Alpha-root Processes for Derivatives pricing

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Abstract

A class of mean reverting positive stochastic processes driven by $\alpha-$stable distributions, $1 \leq \alpha < 2$, are discussed. They are referred to as $\alpha-$root processes in analogy to the square root process or the Cox-Ingersoll-Ross process derived from the Brownian motion. They are affine models in the same sense as the square root process, providing semi-analytical results for the implied term structures as well as for the characteristic exponents for their associated distributions. Though likely that they have caught the attention of researchers in the field, their use has not been appreciated perhaps due to lack of an efficient numerical algorithm to supplement their semi-analytical results. The present article introduces a formulation that admits an efficient simulation algorithm to enable an extensive investigation of their properties.

Stochastic processes are the building blocks of modeling discipline. Though Brownian motion has been largely successful in this regard, there are certain areas where more advanced processes could be helpful. This is especially so in mathematical finance wherein alternate processes have been utilized, in particular to provide an explanation to parameter smiles, such as volatility smiles or correlation smiles. Among other approaches, a class of stochastic processes called $\alpha-$stable Lévy processes have been used for this purpose with encouraging results. Because applicable α usually lies between 1 and 2, and the associated stable processes can have negative values, their use has been largely limited to their exponentials as stochastic variables of interest. This makes them analytically intractable for many objects of interest, such as term structures of discount factors in interest rate modeling or survival probabilities in credit risk modeling.

It is known that the Cox-Ingersoll-Ross process, also known as the square-root process, though confined only to the positive real axis, admits semi-analytical results useful for term structure modeling. It belongs to a class of affine models, the spot rate in interest rate modeling being related affinely to the short rate. The square-root process is driven by a Brownian motion, which in the language of stable processes has $\alpha = 2$. A natural question then arises as to whether there exist $\alpha-$root processes driven by $\alpha-$stable Lévy processes, and whether they too exhibit the affine property. As it turns out, the answer to this question is pleasantly in the affirmative. $\alpha-$root processes thus provide a natural and appealing

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alternative to affine jump diffusion processes by incorporating jumps into the diffusion component itself to turn it into an $\alpha-$root process, rather than extending the process to include a jump component.

The class of affine Markov processes has been characterized in generality by Duffie, Filipovic and Schachermayer [2003]. However, this class being rather large, identification of specific affine processes for their usefulness is important in itself. Being a natural extension of the square-root process, $\alpha-$root processes have very likely caught the attention of researchers in the field. For instance, they are briefly touched upon by Carr and Wu [2004] as an activity process for generating random time. Their use has not been appreciated in the field perhaps due to lack of an efficient numerical algorithm to supplement their semi-analytical results. In the present article, an efficient Monte Carlo simulation algorithm is presented based on a convenient formulation of the process that enables an extensive investigation of their properties.

Section 1 presents the $\alpha-$root process in the form of a mean-reverting pure-jump process of infinite activity and presents a semi-analytical solution for the implied term structure. Section 2 presents analytical expressions for the Laplace exponents in some special cases. Section 3 presents an efficient Monte Carlo simulation algorithm that enables a numerical investigation of the process. Section 4 discusses the simulation results and concludes with a brief summary. Semi-analytical solutions are derived to Appendix A. The results of a numerical investigation are presented in Figures 1-9.

1 Alpha-Root Process

Let us start with the following pure jump process for a positive stochastic variable $r(t)$,

$$dr(t) = [\phi(t) - mr(t)]\, dt + \int_{z=0}^{\infty} h(z/r(t)) dM(dz, t). \tag{1}$$

Here $dM(dz,t) = dN(dz,t) - dzdt$ where $N(dz,t)$s are independent Poisson processes. Process $N(dz,t)$ is of intensity dz and is associated with the interval $(z, z+dz)$. If $N(dz,t)$ jumps up by one at time t, $dN(dz,t)$ causes $r(t)$ to jump up by $h(z/r(t_-))$ where t_- is just prior to t. We may refer to $h(x)$ as the jump function. It is taken to be a non-increasing function of its argument x going to zero as $x \to \infty$. $M(dz,t)$ is the compensated Poisson process (a Martingale). Parameter m is the mean reversion rate. Drift $\phi(t)$ is assumed to positive. The total intensity of the Poisson processes is infinite and hence the stochastic process is of infinite activity (note that the effective intensity depends on the jump function and is not necessarily infinite).

An attractive feature of the above process is that it is an affine model, just as the well-known square-root process is. Being an affine model, it admits semi-analytical results for the implied term structures as well as for the characteristic exponents for their associated distributions. The following result is derived in Appendix A,

$$\mathrm{E}_t \left\{ \exp\left[-\int_t^T dsu(T-s)r(s) \right] \right\} = \exp\left[-\int_t^T ds\phi(s)B(T-s) - B(T-t)r(t) \right], \tag{2}$$

where $u(\tau)$ is some deterministic function and $B(\tau)$, satisfying $B(0) = 0$, is a solution of

$$\frac{dB(\tau)}{d\tau} + mB(\tau) = u(\tau) + \int_0^\infty dx \left\{1 - h(x)B(\tau) - \exp\left[-h(x)B(\tau)\right]\right\}. \tag{3}$$

For term structure modeling, one is interested in solving this equation with $u(\tau) = 1$. If $u(\tau) = u\delta(\tau - 0_+)$ where $\delta(\tau - 0_+)$ is a Dirac-delta function sitting just above $\tau = 0$, one obtains the Laplace transform $\mathrm{E}_t\left\{\exp\left[-ur(T)\right]\right\}$ of the distribution of $r(T)$, or its negative logarithm known as the Laplace exponent. For this, the above equation is solved in the absence of $u(\tau)$, but under the initial condition $B(0) = u$.

