# Optimal Stopping of a Risk Process when Claims are Covered immediately

Muciek, Bogdan K. and Szajowski, Krzysztof J.

AIG Credit SA, ul. Strzegomska 42c, 53-611 Wrocław, Poland,
Institute of Mathemtaics and Computer Science, Wrocław University
of Technology, Wybrzeze Wyspianskiego 27, 50-370 Wroclaw, Poland

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# OPTIMAL STOPPING OF A RISK PROCESS WHEN CLAIMS ARE COVERED IMMEDIATELY

BOGDAN K. MUCIEK AND KRZYSZTOF J. SZAJOWSKI

ABSTRACT. The optimal stopping problem for the risk process with interests rates and when claims are covered immediately is considered. An insurance company receives premiums and pays out claims which have occured according to a renewal process and which have been recognized by them. The capital of the company is invested at interest rate $\alpha \in \Re^+$, the size of claims increase at rate $\beta \in \Re^+$ according to inflation process. The immediate payment of claims decreases the company investment by rate $\alpha_1$. The aim is to find the stopping time which maximizes the capital of the company. The improvement to the known models by taking into account different scheme of claims payment and the possibility of rejection of the request by the insurance company is made. It leads to essentially new risk process and the solution of optimal stopping problem is different.

## 1. INTRODUCTION

The following problem in collective risk theory (see Rolski et al. (1998)) is considered. An insurance company, endowed with an initial capital $a > 0$, receives premiums and pays out claims that occur according to a renewal process $\{N(t), t \geq 0\}$, where $N(t)$ is the number of losses up till time $t$. The initial capital of the insurance company and received premiums are invested at a constant rate of return $\alpha \in \Re^+$. Let $T_0 = 0$ and let $T_i$ denotes the time of the $i$-th loss, then random variables $\zeta_i = T_i - T_{i-1}$ are independent and identically distributed (i.i.d.) with cumulative distribution function (cdf) $F$, such that $F(0) = 0$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables independent of $\{\zeta_i\}$, with cdf $H$ with $H(0) = 0$. The sequence $\{X_i\}_{i=1}^{\infty}$ represents values of successive claims. Usually the costs of damages elimination increase. It is modelled by the rate $\beta \in \Re^+$. If a claim appears

at moment $T_n$, then the company have to pay $X_n e^{\beta T_n}$. This amount of money decreases the company investment by rate $\alpha_1$ and, as a consequence of that, at the end of investment period $t$ the claim at $T_n$ decreases the capital by $X_n e^{\beta T_n} e^{\alpha_1 (t-T_n)}$. Although it may seem somewhat surprising at first glance, claims of size zero arise in some insurance contexts (see Panjer and Willmot (1992)). If a company records all claims as they are presented to the company and some claims are resisted, refused or a complete recovery of losses is made from another insurer, the net cost of the claim is zero. This effect is modelled by additional sequence of i.i.d random variables $\{\epsilon_i\}_{i=1}^{\infty}$, independent of claims size process and the process of moments of claims. It is assumed that $\mathbf{P}\{\epsilon_n = 1\} = p$ and $\mathbf{P}\{\epsilon_n = 0\} = 1-p$. The investigated process of capital assets of the insurance company is

$$
(1) \qquad U_t = ae^{\alpha t} + \int_0^t ce^{\alpha(t-s)} ds - \sum_{n=0}^{N(t)} \epsilon_n X_n e^{\beta T_n} e^{\alpha_1 (t-T_n)},
$$

where $a > 0$ is the initial capital, $c > 0$ is a constant rate of income from the insurance premiums, $X_0 = 0$ and $N(0) = 0$. The form of capital assets (1) reduces to

$$
(2) \qquad U_t = ae^{\alpha t} + ce^{\alpha t} \frac{1 - e^{-\alpha t}}{\alpha} - e^{\alpha_1 t} \sum_{n=0}^{N(t)} \epsilon_n X_n e^{\beta_1 T_n}
$$

where $\beta_1 = \beta - \alpha_1$. Let $g(u,t) = g_1(u) \mathrm{I}_{\{t \geq 0\}}$, where $g_1$ is a utility function. The return at time $t$ is $\{Z(t), t \geq 0\}$ and it is given by

$$
(3) \qquad Z(t) = g(U_t, t_0 - t) \prod_{j=0}^{N(t)} \mathrm{I}_{\{U_{T_j} > 0\}} = g(U_t) \mathrm{I}_{\{U_s > 0, s \leq t\}}
$$

