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Abstract. This paper revisits the Levy sections theorem. We extend the scope of the theorem to time series and apply it to historical daily returns of selected dollar exchange rates. The elevated kurtosis usually observed in such series is then explained by their volatility patterns. And the duration of exchange rate pegs explains the extra elevated kurtosis in the exchange rates of emerging markets. In the end our extension of the theorem provides an approach that is simpler than the more common explicit modeling of fat tails and dependence. Our main purpose is to build up a technique based on the sections that allows one to artificially remove the fat tails and dependence present in a data set. By analyzing data through the lenses of the Levy sections theorem one can find common patterns in otherwise very different data sets.

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1. Introduction

Recently the study of complex systems has attracted the attention of a growing number of physicists. Scaling laws, self-organized criticality, self-similarity, and fractals, just to name a few, have been found in fields as diverse as biology and economics. These phenomena have created the need for a general theoretical framework to explain them coherently through a physics of complex systems.

A branch known as "econophysics" attempted to explain the self-similarity and fat tails observed in financial distributions that can be responsible for a variety of behaviors and, in particular, ultraslow convergence to the Gaussian regime [1]. Here one major contribution was Mantegna and Stanley's truncated Levy flight [2], which takes into account both the departures from the classical central limit theorem and the presence of scaling laws.

More recently, we have pursued a different line of research [3, 4]. Rather than looking for underlying probability distributions of financial processes, we focused on the role of nonlinear autocorrelations as well as nonidentically distributed variables. As a result, we could alternatively explain both the ultraslow convergence and scaling laws.

This paper moves forward and suggests an even simpler approach based on the Levy sections theorem [5]. The classical central limit theorem does not take chains of random variables that are dependent into account. Yet the Levy sections theorem is stated under Levy's generalization of the classical central limit theorem to encompass dependent variables. The Levy sections theorem is not to be confused with his stable distribution of infinite variance. Levy also employed his notion of "sections" to outline a proof for the generalization of the classical central limit theorem in order to consider the sums of dependent random variables [6]. This proof was reworked afterward using less restrictive assumptions [7, 8]. A full description of these developments was presented in his subsequent book [9].

Taking Levy sections amounts basically to using the inverse of the predictable quadratic variation as a random time change to transform a given process into a Gaussian one. And every continuous martingale is a time-changed Wiener process, where the time change is the quadratic variation. This is known as the Dambis-Dubins-Schwarz theorem [10, 11, 12]. Also every semimartingale is a time-changed Wiener process [13]. At first, the last result can be employed for discrete time processes (time series). And in particular, asset prices can be considered as time-changed Wiener processes [14, 15]. References on martingale limit theory and the central limit theorem for martingales can be found elsewhere [16, 17, 18].

This paper thus extends the Levy sections theorem's approach to time series. And we take historical daily returns of selected dollar exchange rates from both developed and emerging markets to illustrate our case. By using the Levy sections to account for local volatilities we find universal patterns in the random behavior of actual financial series. Indeed we explain their stylized fact of elevated kurtosis by the volatilities. And the extra elevated kurtosis of emerging markets is explained by the duration of exchange rate pegs. The longer foreign exchange intervention is, the greater the kurtosis. One can then build a gauge of exchange rate peg duration based on the kurtosis. In the end, our extension of the Levy sections theorem provides an approach that is simpler than the more common explicit modeling of fat tails and dependence [3, 4].

The main purpose of this paper is to build up a technique based on the sections that allows one to artificially remove the fat tails and dependence present in a data set, and then compare this set with a Gaussian one, only to realize that both data sets become very similar if analyzed through the lenses of the Levy sections theorem.

The rest of the paper is organized as follows. Section 2 presents building-block definitions and the Levy sections theorem. Section 3 extends the previous definitions to time series. Section 4 illustrates our framework using data from exchange rate returns. Section 5 puts forward a qualitative gauge of foreign exchange intervention using a Gaussian generator. And Section 6 concludes.

2. Definitions and the Levy sections theorem

We consider a chain of random variables denoted by X_n with $n \in \mathbb{N}$. The conditional probability of a given realization x_{n+1} of X_{n+1} is written as $P\left(x_{n+1} \middle| x_1,...,x_n\right)$. This means the probability of x_{n+1} if the random variables $X_1,\ldots,X_n$ follow the particular chain walk $x_1,\ldots,x_n$. The conditional mean and variance of X_{n+1} are

$$\mu_n = \left\langle X_{n+1} \right\rangle_{x_1 \cdots x_n} = \int x_{n+1} P\left(x_{n+1} \middle| x_1,...,x_n\right) dx_{n+1} \tag{1}$$

and

$$m_n^2 = \left\langle X_{n+1}^2 \right\rangle_{x_1 \cdots x_n} - \left\langle X_{n+1} \right\rangle_{x_1 \cdots x_n}^2 = \int x_{n+1}^2 P\left(x_{n+1} \middle| x_1,...,x_n\right) dx_{n+1} - \mu_n^2 . \tag{2}$$