The above result is for a general jump function $h(x)$. For $h(x) = ax^{-1/\alpha}$, $1 < \alpha < 2$, we have $h(z/r) \propto r^{1/\alpha}$ and (1) may be referred to as an $\alpha-$root process. Equation for $B(\tau)$ then becomes

$$\begin{aligned} \frac{dB(\tau)}{d\tau} + mB(\tau) &= u(\tau) - \sigma^\alpha \left[B(\tau)\right]^\alpha, \quad 1 < \alpha < 2, \\ &= u(\tau) - \sigma B(\tau) \ln B(\tau), \qquad \alpha = 1. \end{aligned} \tag{4}$$

where σ is $a\left[\alpha\Gamma(-\alpha)\right]^{1/\alpha}$ for $1 < \alpha < 2$ and is a for $\alpha = 1$. Equation for $\alpha = 1$ is also presented above, though it needs to be treated as a special case. For the Laplace exponent, the above can be solved with $u(\tau) = 0$ and $B(0) = u$ to obtain

$$\begin{aligned} B(\tau) &= e^{-m\tau} \left\{u^{-(\alpha-1)} + \frac{\sigma^\alpha}{m}\left[1 - e^{-(\alpha-1)m\tau}\right]\right\}^{-1/(\alpha-1)}, \quad 1 < \alpha < 2, \\ &= \exp\left[e^{-\sigma\tau}\left(\ln u + \frac{m}{\sigma}\right) - \frac{m}{\sigma}\right], \qquad \alpha = 1. \end{aligned} \tag{5}$$

Case $\alpha < 1$ turns out to be inconsistent. These results have a limit as $\alpha \to 2$ (given a fixed σ) to correspond to the case of the square-root process. Closed form solutions for the Laplace exponent can be obtained in some special cases as discussed in the next section.

Drift $\phi(t)$ has been assumed to be positive. This ensures that the origin is inaccessible. Note that the value of the probability density of $r(T)$ at $r(T) = 0$ can be obtained, as usual in Laplace transforms, by taking the $u \to \infty$ limit of $u\mathrm{E}_t\left\{\exp\left[-ur(T)\right]\right\}$. As can be verified, the leading contribution comes from the integral in (2) near $s = T$,

$$u\mathrm{E}_t\left\{\exp\left[-ur(T)\right]\right\} \sim u\exp\left[-\phi(T)\frac{u^{2-\alpha}}{(2-\alpha)\sigma^\alpha}\right], \quad \text{as } u \to \infty. \tag{6}$$

For $\phi(T) > 0$, this goes to zero as $u \to \infty$. As for $\alpha = 1$, $B(\tau) \to \infty$ as $u \to \infty$ for all τ so that the above quantity goes to zero for any $\phi(T) \geq 0$ (in this case, $\phi(T)$ can be zero).

The behavior $\sim u^{2-\alpha}$ of the Laplace exponent as $u \to \infty$ indicates that, as $r(T) \to 0$, the distribution of $r(T)$ approaches that of a stable distribution of index $2 - \alpha$ and skew parameter one (maximally skewed to the right). Further, its behavior $\sim u^\alpha$ as $u \to 0$, apart from a term linear in u, indicates that, as $r(T) \to \infty$, the distribution approaches that of a stable distribution of index α and skew parameter one. A stable distribution of index > 1, even when maximally skewed to the right, has nonzero probability at negative values of the random variable. The $\alpha-$root distribution, constructed out of a stable distribution of index

> 1, can be viewed as avoiding $r(T) \leq 0$ region by turning itself into a stable distribution of index < 1 as $r(T) \to 0$.

The $\alpha-$root process can be viewed as being driven by an $\alpha-$stable Lévy process. This is analogous to the square root process being driven by the Brownian motion. To see this, consider small $\tau = T - t$ when $B(\tau) \simeq (1 - m\tau)u - \sigma^\alpha \tau u^\alpha$ and the Laplace exponent approximates to

$$[r(t) + (\phi(t) - mr(t))\tau]\, u - \sigma^\alpha r(t) \tau u^\alpha. \tag{7}$$

The u^α term is the Laplace exponent of a stable process of index α and skew parameter one (maximally skewed to the right) with zero location, the term linear in u arising from the deterministic part of the $r-$process. Its scale parameter is $\sigma(r(t)\tau)^{1/\alpha}$ (times $[-\cos(\pi\alpha/2)]^{1/\alpha}$ to be exact), as expected with the $\alpha-$root of $r(t)$ attached (similar analysis can be done for $\alpha = 1$). Given the above infinitesimal result, one can indeed recover the full Laplace exponent using the law of iterated expectations. Note that infinitesimally, the $\alpha-$root process can be viewed as being driven by a time-scaled stable process, τ getting effectively scaled by $r(t)$. This is a stochastic scaling of time, scaling by the stochastic process $r(t)$ itself. This gives us an alternate view of the process (1) for a general jump function $h(x)$. It is this viewpoint that is usually adopted in the literature to construct additional stochastic processes.

The expression for term structure in (2) involves convolution of $\phi(s)$ and $B(s)$ (consider $t = 0$). When modeling term structure models, say for interest rates or credit spreads, one approach is to imply the drift $\phi(t)$ from the given data on discount factors or survival probabilities as the case may be. If this deconvolving procedure is not convenient, one may consider the well-known approach in affine modeling of working with a constant ϕ, but with the stochastic variable $r(t)$ related to the variable of interest by a deterministic shift that is implied from the given data.

