Optimal Cash Management Under Uncertainty

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Abstract

The problem of optimal investment in two types of assets over time is formulated as a stochastic optimal control problem. The two assets considered are a bank account and stock. The earnings derived from stock consist of dividends and capital gains. The randomness in the return on stock is modeled using a standard Brownian motion. Using a stochastic maximum principle, an explicit decision rule of the bang-bang type is derived for optimal management of cash.

Keywords: Cash management, stochastic control, maximum principle, risky assets.

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1 Introduction

Firms need cash to manage day-to-day operations. Demand for cash can be positive or negative. Positive demand consists of accounts payable, whereas negative demand is known as account receivables. Cash can be held in bank account earning interest or invested in risky securities (stocks or bonds) earning possibly a higher mean rate of return than the bank account. For simplicity, we will refer to the securities as simply stock. Transfer of cash from bank to stock and vice versa incurs a broker's commission. The problem of managing the operating cash to meet demand for cash at minimum cost is known as the cash balance or cash management problem.

The cash balance problem has often been treated in the literature as an inventory problem. Deterministic models of inventory type have been developed by Baumol [1] and Tobin [21]. Sethi and Thompson [17] solved a deterministic cash balance problem for a firm with time-varying demand for cash. They considered two kinds of assets in their model and applied the maximum principles in discrete as well as continuous time to maximize the terminal value of the total assets.

Works on stochastic cash balance models include Miller and Orr [15], Dallenbech [7], Eppen and Fama [9, 10], Constantinides [4, 5], and Vickson [23]. These papers consider an inventory approach to cash management where the stochastic nature appears in the demand for money. There are examples of asset prices modeled as stochastic processes, but most of these find applications in portfolio consumption setting rather than in a cash management framework. Examples include Merton [14], Karatzas et al. [13], Davis and Norman [8]; see also Sethi [18].

In this paper, we consider a cash management problem involving two types of assets, namely deposits in a bank account and investments in stock. We present an extension of the Sethi and Thompson [17] model to allow for cash dividends and uncertain capital gains on stock. This formulation applies to a firm that wishes to control its level of cash balance to meet its demand rate $d(t)$ for cash over time at minimum total cost, or equivalently to maximize the terminal value of its cash and stock holdings. It is also useful for a person who needs money at the rate $d(t)$, keeps the money either in stock or in a bank account, and wishes to maximize total holdings at his retirement age.

Sethi and Thompson [17, 19] used Pontryagin's maximum principle to derive optimal policies for cash management. We shall use a stochastic maximum principle to solve the proposed extension. In Section 2, we consider first a deterministic extension by assuming that the stock pays dividends in addition to appreciating at a certain average rate. This is a straightforward extension of the Sethi-Thompson model. In Section 3, we allow the rate of growth in the price of stock to be random. For analysis, we use the stochastic maximum principle proposed by Bensoussan [3] in his unpublished lecture notes.

We should mention that most of the stochastic optimal control problems in the literature use the dynamic programming approach resulting in Hamilton-Jacobi equations to be solved. However, there are a few exceptions where applications of the stochastic maximum principle have resulted in elegant solutions. These applications that include Benes [2] and Haussmann [11] are in the field of engineering or mathematics. Our use of the stochastic maximum principle is perhaps the first such application in the areas of management science and economics. We have accomplished this by keeping our cash balance formulation to be quite simple, and thus our paper also serves as an introduction of the stochastic maximum principle to researchers working in these areas.

2 Model with Dividends on Stock

In this section, we develop a deterministic model that closely resembles the Sethi-Thompson model, by extending theirs to allow for a dividend paying stock.

Consider a firm that invests its cash in stock or in a bank account. The amount invested in the bank account at time t is $x(t)$ and the amount invested in stock is $y(t)$, $t \in [0, T]$, where T defines the length of the planning horizon. The interest rate earned on the bank account is $r_1(t)$ and the returns derived from stock at time t takes two forms:

capital gains rate of $r_2(t)$ and cash dividend rate $r_3(t)$. The firm has a demand rate $d(t)$ for cash at time t. The demand $d(t)$ can be positive or negative.