Both μ_n and m_n depend on $x_1,...,x_n$. To simplify notation, we omit the index associated with the walk dependence. For the chain walk $x_1,\ldots,x_n$ of size n of the random variables $X_1,\ldots,X_n$ we calculate the quantity

$$\lambda_n = \sum_{i=1}^{n} m_i^2$$

where m_i is the conditional variance of $x_1,\ldots,x_i$ for $i=1,\ldots,n$. Consider a positive real number t such that the condition

$$\lambda_{n-1} \le t < \lambda_n \tag{3}$$

is satisfied. We say that the chain walk $x_1,...,x_n$ belongs to the section t, and condition (3) is called the *section condition* t. The λ_{n-1} is calculated for the chain walk $x_1,...,x_{n-1}$, i.e. $\lambda_{n-1} = \sum_{i=1}^{n-1} m_i^2$. For a given chain walk of size n we have $\lambda_i = \lambda_{i-1} + m_i^2, i = 2,\cdots,n$, and $\lambda_1 = m_1^2$.

The section t is made up of all chain walks obeying the section condition t. Note that the chain walks can have different sizes n.

The sum $x_1 + ... + x_n$ of elements in a given chain walk belonging to the section t defines a random variable, denoted by S_t, whose variance is $M_t^2 = \langle S_t^2 \rangle - \langle S_t \rangle^2$. The Levy sections theorem [6–9] is the following.

Theorem. *For the null conditional means* $\mu_n = 0$ $(\forall n \in \mathbb{N})$ *and random variables* X_n $(\forall n \in \mathbb{N})$ *satisfying the Lindeberg conditional condition (see reference [9], section 67, pages 237-246, theorem 67.3), the probability distribution of* $S_t / \sqrt{t}$ *is such that*

$$\lim_{t\to\infty} P\left(S_t / \sqrt{t} < \eta\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta} e^{-\frac{x^2}{2}} dx.$$

Stationarity is not assumed. This theorem extends the classical central limit theorem to consider the chains of *dependent* random variables. The distribution of the variable $S_t / \sqrt{t}$ converges to a Gaussian of zero mean and unity standard deviation as the section t goes to infinity. The normalized variable S_t / M_t, with $M_t = \sqrt{\langle S_t^{\,2} \rangle - \langle S_t \rangle^2}$, also converges to the Gaussian of zero mean and unity standard deviation. For a given section t, the variable $S_t / \sqrt{t}$ (unlike S_t / M_t) has not unity standard deviation. Yet both variables have the same skewness and kurtosis (and the same is true of the other reduced statistical moments). Both converge to a normal distribution of unity standard deviation. While the standard deviation of S_t / M_t remains constant and equal to unity over the convergence process, the standard deviation of $S_t / \sqrt{t}$ changes, yet converging asymptotically to unity.

Given the conditional probability of the random variable X_n, its probability distribution is given by the marginal probability defined as

$$p_n(x_n) = \sum_{x_1 \cdots x_{n-1}} P\left(x_n \middle| x_1, ..., x_{n-1}\right)$$

where the sum considers all possible walks $x_1,...,x_{n-1}$ followed by the random variables $X_1,...,X_{n-1}$. The marginal variance of X_n is calculated from (1), i.e.

$$\nu_n^2 = \langle X_n^2 \rangle - \langle X_n \rangle^2 = \int x_n^2 p_n(x_n) dx_n - \left(\int x_n p_n(x_n) dx_n \right)^2 .$$

Let us define the quantity

$$\sigma_n^2 \equiv \sum_{i=1}^{n} \nu_i^2 \Big/ n$$

which we call the *cumulated average variance* of X_n. Then, let us consider the sum random variable S_n of size n defined in the usual way, i.e.

$$S_n = X_1 + \cdots + X_n . \tag{4}$$

Its variance, $M_n^2 = \langle S_n^2 \rangle - \langle S_n \rangle^2$, satisfies

$$M_n^2 = \sum_{i=1}^{n} \nu_i^2 + \sum_{i \neq j=1}^{n} \mathrm{cor}(X_i, X_j) \nu_i \nu_j ,$$

where $\mathrm{cor}(X_i, X_j)$ is the linear correlation between variables X_i and X_j.

