.5%
6%	62.2%	65.7%	72.9%	42.1%	43.9%	48.2%	42.8%	44.4%	48.4%
9%	63.6%	64.7%	69.8%	42.4%	42.1%	44.6%	43.4%	42.6%	44.8%
12%	69.8%	70.0%	72.8%	46.1%	44.9%	45.2%	47.2%	45.5%	45.4%
22%	-	-	-	63.4%	62.9%	65.3%	65.3%	64.2%	65.9%
60%	-	-	-	74.8%	76.4%	82.3%	84.4%	85.0%	90.3%

Figure 4: iTraxx S9 Base Correlation Surface - 11^{th} March 2009

CDX S9	Constant Recovery			Spot Recovery (1)			Spot Recovery (2)		
Index Strike/Maturity	5Y	7Y	10Y	5Y	7Y	10Y	5Y	7Y	10Y
3%	61.1%	65.7%	67.3%	40.3%	43.6%	44.3%	41.4%	44.3%	44.8%
7%	52.1%	56.7%	58.3%	33.6%	36.6%	37.1%	34.7%	37.2%	37.4%
10%	58.7%	60.8%	59.4%	36.7%	37.3%	35.4%	38.1%	38.0%	35.8%
15%	69.7%	71.8%	68.0%	42.0%	42.2%	38.3%	44.0%	43.3%	38.8%
30%	-	-	-	72.2%	73.9%	69.3%	80.5%	80.6%	75.2%
60%	-	-	-	65.5%	62.7%	62.7%	89.1%	83.2%	81.7%

Figure 5: CDX S9 Base Correlation Surface - 11^{th} March 2009

Tables (4) and (5) give the base correlation surfaces for the different model specifications. First, these results confirm that it is not possible to calibrate standard tranches using constant recovery. With this assumption the senior tranches - 12%-22% and 15%-30% respectively for iTraxx and CDX - are overpriced compared to market level.

Both versions of the spot recovery model are flexible enough to calibrate iTraxx and CDX, including the 60%-100% tranches. As with other stochastic recovery models, the correlation smile is lower and significantly flatter than in the constant recovery case. The two specifications have similar correlations for the junior tranches whereas differences appear for the senior part of the capital structure. This is directly linked to the particular choice for $r(0^+, x)$. Further analysis is provided in the sections below.

6.2. Implied Term Structure of Recovery.

To illustrate the behaviour of the spot recovery function, we take one particular single name curve of the iTraxx S9 index and sample the conditional expected loss function at different maturity dates for both specifications of the spot recovery model. The same correlation is used for all maturities and both parameterizations, with $\rho = 40\%$.