The control variable is the sale of stock at a rate $u(t)$ at time t, where a negative sale represents a purchase. Moreover, the control $u(t)$ is bounded, i.e.,

$$-M_1(t) \leq u(t) \leq M_2(t), \text{ where } M_1(t) > 0 \text{ and } M_2(t) > 0.$$

For each unit of stock that is bought or sold, the firm pays a broker's commission at a cost of α dollars per dollar worth of stock, $0 < \alpha < 1$. We can now write the state equations as follows:

$$\frac{dx}{dt} = r_1(t)x(t) - d(t) + u(t) - \alpha|u| + r_3(t)y(t), \qquad x(0) = x_0, \tag{1}$$

$$\frac{dy}{dt} = r_2(t)y(t) - u(t), \qquad y(0) = y_0. \tag{2}$$

We do not impose any constraints on the state variables $x(t)$ and $y(t)$. This means that overdrafts on cash and short-selling of stock are allowed.

The objective function is:

$$\text{Max} \quad [x(T) + y(T)]. \tag{3}$$

This objective function is equivalent to minimizing the net cost of managing the cash balances during the planning period $[0, T]$.

Next we introduce the adjoint functions $p_1(t)$ and $p_2(t)$ to define the Hamiltonian

$$H = p_1(r_1 x - d + u - \alpha|u|r_3 y) + p_2(r_2 y - u), \tag{4}$$

where $p_1(t)$ and $p_2(t)$ satisfy the adjoint equations

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x} = -p_1(t)r_1(t), \qquad p_1(T) = 1, \tag{5}$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial y} = -p_2(t)r_2(t) - p_1(t)r_3(t), \qquad p_2(T) = 1. \tag{6}$$

According to Pontryagin's maximum principle (see, e.g., Sethi and Thompson [19]), we solve the problem (1)-(6) by maximizing (4) with respect to u. From (5) we can solve for $p_1(t)$:

$$p_1(t) = e^{\int_t^T r_1(s)\, ds}, \tag{7}$$

and then we can use (6) and (7) to solve for $p_2(t)$

$$p_2(t) = e^{\int_t^T r_2(s)\, ds} - \int_t^T d\tau\, r_3(\tau)\, e^{\left(\int_\tau^T r_1(s)\, ds + \int_t^\tau r_2(s)\, ds\right)}. \tag{8}$$

Remark. When there are no dividends ($r_3(t) = 0$), our model reduces to the Sethi-Thompson model.

The optimal policy is now derived by selecting the control variable such that the Hamiltonian (4) is maximized. Since the Hamiltonian is a linear function of u, the solution will be of "bang-bang" type. We see from (4) that the only terms involving u in the Hamiltonian are:

$$W = p_1 u - p_1 \alpha |u| - p_2 u. \tag{9}$$

For positive values of u, the right-hand side of (9) can be written as $u(p_1 - p_1\alpha - p_2)$, and for $p_1(1-\alpha) > p_2$, we would make u as large as possible in order to make (9) as large as possible. For negative values of u, the right hand side of (9) can be written as $u(p_1 + p_1\alpha - p_2)$, and if $p_1(1+\alpha) < p_2$, we would make u as small as possible.

We therefore obtain the following decision rule:

$$u(t) = \begin{cases} -M_1 & \text{if } p_1(t)(1-\alpha) > p_2(t), \\ 0 & \text{if } p_1(t)\,(1-\alpha) < p_2(t) < p_1(t)(1+\alpha), \\ M_2 & \text{if } p_1(t)(1+\alpha) < p_2(t). \end{cases} \tag{10}$$

It is interesting to observe that the adjoint variable $p_1(t)$ denotes the marginal return in the interval $[t, T]$ from a dollar invested in bank account at time t. Similarly, $p_2(t)$ denotes the same for a dollar invested in stock at time t. Thus, the optimal policy requires us to sell stock at the maximum rate if the future value of $(1-\alpha)$ dollars in bank account is more than that of 1 dollar in stock. Similarly, if the future value of one dollar in stock is more than the future value of the cash used for purchasing one dollar of stock $(1+\alpha)$, then it is optimal to buy stock at the maximum rate.