The optimal stopping problem for the process $Z(t)$ is investigated. The model with $\alpha = \beta = 0$ have been considered by Ferenstein and Sierociński (1997). Jensen (1997) investigated a similar model with a claim process modulated by periodic Markovian processes but without care for time value of money, formulated in (2). When the claims are paid from the capital of the company, it can be assumed that $\alpha = \alpha_1$. Muciek (2002) investigated the model given by (1) with $\alpha_1 = 0$ which described the case when the claims were paid at the end of the investing period. The improvement introduced here, which takes into account the consequence of the immediate payment of claims, change the considered risk process essentially. The model admitted will have an impact on the form of the strong generator for the process (2) as well as on the form of the dynamic programming equations, which are the tools for describing the solution of the optimal stopping problem for (3).

The considered process $Z(t)$ is the piecewise-deterministic process. The methods of solving the optimal stopping problem for such processes can by found in papers by Boshuizen and Gouweleeuw (1993) and Jensen (1997); Jensen and Hsu (1993) and

the monography by Davis (1993). Muciek (2002) has solved the optimal stopping problem for process (3) with $\alpha_1 = 0$ which is not direct consequence of the optimal stopping problem solution for model (1) with $\alpha_1 \neq 0$.

The organization of the paper is following. In the next section the optimal stopping problem for the process (3) is formulated. The case of the optimal stopping up to the fixed number of claims is the subject of investigation in the section 3. The solution of the problem for the infinite number of claims is given in the section 4.

## 2. The optimization problem

In this section we define an optimization problem for the model introduced in the previous section. This optimization problem will be solved in the next section.

Let $\mathcal{F}(t) = \sigma(U_s, s \leq t) = \sigma(X_1, \epsilon_1, T_1, \ldots, X_{N(t)}, \epsilon_{N(t)}, T_{N(t)})$ be the $\sigma$-field generated by all the events up to time $t \geq 0$. Let $\mathcal{T}$ be the set of all stopping times with respect to the family $\{\mathcal{F}(t), t \geq 0\}$. Furthermore, for fixed $K$ and for $n = 0, 1, \ldots, k < K$ let $\mathcal{T}_{n,K}$ denote the subset of $\mathcal{T}$, such that $\tau \in \mathcal{T}_{n,K}$ if and only if $T_n \leq \tau \leq T_K$    a.s.

Let $\mathcal{F}_n = \mathcal{F}(T_n)$. The essence of the considerations in the next chapter will be to find the optimal stopping time $\tau_K^*$, such that $\mathsf{E}Z(\tau_K^*) = \sup\{\mathsf{E}Z(\tau) : \tau \in \mathcal{T}_{0,K}\}$. In order to find the optimal stopping time $\tau_K^*$, we first consider optimal stopping times $\tau_{n,K}^*$, such that

$$(4) \qquad \mathsf{E}(Z(\tau_{n,K}^*)|\mathcal{F}_n) = \operatorname{ess\,sup}\{\mathsf{E}(Z(\tau)|\mathcal{F}_n) : \tau \in \mathcal{T}_{n,K}\}$$

and using backward induction as in dynamic programming, we will obtain $\tau_K^* = \tau_{0,K}^*$

After finding the optimal stopping time $\tau_K^*$ for fixed $K$ we will deal with unlimited number of claims and the aim will be to find the optimal stopping time $\tau^*$, such that

$$(5) \qquad \mathsf{E}Z(\tau^*) = \sup\{\mathsf{E}Z(\tau) : \tau \in \mathcal{T}\}$$

is fulfilled. It will be shown that $\tau^*$ can be defined as the limit of the finite horizon optimal stopping times. Such a stopping time in an insurance company management can be used as the best moment to recalculate premium rate.