2 Laplace Exponents

The Laplace exponent of the distribution of $r(T)$ can be obtained given the solution (5) for $B(\tau)$. For constant drift ϕ and for $1 < \alpha < 2$, this gives for the exponent

$$\frac{\nu\phi}{mq^\nu} \int_1^{1+pu^{1/\nu}} dx x^{-\nu} \left(1 + qu^{1/\nu} - x\right)^{\nu-1} + \frac{r(t)e^{-m(T-t)}u}{\left(1 + pu^{1/\nu}\right)^\nu}, \tag{8}$$

where $\nu = 1/(\alpha - 1)$, $q = \sigma^\alpha/m$ and $p = q(1 - e^{-(\alpha-1)m(T-t)})$. The integral can be expressed in terms of incomplete beta functions. For small u, the exponent has the expansion

$$\left[\frac{\phi}{m}(1-s) + r(t)s\right] u - \left\{\frac{\phi}{m\alpha}\left[q(1-s) - p\nu s\right] + p\nu r(t)s\right\} u^\alpha, \tag{9}$$

where $s = e^{-m(T-t)}$. This gives the mean, and the scale parameter for the large $r(T)$ behavior. Closed form solution for the exponent can be obtained if $m = 0$, that reads

$$\frac{\phi u^{2-\alpha}}{(2-\alpha)\sigma^\alpha}\left[1 - \left(1 + pu^{\alpha-1}\right)^{-(2-\alpha)/(\alpha-1)}\right] + \frac{r(t)u}{\left(1 + pu^{\alpha-1}\right)^{1/(\alpha-1)}}, \tag{10}$$

where $p = (\alpha - 1)\sigma^\alpha (T - t)$. If $m \neq 0$, closed form solutions can be obtained for some special values of α. For $\alpha = 2$, we obtain the well-known result

$$\frac{\phi}{\sigma^2} \ln(1 + pu) + \frac{r(t)e^{-m(T-t)}u}{1 + pu}, \tag{11}$$

where $p = (\sigma^2/m)(1 - e^{-m(T-t)})$. This is the exponent of the non-central chi-square distribution. Because of our scale convention, the volatility of the square root process turns out to be $\sigma\sqrt{2}$. For $\alpha = 3/2$, one obtains

$$\frac{2\phi}{mq^2} \left\{ \frac{p\sqrt{u}(1 + q\sqrt{u})}{1 + p\sqrt{u}} - \ln\left(1 + p\sqrt{u}\right) \right\} + \frac{r(t)e^{-m(T-t)}u}{\left(1 + p\sqrt{u}\right)^2}, \tag{12}$$

where $q = \sigma^{3/2}/m$ and $p = q(1 - e^{-m(T-t)/2})$. For $\alpha = 4/3$, the exponent is

$$\frac{3\phi}{mq^3} \left\{ \frac{pu^{1/3}(1 + qu^{1/3})}{P(u)} \left[\frac{q}{2} u^{1/3} \left(1 + \frac{e^{-m(T-t)/3}}{P(u)} \right) - 1 \right] + \ln\left(P(u)\right) \right\} + \frac{r(t)e^{-m(T-t)}u}{\left(P(u)\right)^3}, \tag{13}$$

where $q = \sigma^{4/3}/m$ and $p = q(1 - e^{-m(T-t)/3})$ and $P(u) = 1 + pu^{1/3}$. Another integrable case is $\alpha = 5/3$ that gives

$$\frac{3\phi}{mq\sqrt{q}} \left\{ \frac{\sqrt{q}u^{1/3}R(u)}{\sqrt{1 + pu^{2/3}}} - \mathrm{Sin}^{-1} \left[\frac{\sqrt{q}u^{1/3}R(u)}{1 + qu^{2/3}} \right] \right\} + \frac{r(t)e^{-m(T-t)}u}{(1 + pu^{2/3})^{3/2}}. \tag{14}$$

Here $q = \sigma^{5/3}/m$, $p = q(1 - e^{-2m(T-t)/3})$ and $R(u) = \sqrt{1 + pu^{2/3}} - e^{-m(T-t)/3}$. Closed form solutions can be obtained more generally for $\alpha = 1 + 2/k$ where $k \geq 2$ is an integer.

3 Monte Carlo Simulation

Process (1) is of infinite activity as presented. The integral over z needs to be cutoff at the higher end to render the total intensity of the Poisson processes finite for simulation purpose. This can be done by forcing $h(x) = 0$ for $x > X$ given a sufficiently large X. Process (1) can now be viewed as being driven by a compound Poisson process of stochastic total intensity $r(t)X$. It can be simulated starting with a more convenient form,

$$d[r(t) - c_X(t)] = -\mu_X[r(t) - c_X(t)]dt + \int_{z=0}^{r(t)X} h(z/r(t))dN(dz, t). \tag{15}$$

Here $\mu_X = m + \int_0^X dxh(x)$ and c_X is introduced via $\phi(t) = dc_X(t)/dt + \mu_X c_X(t)$. Since $\phi(t)$ is taken to be positive, $c_X(t)$ solves to be positive. $c_X(0)$ can be conveniently chosen, say as $r(0)$ or $\phi(0)/\mu_X$. The algorithm reads as follows.