3 Model with Uncertain Capital Gains

We now introduce one main factor of randomness in the model presented in the previous section, namely, that the capital gains rate on stock is a stochastic process. To distinguish it from $r_2(t)$ in Section 2, we denote it as $R_2(t)$ in this section. Thus, we can modify the state equations (1) and (2) as follows:

$$\frac{dx}{dt} = r_1 x - d + u - \alpha|u| + r_3 y, x(0) = x_0, \quad (11)$$

$$\frac{dy}{dt} = R_2(t)y(t) - u(t), y(0) = y_0, \quad (12)$$

where $R_2(t)$ is a stochastic process. We assume that the system is fully observed, so that at any time t, we know the realization of $x(t)$, $y(t)$ and $R_2(t)$. The objective function (3) changes to

$$\text{Max } E\left[x(T) + y(T)\right]. \quad (13)$$

Next, we develop a model to describe the process $R_2(t)$. For this, we look into the empirical research on the stock price variations carried by various researchers. Nearly a century ago, Bachelier (see Cootner [6]) suggested that the first differences of the prices should be normally distributed with zero mean. Simmons [20] tested the model

$$G_t = \gamma + \beta_1 G_{t-1} + \cdots + \beta_m G_{t-m} + \epsilon_t, \quad (14)$$

where G_t is the relative increase in the stock price in period t, γ and β are constants, and ϵ_t is a random disturbance term with zero mean. He found that G_t is not independent of the previous price changes.

For our formulation, we take as its basis a first approximation of [14], namely,

$$G_t - G_{t-1} = \gamma_t + (\beta_t - 1)G_{t-1} + \bar{\epsilon}_t, \quad (15)$$

where $\bar{\epsilon}_t$ is a normally distributed disturbance with zero mean and where we, furthermore, allow γ_t and β_t to depend on time t to allow for business cycles. Thus, e.g., γ_t can be negative in bearish times. By the definition of $G_t = \frac{(P_t - P_{t-1})}{P_{t-1}}$, where P_t is the stock price at time t, we see that G_t is the discrete-time analog of $R_2(t)$. Thus, we express the following differential equation in $R_2(t)$ as the continuous-time analog of (15):

$$\frac{dR_2(t)}{dt} = \beta(t)R_2(t) + f(t) + \eta(t), \qquad R_2(0) = R_2^0, \tag{16}$$

where $\beta(t)$ and $f(t)$ are known functions of time, and $\eta(t)$ is a random disturbance with zero mean and variance $q(t)$. We assume further that $\eta(t)$ is the white noise process, a gaussian process non-correlated in time, i.e., for any $t_1, t_2,\ t_2 > t_1$, we have

$$E\eta(t_1)\eta(t_2) = q(t_1)\delta(t_2 - t_1), \tag{17}$$

where δ is the Dirac function. Equivalently, we can write the state equation (16) as

$$dR_2(t) = (f(t) + \beta(t)R_2(t))dt + \sigma(t)\,dW(t), \quad R_2(0) = R_2^0, \tag{18}$$

where $W(t)$ is the standard Weiner process and $\sigma^2(t) = q(t)$.

Note that this modeling of $R_2(t)$ is very similar to the modeling of stochastic interest rates in the literature (see, e.g., Vasicek [22] and Hull and White [12]).

3.1 A Stochastic Maximum Principle

In this section, we briefly recapitulate the stochastic maximum principle developed by Bensoussan [3]. To formulate a general stochastic control problem, let (Ω, F, P) be the probability space and $\mathfrak{F}^t$ be an increasing family of sub σ-algebras of F. Thus, $\mathfrak{F}^t$ represents the information up to time t. Since we have a finite horizon of length T, we have $\mathfrak{F}^T \subset F$. Let $W(t)$ be a standard $\mathfrak{F}^t$ Weiner process with values in $\mathbb{R}^n$. Note that $W(t)$ is an $\mathfrak{F}^t$ martingale.

The state equation is given by the Ito process

$$dx(t) = g(x(t), u(t), t)dt + \sigma(x(t), t)dW(t), \qquad x(0) = x_0, \tag{19}$$

where $x(t)$ and $u(t)$ denote the state and control variables, respectively, x_0 is a deterministic initial condition, $g : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}^n$, and $\sigma : \mathbb{R}^n \times [0,T] \to \mathfrak{L}(\mathbb{R}^n; \mathbb{R}^n)$.