## 3. Case with fixed number of claims

In this section we find the form of optimal stopping time in the finite horizon case, which means the optimal stopping time in the class $\mathcal{T}_{0,K}$, where $K$ is finite and fixed (the number of claims is fixed, but the time of the $K$th claim, ie. time horizon, remains non-deterministic). This is a technical assumption which allows

calculations for finite number of claims and generalize this result to the infinite number of claims. First we present dynamic programming equations satisfying $\Gamma_{n,K} = \operatorname{ess\,sup}\{\mathsf{E}(Z(\tau)|\mathcal{F}_n) : \tau \in \mathcal{T}_{n,K}\}, \qquad n = K, K-1, \ldots, 1$. Then in Corollary 3.3 we find optimal stopping times $\tau_{n,K}^*$ and $\tau_K^*$ and optimal mean values of return related to them.

The following representation lemma (see for example Davis (1976)) plays the crucial role in consequent considerations:

**Lemma 3.1.** *If $\tau \in \mathcal{T}_{n,K}$, there exists a positive, $\mathcal{F}_n$-measurable random variable $\xi$, such that $\tau \wedge T_{n+1} = (T_n + \xi) \wedge T_{n+1}$ a.s.*

Let $\mu_0 = 1$ and $\mu_n = \prod_{j=1}^{n} \mathrm{I}_{\{U_{T_j} > 0\}}$. Then $\Gamma_{K,K} = Z(T_K) = g(U_{T_K}, t_0 - T_K)\mu_K$. Note that the sum of claims from (2) can be expressed as

$$(6) \qquad \sum_{n=0}^{N(t)} \epsilon_n X_n e^{\beta_1 T_n} = \left(ae^{\alpha t} + \frac{c}{\alpha}(e^{\alpha t} - 1) - U_t\right)e^{-\alpha_1 t}$$

Let us define for $\xi > 0$ such that there is no jump between $t$ and $t + \xi$

$$(7) \qquad d(t, \xi, U_t) \quad = \quad U_{t+\xi} - U_t = e^{\alpha t}\left(a + \frac{c}{\alpha}\right)\left(e^{\alpha \xi} - e^{\alpha_1 \xi}\right)$$
$$+ \frac{c}{\alpha}\left(e^{\alpha_1 \xi} - 1\right) + \left(e^{\alpha_1 \xi} - 1\right)U_t,$$

then we have

$$(8) \qquad \mu_K = \mu_{K-1}\mathrm{I}_{\{U_{T_{K-1}} + d(T_{K-1}, \zeta_K, U_{T_{K-1}}) - \epsilon_K X_K e^{\beta(T_{K-1}+\zeta_K)} > 0\}}.$$

Similarly as in Muciek (2002), Theorem 1, from (6) and from (7) we get the following dynamic programming equations:

**(i):** For $n = K - 1, K - 2, \ldots, 0$,

$$\Gamma_{n,K} \quad = \quad \operatorname{ess\,sup}\left\{\mu_n \bar{F}(\xi)g\left(U_{T_n} + d(T_n, \xi, U_{T_n}), t_0 - T_n - \xi\right)\right.$$
$$\left. + \mathsf{E}(\mathrm{I}_{\{\xi \geq \zeta_{n+1}\}}\Gamma_{n+1,K}|\mathcal{F}_n) : \xi \geq 0 \text{ is } \mathcal{F}_n\text{-measurable}\right\} \quad \text{a.s.,}$$

where $\bar{F} = 1 - F$ is the survival function.