1. Set $t_o = 0$ and $r_+ = r(0)$.

2. Draw an independent exponentially distributed unit mean random number v. Set t to the next event arrival time $t_o + v/\zeta$ where $\zeta = r_+X$, or the time horizon whichever is earlier.

3. Update r_+ to r_- given by

$$r_- = (r_+ - c_X(t_o))e^{-\mu_X(t-t_o)} + c_X(t). \tag{16}$$

4. If t is the time-horizon, go to step 6.

5. Draw an independent uniformly distributed random number $w \in [0, 1]$. Update r_- to

$$r_+ = r_- + h(x), \qquad \text{where } x = w\zeta/r_-. \tag{17}$$

Note that $h(x) = 0$ if $x > X$. Set $t_o = t$ and go to step 2.

6. Collect this sample or value a derivative. For the next scenario, go to step 1.

7. From all the samples thus obtained, determine the distribution, or average the values to obtain a price for the derivative.

Some improvements are possible to ensure that $\zeta \geq r(t)X$ in between Poisson events if $c_X(t)$ increases with t and can make r_- larger than r_+ before the next event arrival time. Note that, since jumps are nonnegative, $r(t)$ never goes below $c_X(t)$ (consider $c_X(0) = r(0)$). Hence, because $c_X(t) > c_\infty(t)$ for any finite X (and $t > 0$), to sample $r(t)$ close to its lower bound of $c_\infty(t)$, X will have to be very large. For the $\alpha-$root process, $c_\infty(t)$ is zero and there will always be some region left unsampled near zero for any finite X. This deficiency is corrected in the updated algorithm discussed below.

For $h(x) = ax^{-1/\alpha}$, $1 < \alpha < 2$, there is an issue of convergence. The $x-$integral in (3), limited to $x < X$, can be approximated as

$$-\alpha\Gamma(-\alpha)(aB)^\alpha + \frac{\alpha}{2(2-\alpha)}(aB)^2X^{1-2/\alpha} - \frac{\alpha}{6(3-\alpha)}(aB)^3X^{1-3/\alpha} + \mathcal{O}\left((aB)^4X^{1-4/\alpha}\right).$$

Note that, as $\alpha \to 2$, the second term tends to be of the same order as the leading contribution. This makes our Monte Carlo not useful near $\alpha = 2$. Fortunately, there is an interesting solution. Consider extending process (1) to include another set of Poisson processes. If identical to the first, but with $h(y) = by^{-1/\omega}$ for some parameters b, ω and a cutoff Y, this adds a $y-$integral to (3) that can be approximated as above. Note that the sign of the second term in its expansion can be made negative by choosing $\omega > 2$, or $\omega > 3$ so that $(bB)^\omega$ term remains farther away. Any $\omega > 3$ could be chosen, in fact, $\omega = \infty$ turns out to be a good choice. For $\omega = \infty$, $h(y) = b$ for $y < Y$ and zero otherwise, and the added process is effectively just one Poisson process. Its $y-$integral is just $(1 - bB - e^{-bB})Y$ that can be expanded in powers of bB. Parameter b can be chosen so as to cancel the troubling term. The x and $y-$integrals then together get approximated to $-\alpha\Gamma(-\alpha)(aB)^\alpha$.

However, convergence is still not satisfactory. Ignoring for the moment the likelihood of getting into negative $r-$values, let us consider extending process (1) with one more Poisson process with $h(y) = -c$ for $y < Y$ and zero otherwise. It is now possible to choose b and c to cancel both the $(aB)^2$ and $(aB)^3$ terms. The equations for b and c turn out to be cubic that can be solved to obtain

$$b = aq(s + d)X^{-1/\alpha}, \quad c = aq(s - d)X^{-1/\alpha}, \tag{18}$$

where

$$s = \sqrt{1/2 - d^2},\ d = \cos((\pi+\theta)/3),\ \theta = \cos^{-1}(p/q^3),\ p = \frac{\alpha X}{(3-\alpha)Y},\ q = \sqrt{\frac{\alpha X}{(2-\alpha)Y}}. \tag{19}$$

As long as $Y/X \leq \alpha(3-\alpha)^2/(2-\alpha)^3$, this gives a solution[1] $b \geq c \geq 0$.

Changes to Monte Carlo are straightforward. There is an additional positive contribution $(b-c)Y$ to μ_X. Total intensity is now $\zeta_X + 2\zeta_Y$ where $\zeta_X = r_+X$ and $\zeta_Y = r_+Y$. Further in step 5, the original process is chosen with probability $X/(X+2Y)$ and the two added processes with probabilities $Y/(X+2Y)$ each, and an appropriate jump is added to r_-. For Y not too small relative to X, c is small relative to b and the likelihood of getting into negative $r-$values is small. If r_+ does end up negative after adding $-c$ in step 5, it is set to zero. For the present simulation results, Y is chosen to be equal to X. To improve efficiency, quasi random sequences such as Sobol sequences are used to generate each of the independent random numbers.

As $\alpha \to 2$, $a = \sigma[\alpha\Gamma(-\alpha)]^{-1/\alpha}$ tends to zero for a given σ, but b and c tends to a nonzero value ensuring that the square root process is simulated appropriately in the limit as trinomial branching. The algorithm should also be applicable for the case of a standard mean reverting process driven by an $\alpha-$stable process. If in process (1) $r(t)$ can be negative, and $h(z/r)$ is replaced by a $r-$independent function $h(z)$, our analysis in section 1 can be carried through to obtain the known results. For simulation purpose, the process can be rewritten in terms of a redefined drift $\phi_X(t)$ instead of a redefined mean reversion μ_X.