Next we define the quantity

$$\tau_n = M_n^2 \big/ \sigma_n^2$$

which we call the *variance time* of S_n. To understand its meaning first consider the example of a chain of independent random variables X_n, where $\mathrm{cor}(X_i, X_j) = 0$ for all $i \neq j = 1, \ldots, n$, and $\tau_n = n$. The variance time is just the "actual" time n.

Another example shedding light on the meaning of the variance time of S_n is a situation where the marginal variance ν_i^2 is stationary, i.e. $\nu_i^2 = \nu^2$ for all $i \in \mathbb{N}$. In this case the variance time becomes

$$\tau_n = \frac{M_n^2}{\nu^2} = n + \sum_{i \neq j=1}^{n} \mathrm{cor}(X_i, X_j) .$$

Note that the presence of linear correlations can lead to delays and advances in the variance time when compared to the actual time n. For some chains (for example, Mandelbrot's fractional Brownian motion) the variance of S_n may follow a scaling law such as $M_n = An^H$, where H is Hurst exponent. Here the variance time is $\tau_n = n^{2H}$. If $H > 1/2$ ($H < 1/2$) the variance time will move ahead (fall behind) the actual time.

We do not attach an index related to the actual time in the random variable S_t because the number of terms in S_t depends on the chain walk. Yet the size n of a chain walk $x_1, \ldots, x_n$ satisfying the section condition t (equation (3)) can be used as a (random) variable related to time.

We denote it by n_t. If the variance of X_n is stationary, the variance time (associated with the section) τ_t of S_t is

$$\tau_t = M_t^2 / \nu^2 .$$

3. Extending the concepts to time series

A time series $(x_i)_{i=1,\dots,N}$, where i is a time counter and N is the series size, can be thought of as a single realization of a random process. We can employ to this series either (1) the technique based on the marginal probability of a chain of identical random variables [3–6] applied to study of the properties of S_n or (2) the properties associated with the sum S_t as defined in Eq. (4). The technique uses the concept of conditional variance m_n of a given chain of random variables as well as the variable S_t introduced in Section 2. To clarify the differences between the two approaches we elaborate further on the definitions associated with S_n and S_t.

For the time series we rewrite the sum S_n for a given $n < N$ as

$$S_n = \left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_{1+i}, \cdots, \sum_{i=1}^{n} x_{N-n+i} \right).$$

Such collection of sums is seen as one realization of $S_n = X_1 + \cdots + X_n$, where the random variables X_i with $i = 1, \dots, n$ are identically distributed. The original time series is only one of the all possible realizations, i.e. $X_i = X = (x_1, x_2, \dots, x_N)$. The marginal variance $\nu^2 = \langle X^2 \rangle - \langle X \rangle^2$ can be straightforwardly reckoned from the list X. This allows one to calculate the variance time $\tau_n = M_n^2 / \nu^2$, where M_n^2 is the variance of S_n. Thanks to the presence of correlations, the normalized S_n does not converge to a Gaussian.

One big difficulty is learning the values taken by the local volatilities m_i^2 since it is impossible to get them from only one realization of the variable, namely the empirical value of x_i taken from the data set. For this reason we need an extra technique to calculate the local volatilities. So consider a positive integer q, and the new time series

$$(y_n)_{n=1,\dots,N-2q},$$

where the first and the last q terms of $(x_n)_{n=1,\dots,N}$ were dropped, i.e. $y_n = x_{n+q}$, $n = 1, \dots, N-2q$. Then the local volatility m_n^2 is

$$m_n^2 = \frac{1}{2q+1} \sum_{i=n}^{n+2q} x_i^2 - \left(\frac{1}{2q+1} \sum_{i=n}^{n+2q} x_i \right)^2 \tag{5}$$

for $n = 1, \dots, N-2q$. Here the local volatility is a measure of the conditional variance associated with a given chain of random variables.

Thus we can extend the concept of the Levy section t for the collection $\left(y_n\right)_{n=1,\dots,N-2q}$. The S_t ends up as the collection of all the sums

$$y_i + y_{i+1} + \ldots + y_{n_i - 1} + y_{n_i}, i \in \{1, \ldots, N-2q\}$$

such that the condition in Eq. (3) is fullfilled, i.e.

$$\lambda_{n_i - 1} = m_i^2 + m_{i+1}^2 + \ldots + m_{n_i - 1}^2 \le t < m_i^2 + m_{i+1}^2 + \ldots + m_{n_i}^2 = \lambda_{n_i},$$

where every m_i^2 is calculated by Eq. (5).