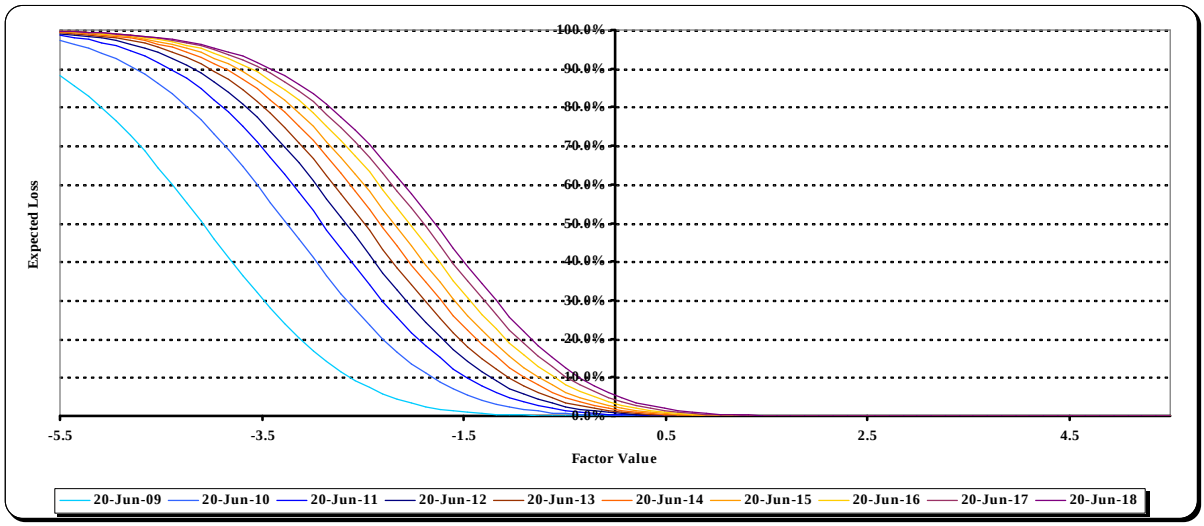

Figure 6: Conditional Expected Loss Profile - Spot Recovery (1)

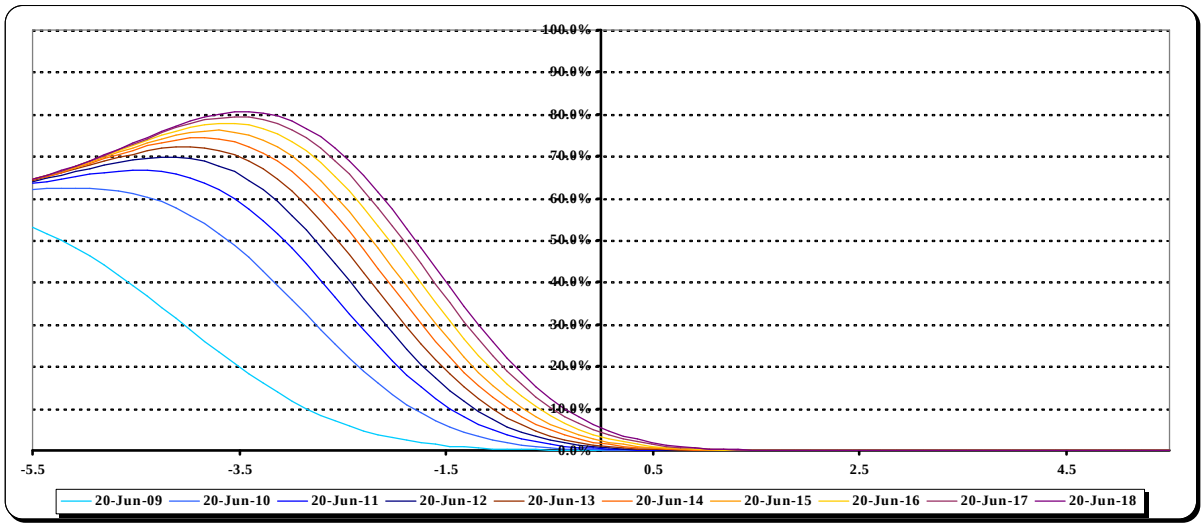

Figure 7: Conditional Expected Loss Profile - Spot Recovery (2)

Figures (6) and (7) display the evolution with maturity of the conditional expected loss as a function of the common factor.

We can observe that the difference between the two profiles comes essentially from the specification of the initial recovery function $r(0^+, x)$. Note that for the same level of correlation, the second specification of the spot recovery model puts less weight in the extreme region, typically above 60% loss. This is directly caused by the initial condition on the recovery, and explains the significant difference in correlation when calibrating both models to the same index tranche market.

6.3. Continuity on Default.

Continuity on default is defined in Section 5.3.2 as the continuity of the price function when going smoothly to default.

Figures (7) and (8) provide numerical examples of default continuity. We take iTraxx S9 portfolio as of 11^{th} March 2009, and use the first issuer curve to simulate the behaviour to default. A flat correlation set to 40% is used. We represent the difference between the final price $P^{-i_0}(T, K_1^{-i_0}, K_2^{-i_0})$ and the price obtained with issuer i_0 set to a given spread level. We choose to report the corresponding 3M default probability, which is more useful in this particular case. We choose two different parts of the capital structure with a 5Y equity tranche and a 5Y 12%-22% tranche. We report the difference in upfront terms for the equity and in basis points for the senior tranche.