We now define the cost functional as follows. Let $l(x,u,t) : \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \to \mathbb{R}$ be a Borel and continuously differentiable function with respect to (x,u). Let $h(x)$ be continuously differentiable with $|h(x)| \le c(|x|+1)$. The objective to be minimized is defined as

$$J(u) = E\left[\int_0^T l(x(t),u(t),t)\,dt + h(x(T))\right]. \tag{20}$$

The adjoint equations for this system can be written as the stochastic differential equation

$$-\frac{dp}{dt} = \Phi'\, g_x'\, \Psi'\, p - \Phi'\, l_x(x(t),u^*(t),t), \;\; p(T) = -\Phi'(T)\, h_x(x(T)), \tag{21}$$

where Φ and Ψ satisfy the matrix stochastic differential equations

$$d\Phi = \langle \sigma_x\, \Phi, dW \rangle, \qquad \Phi_0 = I, \tag{22}$$

$$d\Psi = -\langle \Psi\, \sigma_x, dW \rangle + \langle \Psi\, \sigma_x, \sigma_x \rangle\, dt, \qquad \Psi_0 = I. \tag{23}$$

In this notation A' denotes the transpose of matrix A , f_x denotes the derivative of function f with respect to x, and $< .,. >$ denotes the matrix inner product. We define the adjoint variable to be

$$\Pi(t) = \Psi'(t)\, E^{\mathfrak{F}^t} p(t), \tag{24}$$

where the conditional expectation $E^{\mathfrak{F}^t} p(t) = E[p(t)/\mathfrak{F}^t]$.

According to the maximum principle, the necessary condition for u^* to be an optimal control for the minimization of $J(u)$ is

$$l(y(t),u^*(t),t) + \Pi(t)\, g(y(t),u^*(t),t) \le l(y(t),u(t),t) + \Pi(t)\, g(y(t),u(t),t), \tag{25}$$

a.e. t, a.s. u, where $y(.)$ denotes the trajectory of the state variable obtained by using the control u^*.

We now apply this maximum principle to our problem defined by equations (11)-(13) and (16)-(18). The adjoint equations can be written as

$$-\frac{dp_1}{dt} = p_1(t)r_1(t), p_1(T) = 1, \tag{26}$$

$$-\frac{dp_2}{dt} = p_1(t)r_3(t) + z(t)p_2(t), p_2(T) = 1, \tag{27}$$

$$-\frac{dp_3}{dt} = p_2(t)y(t) + p_3(t)\beta(t), p_3(T) = 0, \tag{28}$$

where $\sigma(x,t)$ is a (3×1) matrix $(0, 0, \sigma(t))'$, and thus Φ and Ψ reduce to $\Phi(t) = I$ and $\Psi(t) = I$. Clearly, $p_1(t)$ is deterministic and is given by exactly the same function as in the deterministic case treated in Section 2. However, $p_2(t)$ and $p_3(t)$ are stochastic in nature. The adjoint variables are

$$\pi_1(t) = p_1(t), \quad \pi_2(t) = E^{\mathfrak{F}^t} p_2(t), \quad \text{and} \quad \pi_3(t) = E^{\mathfrak{F}^t} p_3(t).$$

The stochastic nature of $p_2(t)$ depends only on R_2 between t and T. At time t, we can observe the realizations of x, y and R_2 on $(0,t]$. But x and y get their stochastic nature only from R_2 through the feedback law. In other words, for a given feedback law, $x(0,t)$ and $y(0,t)$ are functions of $R_2(0,t)$. Here $x(0,t)$, $y(0,t)$, and $R_2(0,t)$ denote the trajectories of x, y, and R_2, respectively, from time 0 to t. Thus, if $R_2(0,t)$ is known, the observations of $x(0,t)$ and $y(0,t)$ add no further information. Since $p_2(t)$ is a function of $R_2(t,T)$ and since the realizations of $R_2(t,T)$ depend only on the value $R_2(t)$ at time t, we can therefore write $E^{\mathfrak{F}^t} p_2(t) = E^{R_2(0,t)} p_2(t) = E^{R_2(t)} p_2(t)$.
Thus, our optimization problem reduces to

$$\mathrm{Max}_{[u| \; -M_1 \le u \le M_2]} \; p_1(t)(u - \alpha|u|) - u\, E^{R_2(t)} p_2(t). \tag{29}$$

The decision rule arising from the problem is of "bang-bang" type and is similar to the one in Section 2. It can be expressed as follows:

$$u(t) = \begin{cases} -M_1 & \text{if } p_1(t)(1-\alpha) > E^{R_2(t)} p_2(t), \\ 0 & \text{if } p_1(t)(1-\alpha) < E^{R_2(t)} p_2(t) < p_1(t)(1+\alpha), \\ M_2 & \text{if } p_1(t)(1+\alpha) < E^{R_2(t)} p_2(t). \end{cases} \tag{30}$$

Compared to the deterministic setting in Section 2, the only difference consists in replacing $p_2(t)$ by $E^{R_2(t)}p_2(t)$ and then applying the same decision rule.