The local volatility definition implies the existence of an integer $j_t \in [0, N-2q]$ such that the section condition t is not fulfilled for $i > j_t$. Indeed, j_t is the number of elements belonging to the collection S_t which can be rewritten as

$$S_t = \left(\sum_{i=1}^{n_1} y_i, \sum_{i=1}^{n_2} y_{1+i}, \ldots, \sum_{i=1}^{n_{j_t}} y_{j_t - 1 + i} \right).$$

For every section t we can define the collection $n_t = (n_1, n_2, \dots, n_{j_t})$ made up of the number of terms in every sum belonging to the collection S_t.

For the time series, the variance of S_t is $M_t^2 = \langle S_t^2 \rangle - \langle S_t \rangle^2$, and the variance time of S_t is $\tau_t = M_t^2 / v^2$. Also, the average number of terms associated with S_t is $\langle n_t \rangle = (n_1 + \cdots + n_{j_t}) / j_t$. The purpose of the definition of variance time is to compare the time evolution of both S_n and S_t. Unlike S_n, the S_t is not indexed to actual time, i.e. no particular time is associated with it. The scale of the variance time, however, allows one to compare the two. Although other scales can be imagined, in the one suggested here the variance of both S_n and S_t is the same for every variance time. So we can assess the evolution of S_n and S_t by considering not actual time, but how their respective variances evolve.

We assume that the time series is stationary when doing the sum procedures above. Though the stationarity assumption for a chain of random variables is not made in the Levy sections theorem, our sum procedures to obtain S_t for an empirical time series make sense only if the series is stationary. So our sum procedure is to be blamed in the event of a possible failure of the extension of the Levy sections theorem to time series.

4. Illustrating with exchange rate returns

Now we take historical daily returns of selected dollar exchange rates from six countries, namely Britain, France, Canada, India, Sri Lanka, and the People's Republic of China. These are absolute returns, i.e. $x_n = r_n - r_{n-1}$, where r_n is an exchange rate (dollar price of a foreign currency) at date n. The data are collected from the Federal Reserve website. Table 1 gives more details.

We reckon the local volatility of trading weeks (5-day weeks), which means $q = 2$ in Section 3's formulas. Figure 1 shows the kurtosis K as a function of the variance time. Dashed lines are the kurtosis' evolution of the conventionally ordered series S_n as a function of τ_n. The continuous lines are the kurtosis' evolution of S_t as a function of τ_t.

To display the kurtosis behavior of the sections sums, we start with the initial (very small) section $t = 10^{-15}$, and then calculate the sections $t + i\Delta t$, for $i = 1,\ldots,99$. We cannot pick the section $t = 0$ to begin with because of computational limitations. The values of Δt are arbitrarily chosen to enable one to see smooth variations of the kurtosis as well as the transient period of kurtosis evolution. We restrict the calculations to 100 steps because this is enough to assuring the asymptotical convergence of the kurtosis. And also because this allows one to keep the number of terms of the sums in S_t small if compared to the original number of terms in an empirical time series. This prevents introducing spurious correlations among the terms in sequence S_t. The values of Δt used in every currency are in Table 2. The key features shown in Figure 1 are as follows.

(A) There is kurtosis convergence in the sections sums S_t of the currencies toward a well defined asymptotic state. This does not hold in the sums S_n of the conventionally ordered exchange rate time series.

(B) The variance time of kurtosis convergence for the sections sums is short. Unlike in the conventionally ordered sums, the kurtosis convergence for the sections is similar for all rates. All the sections kurtosis practically reached the limit at the variance time $\tau_t = 10$.

(C) The kurtosis convergence approaches zero. Developed countries' currencies present slightly negative kurtosis and emerging countries' currencies have slightly positive kurtosis. Unlike in the conventionally ordered series, the sections sums converge to a distribution resembling the Gaussian.

(D) The sections' kurtosis evolution presents a universal behavior for the currencies studied, regardless of the fact that a country is developed or not.

What happens from the perspective of actual time? Assuming the variance time $\tau = 10$ as an equilibrium benchmark, we can take the section t corresponding to that time for every currency. Table 3 lists the values of t for the exchange rate series. We can obtain the collection n_t as defined in Section 3, and also calculate $\langle n_t \rangle$: the average number of terms of the sums of section t. Figure 2 shows histograms of the collections n_t, and Table 4 presents $\langle n_t \rangle$ for the exchange rates.