4 Results and Conclusions

Results of the Monte Carlo simulation for a choice of parameters are presented Figures 1-9. Figures 1-4 present the dependence of the probability distribution of $r(T)$ at $T = 5$ on the parameters α, σ and m (t is set to zero). Figure 5 shows the dependence on T itself. As can be seen from Figure 6, X need not be too large. To understand the order of magnitude of X, note that the total intensity of Poisson processes, $\zeta(t) = 3r(t)X$, is about 10 for $r(0) = 0.03$ and $X = 100$, and corresponds to a time-step of about 0.1. To confirm the accuracy, the Laplace exponent is computed and displayed in Figure 7 for $\alpha = 3/2, 5/3$ and 2 for which closed form expressions are available from section 2.

An usual approach to understanding the distribution of a positive random variable is to compare it to a lognormal one. This can be done by computing the implied Black-Scholes volatility for a call or a put option on $r(T)$ at various strikes, ignoring discounting and setting the underlying to $\mathrm{E}_0(r(T))$. The resulting volatility smile is plotted in Figure 8 for different values of α. Figure 9 shows its dependence on T. The smile features are encouraging and could be applicable for instance for modeling interest rates, and further study is needed to confirm their usefulness.

To conclude, the article introduces a formulation defining $\alpha-$root processes driven by $\alpha-$stable Lévy processes as a natural extension of the square root process driven by the Brownian motion. It analyses their affine properties for use in term structure modeling and their Laplace exponents for an understanding of their distributions. The formulation admits an efficient Monte Carlo simulation algorithm to supplement their semi-analytical results. The results of a numerical investigation are displayed in Figures 1-9.

[1] It is possible to cancel the $(aB)^4$ term as well with an appropriate choice of Y (consider $Y \propto X/(2-\alpha)$). However, this results in a Y that is large relative to X, especially as $\alpha \to 2$.

A Semi-Analytics

Because affine models have been well-studied, analytics of an $\alpha-$root process can be written down as a special case. However, for our purpose, it is simpler and more illuminating to derive the same starting with the pure-jump process

$$dr(t) = [\phi(t) - \mu r(t)]\, dt + \int_{z=0}^{\infty} h(z/r(t)) dN(dz,t). \tag{20}$$

The object of interest is the following expectation value

$$F_T(t, r(t)) \equiv \mathrm{E}_t \left\{ \exp\left[- \int_t^T ds u_T(s) r(s) \right] \right\}. \tag{21}$$

Its differential can be written down using Ito's calculus leading to

$$\frac{\partial F_T}{\partial t} + (\phi - \mu r)\frac{\partial F_T}{\partial r} - urF_T + r\int_0^{\infty} dx\, [F_T(t, r + h(x)) - F_T(t,r)] = 0. \tag{22}$$

Integration variable z is scaled to $x = z/r(t)$. The above can be solved with the ansatz

$$F_T(t, r(t)) = \exp\left[-A_T(t) - B_T(t) r(t)\right]. \tag{23}$$

Equating coefficients of F_T independent of r and those linear in r separately gives

$$\begin{aligned} \frac{dA_T(t)}{dt} + \phi(t) B_T(t) &= 0, \\ \frac{dB_T(t)}{dt} - \mu B_T(t) + u_T(t) + \int_0^{\infty} dx\, \{1 - \exp\left[-h(x) B_T(t)\right]\} &= 0. \end{aligned} \tag{24}$$

Consider now $u_T(t) = u(\tau)$ as a function of $\tau = T - t$ only. Then $B_T(t) = B(\tau)$ is also a function of τ only, satisfying $B(0) = 0$ and the differential equation

$$\frac{dB(\tau)}{d\tau} + \mu B(\tau) = u(\tau) + \int_0^{\infty} dx\, \{1 - \exp\left[-h(x) B(\tau)\right]\}. \tag{25}$$

Given $B(\tau)$, and $A_T(t)$ in terms of $B(\tau)$, solution for $F_T(t, r(t))$ is as given in (2). Consider now replacing μ by $m + \int_0^{\infty} dx h(x)$. Process (20) can now be rewritten in terms of m and $M(dz,t)$ given by $dM(dz,t) = dN(dz,t) - dzdt$ to obtain process (1). Equation for $B(\tau)$ then reads as in (3). Solution for $F_T(t, r(t))$ remains the same.

We have assumed that the integral over x is finite. Since $h(x)$ is assumed to be a non-increasing function of x with $h(\infty) = 0$, divergence can arise as $x \to \infty$ for a certain class of jump functions. If so, consider defining process (20) with a cutoff X that forces $h(x) = 0$ for $x > X$ with an $X-$dependent parameter μ_X. Consider next replacing μ_X by $m + \int_0^X dx h(x)$ (assuming no divergence arises near $x = 0$) given some $X-$independent parameter m. Process (20) can now be rewritten in terms of m and $M(dz,t)$ and the limit $X \to \infty$ taken. The $x-$integral in (3) is better behaved in the limit $X \to \infty$ and can be finite for a wider class of jump functions.

For $h(x) = ax^{-1/\alpha}$, $1 < \alpha < 2$, the equation for $B(\tau)$ reads as in (4). The $x-$integral in (3) is $-\alpha\Gamma(-\alpha)a^\alpha\,[B(\tau)]^\alpha$. Note that $\int_0^X dxh(x) = a\alpha X^{(\alpha-1)/\alpha}/(\alpha-1)$ diverges as $X \to \infty$, but gets absorbed into μ_X. The $\alpha = 1$ case is special. The $x-$integral in (25) is $-aB(\tau)\ln B(\tau)$ up to terms linear in $B(\tau)$ that are taken care of by $\mu_X = m + a\ln(X/a) + a(1-\gamma)$ where γ is the Euler's constant.