By definition, the adjoint variables corresponding to $x(t)$ and $y(t)$ are $p_1(t)$ and $E^{\mathfrak{F}^t}p_2(t)$, respectively. Thus, the economic interpretation of the optimal policy is similar to that in deterministic case, the difference being that now the *expected* value of investing one dollar in stock must be considered. If the expected return by selling 1 dollar of stock (which is same as having $(1-\alpha)$ dollars in bank) is more than keeping the dollar in stock at any time t, then sell at the maximum rate at that time. Conversely, if the expected return from buying 1 dollar worth of stock is more than keeping $(1+\alpha)$ dollars in bank at any time t, then buy stock at the maximum rate at that time.

3.2 Calculation of $E^{R_2(t)}p_2(t)$

Let us introduce $r_2(t)$ as the solution of

$$\frac{dr_2}{dt} = \beta r_2(t) + f(t), \qquad r_2(0) = r_2^0, \tag{31}$$

which is thus the mean value of $R_2(t)$. Therefore, one may write $R_2(t) = r_2(t) + \xi(t)$, where $\xi(t)$ is the solution of

$$\frac{d\xi}{dt} = \beta(t)\xi(t) + \eta(t), \qquad \xi(0) = 0. \tag{32}$$

Since $r_2(t)$ is a known function, observation of $R_2(t)$ is equivalent to the observation of $\xi(t)$. The problem amounts to the following. Let us consider the process p_2 on $(t, T]$ as the solution of

$$-\frac{dp_2}{d\tau} = r_3(\tau)p_1(\tau) + r_2(\tau)p_2(\tau) + \xi(\tau)p_2(\tau), \;\; p_2(T) = 1, \tag{33}$$

where $\xi(\tau)$ is the solution of

$$\frac{d\xi}{d\tau} = \beta\xi + \eta, \;\; \tau \in (t, \tau), \;\; \xi(t) = \bar{\xi}_t \;\; \text{(given)}. \tag{34}$$

In order to calculate $E^{\bar{\xi}_t} p_2(t)$, let us set $g(t) = r_3(t)p_1(t)$. It is easy to show that $p_2(t)$ is given by the formula $p_2(t) = e^{\int_t^T (r_2(s)+\xi(s))\,ds} - \int_t^T d\tau g(\tau) e^{\int_t^T (r_2(s)+\xi(s))\,ds}$. If we introduce $\lambda(t,\tau) = e^{\int_t^\tau (r_2(s)+\xi(s))\,ds}$, we can rewrite $p_2(t) = \lambda(t,T) - \int_t^T d\tau\, g(\tau)\, \lambda(t,\tau)$. Since g is deterministic, it then follows that

$$E^{\bar{\xi}_t} p_2(t) = E^{\bar{\xi}_t} \lambda(t,T) - \int_t^T d\tau\, g(\tau)\, E^{\bar{\xi}_t}\, \lambda(t,\tau). \tag{35}$$

Now, our problem amounts to finding the value of $E^{\bar{\xi}_t} \lambda(t,T)$, for any pair (t,τ), $\tau \geq t$. For this, we decompose $\xi(s)$ as $\xi(s) = \xi_1(s) + \xi_2(s)$, where $\xi_1(s)$ is the solution of

$$\frac{d\xi_1(s)}{ds} = \beta\, \xi_1(s), \quad s \in (t,T], \quad \xi_1(t) = \bar{\xi}_t, \tag{36}$$

and $\xi_2(s)$ is the solution of

$$\frac{d\xi_2(s)}{ds} = \eta(s), \quad s \in (t,T], \;\; \xi_2(t) = 0. \tag{37}$$