Compared to the histograms of emerging markets' currencies, the histograms of developed markets' currencies tend to cluster in a near-zero value. And the average number of days $\langle n_t \rangle$ corresponding to the stationary limit $\sigma_t = 10$ of the sections t of developed countries' currencies is smaller than that of emerging markets' currencies (the values of t are those displayed in Table 3). These features may be related to the degree of government intervention in the emerging markets' currencies. A fixed exchange rate regime would mean zero volatility (constant rate) and a return series dominated by zeros. China, for instance, kept an 11-year-old peg of its currency, the yuan, at 8.28 to the dollar. But there were also four big episodes of revaluation in the yuan-dollar returns' series considered. This caused an interesting effect. Because volatility nears zero most days, one needs to accumulate more days to fullfil a given section condition t. Table 5 shows the yuan's

$\langle n_t \rangle$ greater than that of the other currencies. Indian and Sri Lankan rupees present smaller values but still greater than those of the pound, French franc, and Canadian dollar. The developed countries' currencies exhibit very similar $\langle n_t \rangle$.

Figure 3 shows histograms related to the currencies' local volatility. The yuan's volatility clusters in zero, unlike those of developed countries' currencies. This explains the observed patterns in the histogram of n_t (Figure 2).

5. A suggested gauge of exchange rate control

As an exercise, we put forward a qualitative gauge of foreign exchange intervention using a Gaussian generator. Consider a Gaussian random generator of reduced variables that are independent and identically distributed (IIDR) [4]. Then consider the sequence $z_n = m_n g_n$, $n = 1,...,N-2q$ (with q = 2 in the empirical example), where g_n is generated by a normal distribution, and m_n is the local volatility. What is special here is that the volatility process is not modeled, but taken from the data. If m_n is constant, the distribution of $z_n = m_n g_n$ collapses to a Gaussian. The column in the middle of Table 5 shows the kurtosis of the IIDR applied to the exchange rates. The right hand side column shows the kurtosis of the original series of daily returns. The effect of the local volatilities is unambiguous. Because the generator is Gaussian, the elevated kurtosis should be explained by the volatilities.

Thanks to exchange rate pegs, return dispersion is low at the days a rate is fixed. Thus a number of return observations fall out of the variance interval (by variance interval we mean the symmetric interval around the mean that is two standard deviation wide, and with respect to the original returns series; and this without taking the sections into account). The elevated kurtosis in emerging markets' exchange rates can then be explained by too many observations outside the variance interval. This rationale is simpler than the more usual ones based on fat tails and dependence. The Levy sections filter the effects on the local volatility so that the return series present a near-Gaussian universal pattern.

Exchange rate time series are commonly believed to be modeled by a Gaussian whenever government intervention is absent. This is because government intervention introduces patterns in the series that can be exploited by market participants to improve their forecasts. With free float the market is more likely to be efficient in the sense that the properly anticipated prices fluctuate randomly [19]. Our results show that foreign exchange intervention provokes departures from the Gaussian in that it biases the volatility evolution. So the greater the control is, the greater the kurtosis. This is so because the pegs tend to bring a series' dispersion closer to zero, thereby rendering many observations out of the distribution's variance interval. Thus the kurtosis reckoned in the IIDR can be seen as a gauge of peg duration. Normalizing the pound-dollar's kurtosis to unity, we can get a relative intervention scale (Table 6). Note that this gauge is qualitative in that no quantitative relation between the kurtosis ratios and the peg durations are provided. This might be one interesting topic for future research.

6. Conclusion

Levy's notion of sections was a tool for him to outline a proof for the generalization of the classical central limit theorem to consider the sums of dependent random variables [6]. This paper extends his technique to time series. Though the Levy sections do not consider actual time, the notion of a variance time for their sum that converges to a Gaussian can be useful for our purposes. So the sections can be designed to consider only the local volatility. Employing historical daily returns of selected dollar exchange rates, we calculate the local volatilities of their trading weeks. Doing so, we find a universal behavior in the actual series.

Unlike in the conventionally ordered exchange rate time series, we find kurtosis convergence toward a well defined asymptotic state in their correspondent Levy sections. We also find the time of kurtosis convergence to be short. This is similar for the currencies considered. The kurtosis convergence approaches zero. And in the Levy sections, the convergence occurs toward a distribution resembling the Gaussian.

As an exercise, we employ our approach to show that the extra elevated kurtosis of emerging markets' exchange rates can be explained by too many observations outside the variance interval. This is so thanks to the duration of exchange rate pegs. Foreign exchange intervention provokes departures from the Gaussian in that it biases the volatility evolution. So the greater the control is, the greater the kurtosis.

We finally suggest a qualitative gauge of peg duration based on the kurtosis reckoned in the Gaussian generator, and leave the search for a quantitative gauge for future research.

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Table 1. Description of data.