One may wonder whether an $\alpha-$root process can be defined for $\alpha < 1$ as well. After all, the $x-$integral in (25) is then finite and is $-\alpha\Gamma(-\alpha)a^\alpha\,[B(\tau)]^\alpha$. However, the integral dominates the $\mu B(\tau)$ term as $B(\tau) \to 0$, and solving (25) directly for the Laplace exponent with $u(\tau) = 0$ and $B(0) = u$ yields a $B(\tau)$ that does not go to zero as $u \to 0$.

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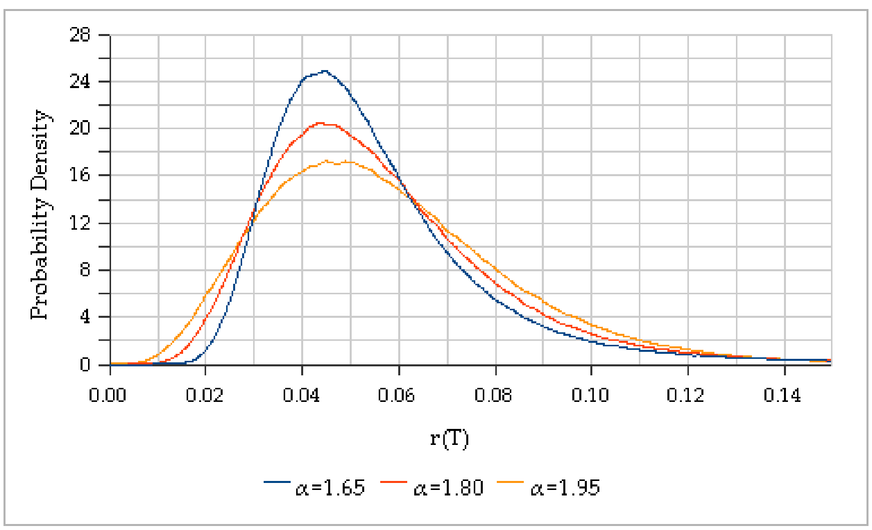

Figure 1: Plots of the probability density functions at $T = 5$ for $\alpha = 1.65, 1.80$ and 1.95. Other parameters chosen are $\sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is one million and cutoff X is 100.

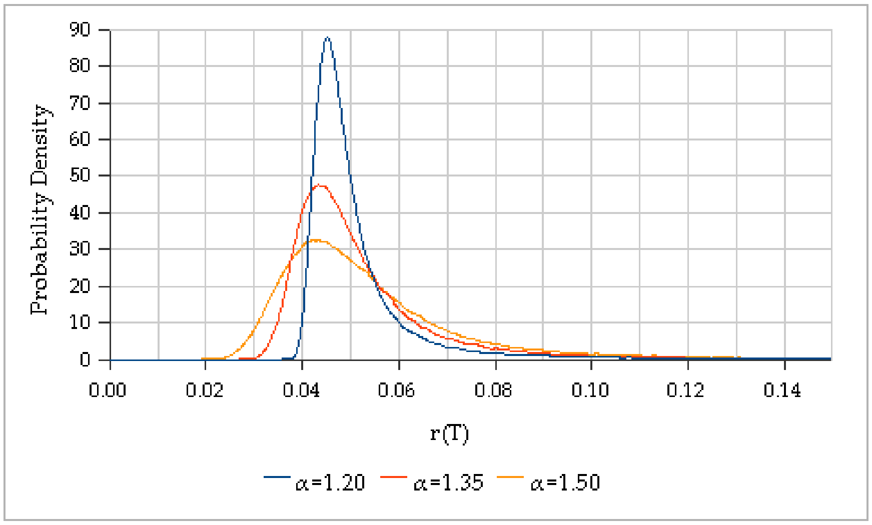

Figure 2: Plots of the probability density functions at $T = 5$ for $\alpha = 1.20, 1.35$ and 1.50. Other parameters chosen are $\sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

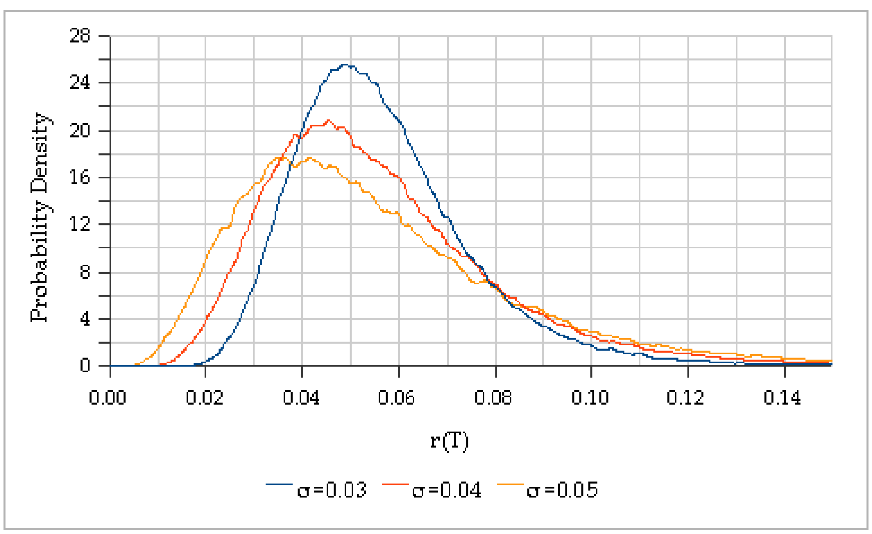

Figure 3: Plots of the probability density functions at $T = 5$ for $\sigma = 0.03, 0.04$ and 0.05. Other parameters chosen are $\alpha = 1.80, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