Moreover, $\xi_2(s)$, $s \geq t$, is independent of $\bar{\xi}_t$, since $\eta(s)$, $s \geq t$, is indeed independent of $\bar{\xi}_t$. Thus, going back to the definition of $\lambda(t,\tau)$, we get

$$E^{\bar{\xi}_t} \lambda(t,\tau) = E^{\bar{\xi}_t}\, e^{\int_t^\tau r_2(s)\,ds}\, e^{\int_t^\tau \xi_1(s)\,ds}\, e^{\int_t^\tau \xi_2(s)\,ds}. \tag{38}$$

Now in (38), $e^{\int_t^\tau \xi_1(s)ds}$ and $e^{\int_t^\tau \xi_2(s)ds}$ are two independent variables and $e^{\int_t^\tau \xi_1(s)ds}$ is a function of $\bar{\xi}_t$. Thus, we get

$$E^{\bar{\xi}_t} \lambda(t,\tau) = e^{\int_t^\tau (r_2(s)+\xi_1(s))ds}\; E\, e^{\int_t^\tau \xi_2(s)ds}. \tag{39}$$

Let us compute $E\, e^{\int_t^\tau \xi_2(s)ds}$. First we solve (37) to obtain

$$\xi_2(s) = \int_t^s d\sigma\, \eta(\sigma)\, exp \int_\sigma^s \beta(\theta)\, d\theta.$$

Then, we set $X = \int_t^\tau \xi_2(s)ds$ and obtain

$$X = \int_t^\tau ds \int_t^s d\sigma\, \eta(\sigma)\, exp \int_\sigma^s \beta(\theta)\, d\theta \;= \int_t^\tau d(\sigma)\, \eta(\sigma) \int_\sigma^t ds\, exp \int_\sigma^s \beta(\theta)\, d\theta. \tag{40}$$

By setting

$$h(\sigma,\tau)=\int_\sigma^\tau ds\; exp\int_\sigma^s \beta(\theta)d\theta,$$

we get $X=\int_t^\tau d\sigma\,\eta(\sigma)h(\sigma,\tau)$. Note that X is a Gaussian variable with mean 0 and variance equal to

$$EX^2=\int_t^\tau d\sigma\, q(\sigma)h^2(\sigma,\tau).$$

Thus, we obtain

$$\begin{aligned} E\,e^X &= \frac{1}{\sqrt{2\pi EX^2}}\int_{-\infty}^{+\infty} e^x\; e^{-\frac{x^2}{2EX^2}}\,dx \;=\; e^{\frac{EX^2}{2}}\,\frac{1}{\sqrt{2\pi EX^2}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2EX^2}}(x-EX^2)^2dx \\ &= e^{\frac{EX^2}{2}}. \end{aligned}$$

We thus get the formula

$$E^{\bar{\xi}_t}\lambda(t,\tau)=e^{\left[\int_t^\tau (r_2(s)+\xi_1(s))ds+\frac{1}{2}\int_t^\tau h^2(s,\tau)q(s)\,ds\right]}. \tag{41}$$

Finally, from (35), we obtain

$$E^{\bar{\xi}_t}p_2(t)=e^{\int_t^T (r_2(s)+\xi_1(s)+\frac{1}{2}h^2(s,T)q(s))ds}-\int_t^T d\tau\, g(\tau)\; e^{\int_t^\tau (r_2(s)+\xi_1(s)+\frac{1}{2}h^2(s,\tau)q(s))ds}. \tag{42}$$

Note that the corresponding deterministic formula is

$$e^{\int_t^T r_2(s)ds}-\int_t^T d\tau\, g(\tau)\; e^{\int_t^\tau r_2(s)ds}, \tag{43}$$

which can be compared with (42).

The functions g, r_2, ξ_1 and h are relatively easy to calculate when $f(t)$ and $\beta(t)$ are known. Furthermore, since q, the variance of η, is known, (42) is completely determined.

4 Conclusion

In this paper, we have extended the Sethi and Thompson [17] cash balance model to the case when there are dividends and uncertain capital gains arising from investing in stock. We use a stochastic maximum principle to obtain an explicit optimal decision rule when the rate of return on the stock is modeled by a diffusion process.

The respective adjoint variables resulting from the application of the maximum principle at any given time represent the value (expected value in the stochastic case) at the terminal time of having a dollar in a bank account and stock, at that time, respectively. Finally, the paper serves as an introduction of the stochastic maximum principle for addressing stochastic control problems in the areas of management science and economics.

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