**(ii):** For $n = K, K-1, \ldots, 0$, $\Gamma_{n,K} = \mu_n \gamma_{K-n}(U_{T_n}, T_n)$ a.s., where the sequence of functions $\{\gamma_j(u, t), u \in \mathbb{R}, t \geq 0\}$, using (7), (6) and (8) is defined as follows

$$\gamma_0(u, t) = g(u, t_0 - t),$$
$$\gamma_j(u, t) = \sup_{r \geq 0}\left[\bar{F}(r)g(u + d(t, r, u), t_0 - t - r)\right.$$
$$+ p\int_0^r dF(s)\int_0^{e^{-\beta(t+s)}(u+d(t,s,u))} \gamma_{j-1}\left(u + d(t, s, u) - xe^{\beta(t+s)}, t + s\right)dH(x)$$
$$\left. + (1-p)\int_0^r \gamma_{j-1}\left(u + d(t, s, u), t + s\right)H(e^{-\beta(t+s)}(u + d(t, s, u)))dF(s)\right]$$
$$j = 1, 2, \ldots$$

The above equations differ from the ones in Theorem 1 in Muciek (2002) as a result of a different form of the capital assets process $U_t$.

The next step is to find the optimal stopping time $\tau_K^*$. To do this we should analyze the properties of the sequence of functions $\{\gamma_n, n \geq 0\}$. Let $B = B[(-\infty, +\infty) \times [0, +\infty)]$ be the space of all bounded and continuous functions with the norm $||\delta|| = \sup_{u,t} |\delta(u,t)|$ and let $B^0 = \{\delta : \delta(u,t) = \delta_1(u,t) \mathrm{I}_{\{t \leq t_0\}} \text{ and } \delta_1 \in B\}$. One should notice that the functions $\{\gamma_n, n \geq 0\}$ are included in $B^0$. For each $\delta \in B^0$ and any $u \in \mathbb{R}$, $t, r \geq 0$ let

$$
\begin{aligned}
\phi_\delta(r, u, t) \quad = \quad & \bar{F}(r) g(u + d(t, r, u), t_0 - t - r) + \\
& + (1 - p) \int_0^r \delta(u + d(t, s, u), t + s) H(e^{-\beta(t+s)}(u + d(t, s, u))) dF(s) \\
& + p \int_0^r dF(s) \int_0^{e^{-\beta(t+s)}(u+d(t,s,u))} \delta(u + d(t, s, u) - x e^{\beta(t+s)}, t + s) dH(x).
\end{aligned}
$$

From the properties of the cumulative distribution function $F$ we know that $\phi_\delta(r, u, t)$ has at most a countable number of points of discontinuity according to $r$ and is continuous according to $(u, t)$ in the case of $g_1(\cdot)$ being continuous and $t \neq t_0 - r$. Therefore, for further considerations we assume that the function $g_1(\cdot)$ is bounded and continuous.

For each $\delta \in B^0$ let

(9)
$$(\Phi\delta)(u, t) = \sup_{r \geq 0}\{\phi_\delta(r, u, t)\}.$$

**Lemma 3.2.** *For each $\delta \in B^0$ we have*

$$(\Phi\delta)(u, t) = \max_{0 \leq r \leq t_0 - t}\{\phi_\delta(r, u, t)\} \in B^0$$

*and there exists a function $r_\delta(u, t)$ such that $(\Phi\delta)(u, t) = \phi_\delta(r_\delta(u, t), u, t)$.*

In subsequent considerations more properties of $\Phi$ will be presented.