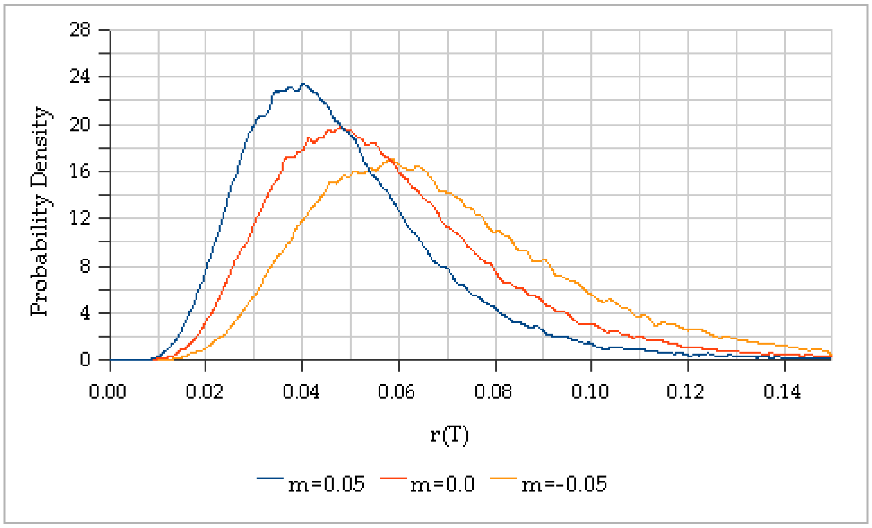

Figure 4: Plots of the probability density functions at $T = 5$ for $m = 0.05, 0.0$ and -0.05. Other parameters chosen are $\alpha = 1.80, \sigma = 0.04, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

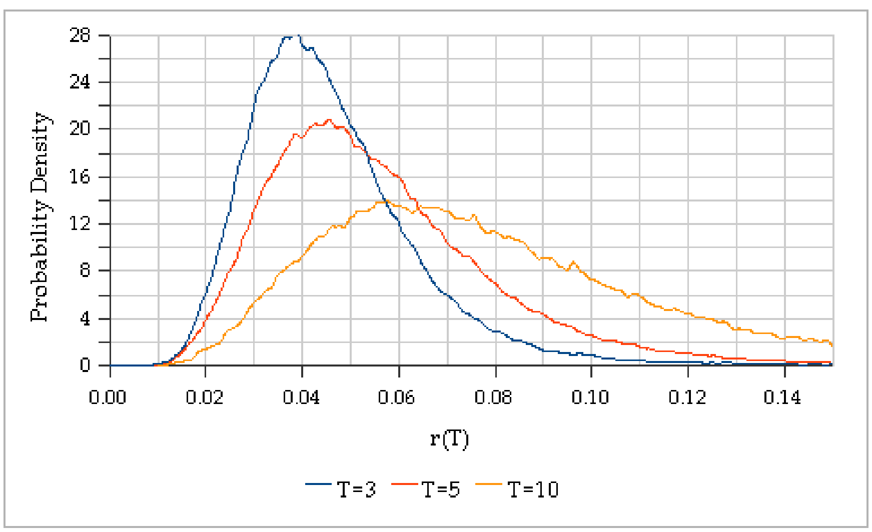

Figure 5: Plots of the probability density functions for $T = 3, 5$ and 10. Other parameters chosen are $\alpha = 1.80, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

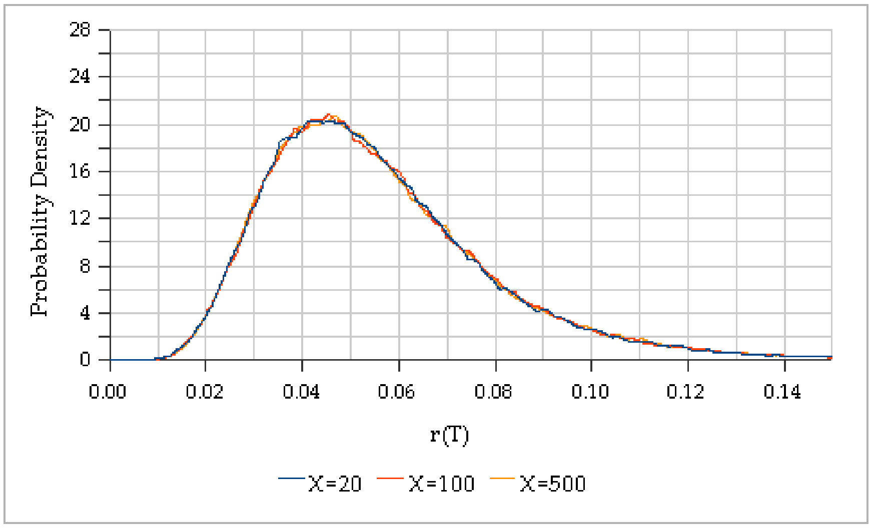

Figure 6: Plots of the probability density functions at $T = 5$ for cutoff $X = 20, 100$ and 500. Other parameters chosen are $\alpha = 1.80, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000.