Country	Currency	Time Period	Observations
Britain	Pound	4 Jan 71 – 10 Jan 03	8031
France	French Franc	4 Jan 71 – 31 Dec 98	7020
Canada	Canadian Dollar	4 Jan 71 – 10 Jan 03	8037
India	Indian Rupee	2 Jan 73 – 10 Jan 03	7525
Sri Lanka	Sri Lankan Rupee	2 Jan 73 – 10 Jan 03	7171
China	Yuan	2 Jan 81 – 10 Jan 03	5471

Table 2. Values for steps Δt.

Currency	Δt
Pound	0.00002
French Franc	0.00031
Canadian Dollar	0.0000023
Indian Rupee	0.054
Sri Lankan Rupee	0.02
Yuan	0.00055

Table 3. Value of t for the section corresponding to $\tau_t = 10$.

Currency	t
Pound	0.00046
French Franc	0.00682
Canadian Dollar	0.0000506
Indian Rupee	0.0584
Sri Lankan Rupee	0.1
Yuan	0.01210

Table 4. Values of n_t for $\tau_t = 10$.

Currency	$\langle n_t \rangle$
Pound	15.04
French Franc	23.13
Canadian Dollar	17.26
Indian Rupee	246.20
Sri Lankan Rupee	79.06
Yuan	323.39

Table 5. Kurtosis of the Gaussian IIDR and of the original series.

Currency	IIDR Series' Kurtosis	Original Series' Kurtosis
Pound	6.76	4.76
French Franc	10.04	8.54
Canadian Dollar	7.73	5.37
Indian Rupee	118.9	118.3
Sri Lankan Rupee	124.3	288.7
Yuan	1547.7	3486.1

Table 6. Intervention scale: IIDR series' kurtosis relative to IIDR pound-dollar return series' kurtosis.

Currency	Intervention Scale
Pound	1.0
French Franc	1.48
Canadian Dollar	1.14
Indian Rupee	17.60
Sri Lankan Rupee	18.39
Yuan	228.97