For $i = 1, 2, \ldots$ and $u \in \mathbb{R}$, $t \geq 0$, $\gamma_i(u, t)$ may be expressed as follows

$$
\gamma_i(u, t) = \begin{cases} (\Phi\gamma_{i-1})(u, t) & \text{if } u \geq 0 \text{ and } t \leq t_0, \\ 0 & \text{otherwise,} \end{cases}
$$

and from Lemma 3.2 there exist functions $r_{\gamma_{i-1}}$ such that

$$
\gamma_i(u, t) = \begin{cases} \phi_{\gamma_{i-1}}(r_{\gamma_{i-1}}, u, t) & \text{if } u \geq 0 \text{ and } t \leq t_0, \\ 0 & \text{otherwise.} \end{cases}
$$

To specify the form of the optimal stopping times $\tau_{n,K}^*$, we need to define the following random variables $R_i^* = r_{\gamma_{K-i+1}}(U_{T_i}, T_i)$ and $\sigma_{n,K} = K \wedge \inf\{i \geq n : R_i^* < \zeta_{i+1}\}$.

Finally in Corollary 3.3 we present the form of the optimal stopping time.

**Corollary 3.3.** *Let*

$$\tau^*_{n,K} = T_{\sigma_{n,K}} + R^*_{\sigma_{n,K}} \quad and \quad \tau^*_K = \tau^*_{0,K},$$

*then for all $0 \le n \le K$ the following hold*

$$\Gamma_{n,K} = \mathsf{E}(Z(\tau^*_{n,K})|\mathcal{F}_n) \ a.s. \quad and \quad \Gamma_{0,K} = \mathsf{E}(Z(\tau^*_K)) = \gamma_K(a,0),$$

*which means $\tau^*_{n,K}$ and $\tau^*_K$ are optimal stopping times in the classes $\mathcal{T}_{n,K}$ and $\mathcal{T}_{0,K}$ respectively.*

## 4. Case with an infinite number of claims

While $\mathcal{T}$ is the set of all stopping times with respect to the family $\{\mathcal{F}(t), t \ge 0\}$, we would like to maximize the mean return (3), i.e. to find the optimal stopping time $\tau^*$, such that

$$(10) \qquad\qquad \mathsf{E}Z(\tau^*) = \sup\{\mathsf{E}Z(\tau) : \tau \in \mathcal{T}\}$$

is fulfilled. It will be shown that $\tau^*$ can be defined as the limit of the finite horizon optimal stopping times.

Let $\tau^*_n$ be an optimal stopping time taken from the set of stopping times which occur not earlier than at $T_n$, the time of $n$-th claim, ie. $\mathsf{E}Z(\tau^*_n|\mathcal{F}_n) = \sup\{\mathsf{E}Z(\tau|\mathcal{F}_n) : \tau \in \mathcal{T} \cap \{\tau : \tau \ge T_n\}\}$.

The solution of this case will be based on the iteration of the operator $\Phi$ defined by (9). By an assumption that the interarrival time is greater than $t_0$ with non-zero probability it can proved that the operator $\Phi$ is a contraction and it has a fixed point.

The essence of this section is contained in the following theorem. The proof is based on the proof of the existence of optimal stopping times for semi-Markov processes presented by Boshuizen and Gouweleeuw (1993).

**Theorem 4.1.** *Assuming the utility function $g_1$ is differentiable and nondecreasing and $F$ has a density function $f$ we have*

- *(i) for $n = 0, 1, \ldots$, the limit $\tilde{\tau}_n := \lim_{K \to \infty} \tau^*_{n,K}$ exists and $\tilde{\tau}_n$ is an optimal stopping time in $\mathcal{T} \cap \{\tau : \ \tau \ge T_n\}$ ($\tau^*_{n,K}$ is a solution of the case with finite number of claims defined by (4)),*
- *(ii) $\mathsf{E}[Z(\tilde{\tau}_n)|\mathcal{F}_n] = \mu_n \gamma(U_{T_n}, T_n) \quad a.s.$*

The optimality of $\tilde{\tau}_n$ may be proved in a similar way as in Boshuizen and Gouweleeuw (1993). The details may be found in Muciek (2002). Let us denote $\eta_t = (t, U_t, Y_t, V_t)$, where $Y_t = t - T_{N(t)}$, $V_t = \mu_{N(t)}$, $t \ge 0$. For $\tilde{g}$ such that