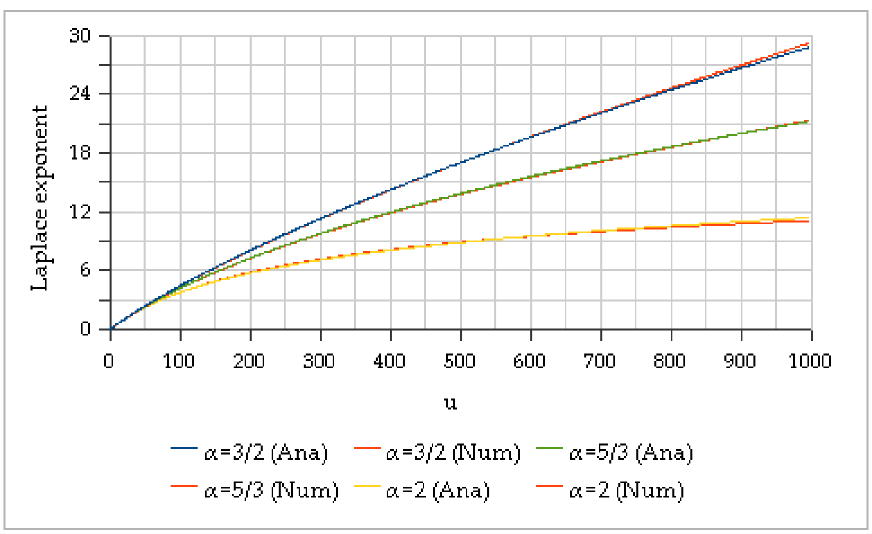

Figure 7: Plots of the Laplace exponents at $T = 5$ computed analytically and numerically for $\alpha = 3/2, 5/3$ and 2. Other parameters chosen are $\sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff X is 100.

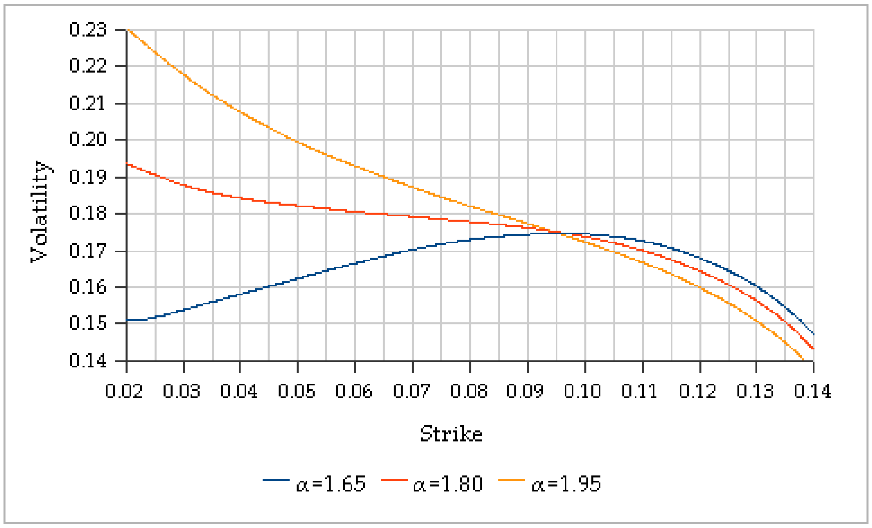

Figure 8: Plots of the volatility smiles at $T = 5$ for $\alpha = 1.65, 1.80$ and 1.95. Other parameters chosen are $\sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff $X = 100$.

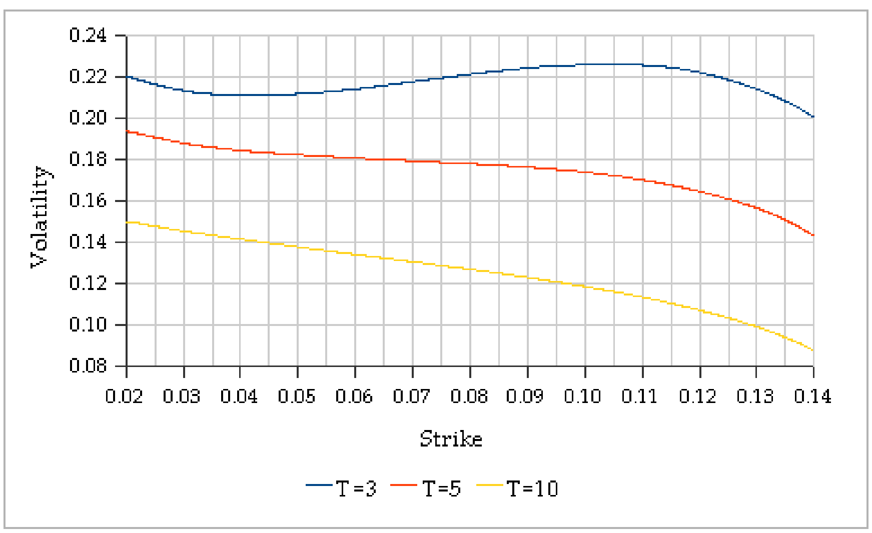

Figure 9: Plots of the volatility smiles for $T = 3, 5$ and 10. Other parameters chosen are $\alpha = 1.80, \sigma = 0.04, m = 0.01, \phi = 0.006$ and $r(0) = 0.03$. Number of Monte Carlo scenarios is 100,000 and cutoff $X = 100$.