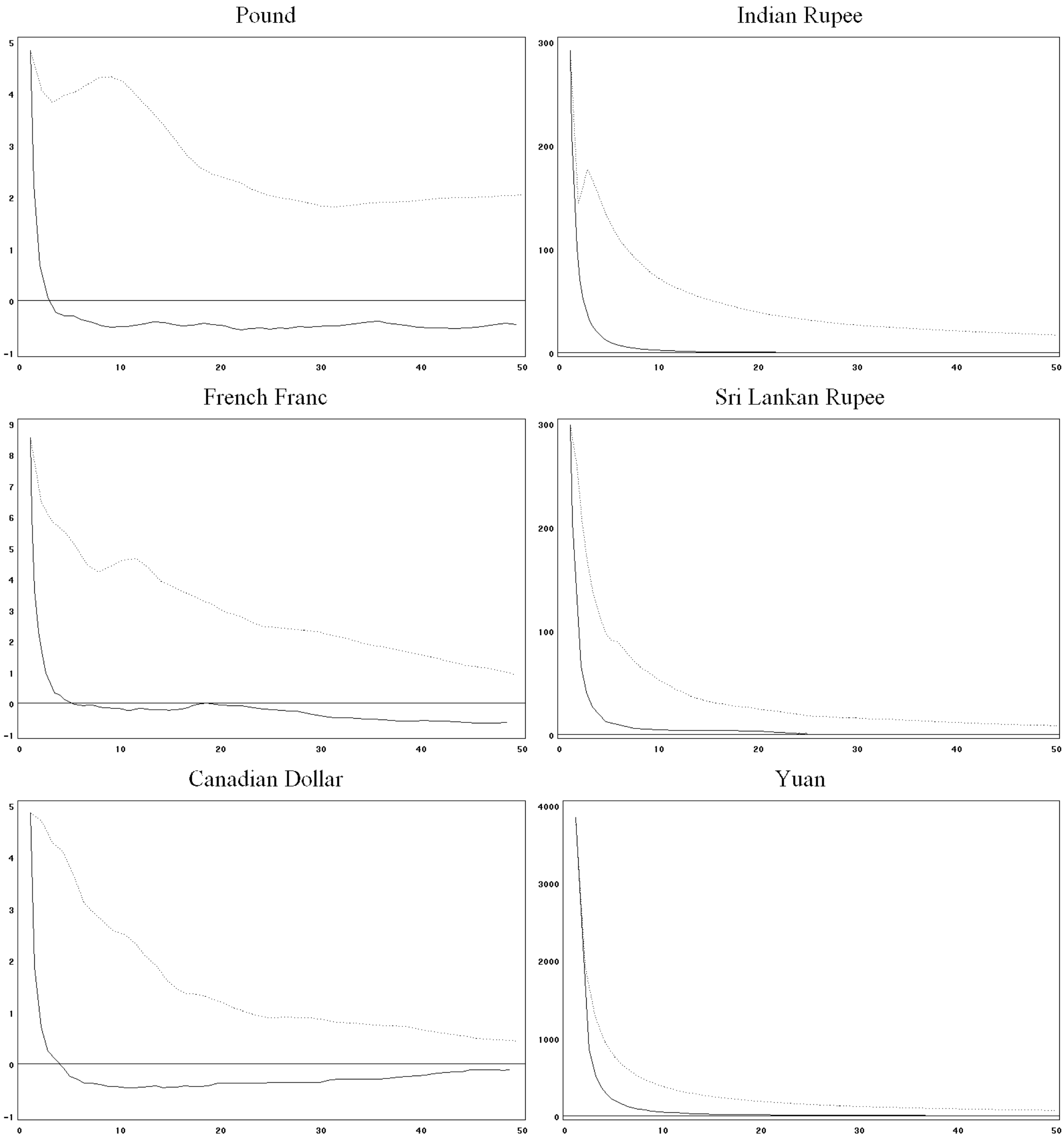

Figure 1: Kurtosis (vertical) vs stochastic time. Dashed lines are for conventionally ordered series and continuous lines are for the Levy sections.

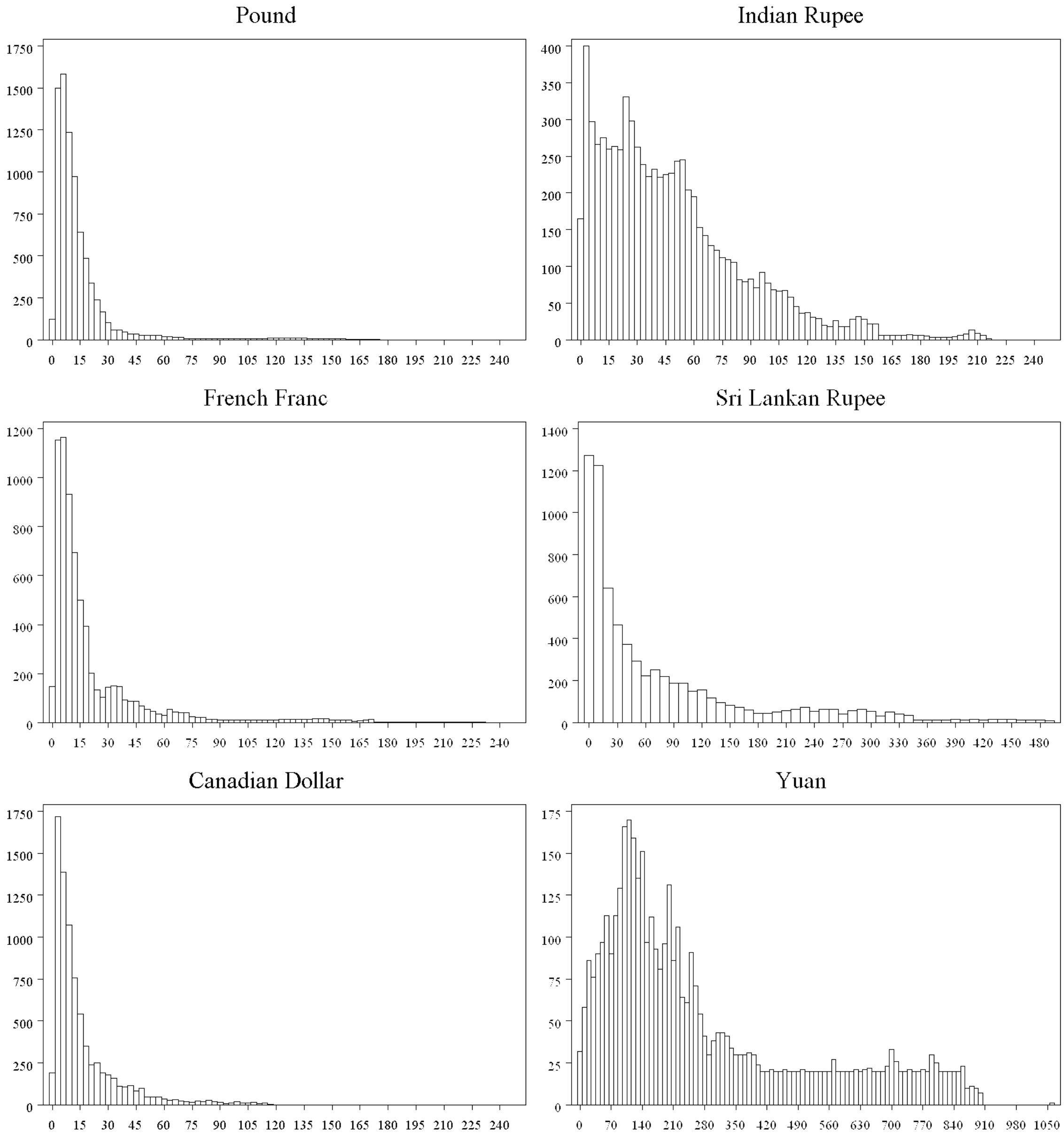

Figure 2. Histograms of n_t. Section t for every currency corresponds to the variance time $\tau_t = 10$.