$Z(t) = \tilde{g}(\eta_t)$ the strong generator $A$ of $\eta_t$ on $\tilde{g}$ has the form (see Gihman and Skorokhod (1975) )

$$
\begin{aligned}
(A\tilde{g})(t,u,y,v) = & \left\{\left[(\alpha a + c)e^{\alpha t} - (\alpha_1 a + c\frac{\alpha_1}{\alpha})e^{\alpha t} + (\frac{\alpha_1}{\alpha}c + \alpha_1 u)\right]g_1'(u) \right. \\
& -\frac{f(y)}{\bar{F}(y)}\left[g_1(u)\bar{H}(ue^{-(\alpha_1 y + \beta(t-y))})\right. \\
& -p\int_0^{ue^{-(\alpha_1 y+\beta(t-y))}} g_1(u - xe^{\alpha_1 y + \beta(t-y)})dH(x) \\
& \left.\left.+pg_1(u)\,H(ue^{-(\alpha_1 y+\beta(t-y))})\right]\right\}v,
\end{aligned}
$$

where $t < t_0$, $y \geq 0$ and $v \in \{0,1\}$. Thus we get

$$
\begin{aligned}
(A\tilde{g})(\eta_s) = & \left\{\left[(\alpha a + c)e^{\alpha s} - (\alpha_1 a + c\frac{\alpha_1}{\alpha})e^{\alpha s} + (\frac{\alpha_1}{\alpha}c + \alpha_1 U_s)\right]g_1'(U_s) \right. \\
& + \frac{f(s - T_{N(s)})}{\bar{F}(s - T_{N(s)})}\left[p\int_0^{U_s e^{-(\alpha_1(s-T_{N(s)})+\beta T_{N(s)})}} g_1(U_s - xe^{\alpha_1(s-T_{N(s)})+\beta T_{N(s)}})dH(x) \right. \\
& -pg_1(U_s)H(U_s e^{-(\alpha_1(s-T_{N(s)})+\beta T_{N(s)})}) \\
& \left.\left.- \bar{H}(U_s e^{-(\alpha_1(s-T_{N(s)})+\beta T_{N(s)})})g_1(U_s)\right]\right\}\mu_{N(s)}.
\end{aligned}
$$

It should also be marked that the limit of optimal stopping times as $K \to \infty$ coincide with the overall optimal stopping time.

## 5. Final remarks

The presented results generalize known solutions of the optimal stopping problems for the risk process (see Ferenstein and Sierociński (1997), Muciek (2002)) to more realistic models of risk reserve processes. For $\alpha = \beta = 0$ the solution presented by Ferenstein and Sierociński (1997) are obtained. The model considered by Muciek (2002) is not direct consequence of model (1). It implies that the solution of the optimal stopping problem for the risk reserve process investigated in Muciek (2002) is not simple conclusion from the formulae of Corollary 3.3 and Section 4.

There are also similar optimal stopping problems considered by Yasuda (1984) and Schöttl (1998). The solution of the problem (5) is not direct consequence of the results from neither Yasuda (1984) nor Schöttl (1998).

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INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE, WROCŁAW UNIVERSITY OF TECHNOLOGY, WYBRZEŻE WYSPIAŃSKIEGO 27, WROCŁAW, POLAND

*Current address*: AIG Credit SA, ul. Strzegomska 42c, 53-611 Wroclaw, Poland

*E-mail address*: `Bogdan.Muciek@aigcredit.pl`

*URL*: `http://www.im.pwr.wroc.pl/~muciek`

(Corresponding author) INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE, WROCŁAW UNIVERSITY OF TECHNOLOGY, WYBRZEŻE WYSPIAŃSKIEGO 27, WROCŁAW, POLAND

*E-mail address*: `Krzysztof.Szajowski@pwr.wroc.pl`

*URL*: `http://neyman.im.pwr.wroc.pl/~szajow`