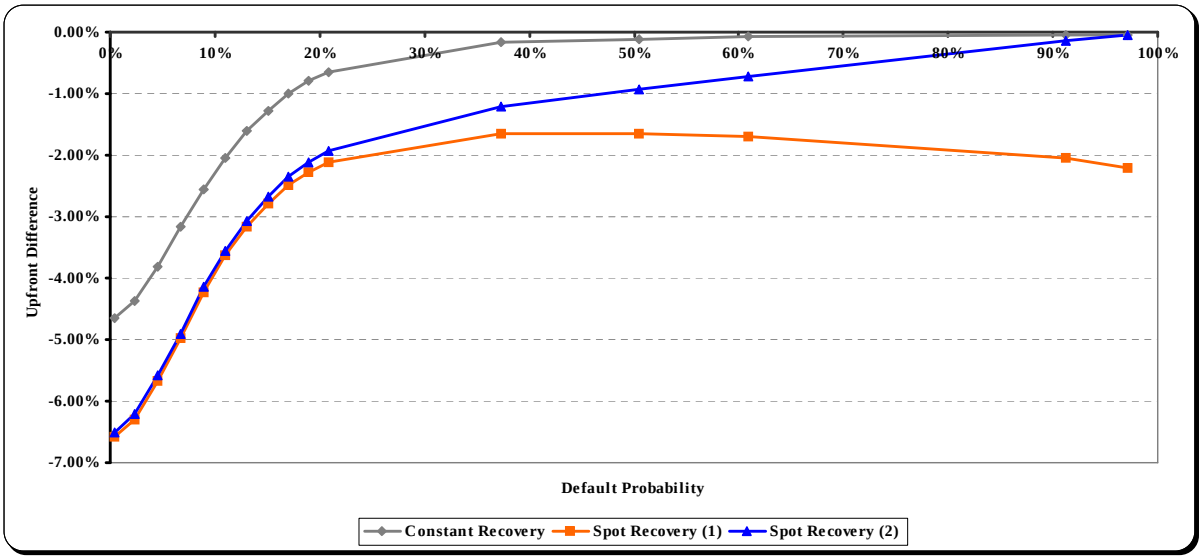

Figure 8: Continuity on default - 0%-3% Tranche

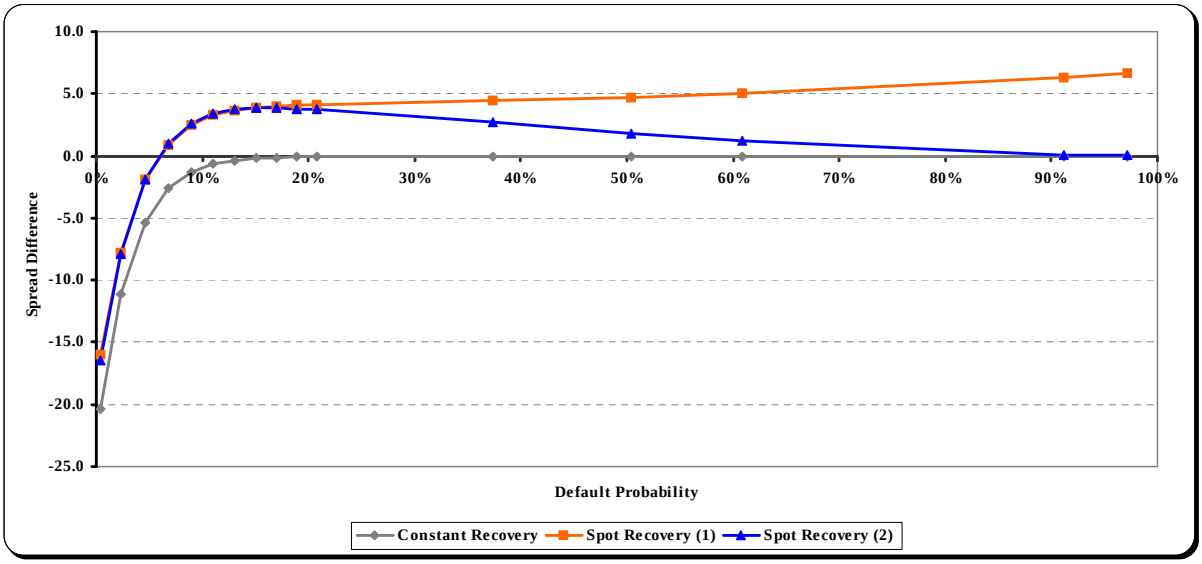

Figure 9: Continuity on default - 12%-22% Tranche

We observe that the second version of the spot recovery model is continuous on default as expected. The difference with the reference price goes to zero as the default probability goes to 1. However, the first specification of the spot recovery is showing significant discontinuities on default, for both tranches, with a particularly large difference for the equity tranche. As the correlation is kept constant, this is only due to the recovery variance not converging to zero as the default probability goes to 1.
It is interesting to note that the two versions of the model have a very similar behaviour when default probability remains below 20%. The standard constant recovery case is also represented to provide a simple reference case.

7. Conclusion.

In this paper, we have presented a simple and tractable extension to the Gaussian copula framework that allows for consistent stochastic recovery. The spot recovery rate has been introduced as a suitable underlying for recovery modelling. General results have been derived, such as time consistency conditions for spot recovery and recovery to maturity. We have shown in particular that existing model specifications using the later are not time consistent.
The modelling framework is tractable and flexible. It can be calibrated to individual credit spread curves, and is able to calibrate standard index tranche markets. Fundamental properties such as continuity on default have been analyzed in details.