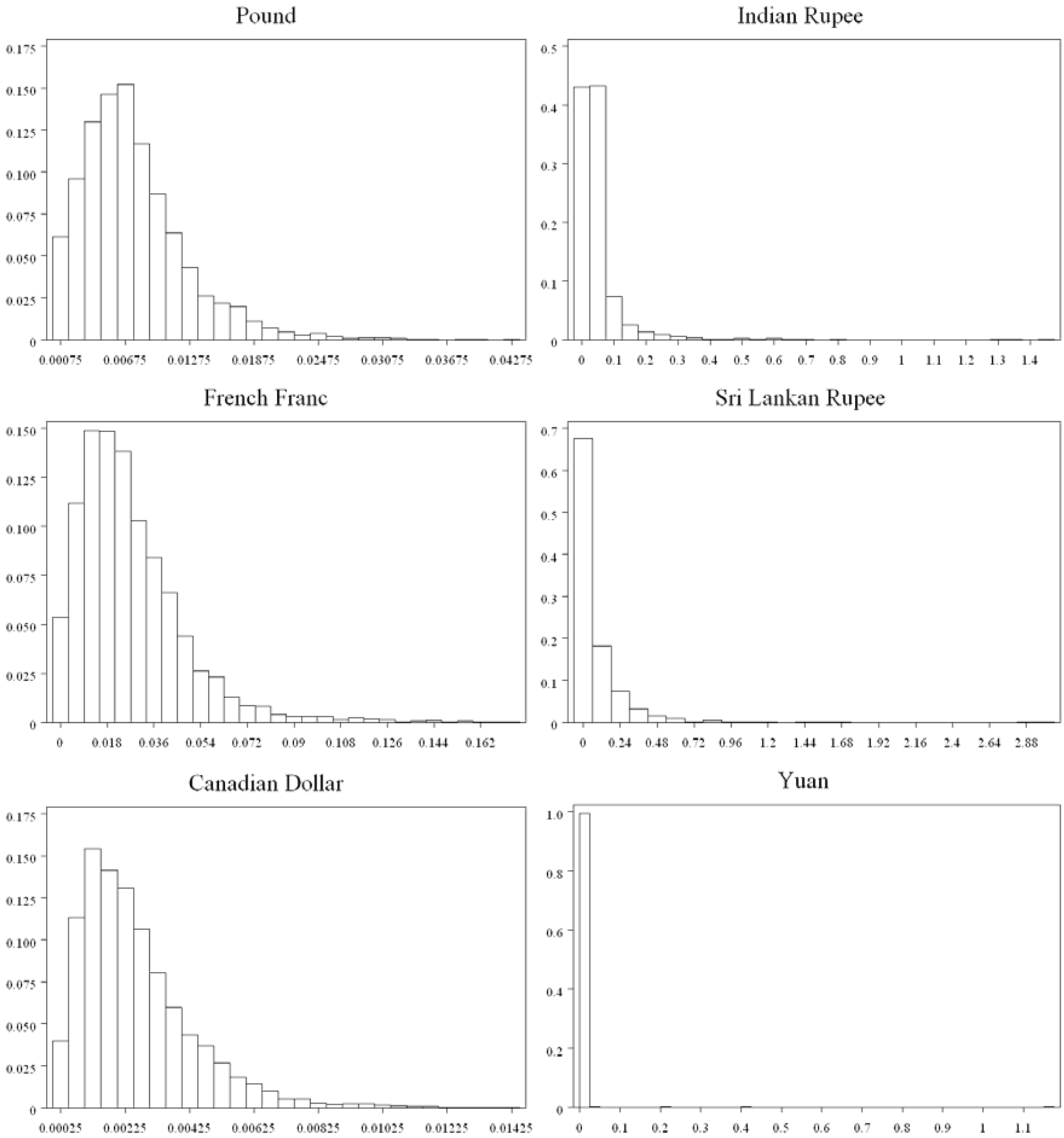

Figure 3. Distributions of the local volatilities of trading weeks ($q = 2$ in the corresponding formulas in Section 3).