Further analysis would lead to an extension of the spot recovery model to more general forms of factor models. More analysis would also be required to understand the true impact of stochastic recovery on hedging, and on the pricing of other portfolio Credit Derivatives.

References

[1] S. Amraoui, S. Hitier (2008). *Optimal Stochastic Recovery for Base Correlation*. Working Paper. BNP Paribas.

[2] L. Andersen, J. Sidenius (2004). *Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings*. Working Paper. Bank of America.

[3] L. Andersen, J. Sidenius (2003). *All your hedges in one basket*. Risk Magazine.

[4] R. Bielecki, M. Rutkowski (2001). *Credit risk: modeling, valuation and hedging*. Springer Finance.

[5] L. McGinty, E. Beinstein, R. Ahluwalia, M.Watts (2004). *Introducing Base Correlations*. Credit Derivatives Strategy. JP Morgan.

[6] Z. Drezner (1994). *Computation of the Trivariate Normal Integral*. Mathematics of Computation, Vol. 62, No. 205, pp. 289-294.

[7] C. Ech-Chatbi (2008). *CDS and CDO Pricing with Stochastic Recovery*. Working Paper. Calyon.

[8] R. Gaspar, I. Slinko (2008). *On recovery and intensitys correlation: a new class of credit risk models*. Journal of Credit Risk.

[9] J.P. Laurent, J. Gregory (2003). *Basket Default Swaps, CDOs and Factor Copulas*. Working Paper.

[10] M. Krekel (2008). *Pricing distressed CDOs with Base Correlation and Stochastic Recovery*. Working Paper. UniCredit Markets and Investment Banking.

[11] Y. Li (2009). *A Dynamic Correlation Modelling Framework with Consistent Stochastic Recovery*. Working Paper. Barclays Capital.

[12] D. Shelton (2004). *Back To Normal*. Working Paper. Citigroup Global Fixed Income Research.