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Market Proxies, Correlation, and Relative Mean-Variance Efficiency: Still Living with the Roll Critique¹

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Abstract

A test of the CAPM is developed conditional on a prior belief about the correlation between the true market return and the proxy return used in the test. Consideration is given to the effect of the proxy's mismeasurement of the market return on the estimation of the market model. Failure to grant this consideration biases tests towards rejection by overstating the inefficiency of the proxy. An extension of the proposed test to a CAPM with conditioning information links mismeasurement of the market return to time-variation in beta.

Keywords: Asset pricing, CAPM, portfolio efficiency, multivariate testing, bootstrap hypothesis testing, triangular systems, endogeneity, identification, GMM, conditional heteroskedasticity, GARCH. JEL codes: C12, C13, C32, G12.

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Disclaimer: The views expressed in this paper are solely those of the author and do not reflect official positions of the Commodity Futures Trading Commission. All errors are my own.

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1. Introduction

A common feature among many asset pricing models in financial economics is the relation of expected returns on risky securities to the covariance between those securities' returns and an economic aggregate like (the marginal utility of) aggregate wealth or consumption. In empirical work, this economic aggregate (central to the pricing model under consideration) is generally unobservable and requires a proxy. Tests of the given model which, by necessity, are based on the proxy are confronted with a joint hypothesis that complicates the interpretation of a rejection of the model's prediction. In particular, does this rejection signal a violation of the model's result or the poor quality of the proxy chosen to render the model "testable"? In specific regard to the capital asset pricing model (CAPM), the existence of this dual hypothesis led Roll (1977) to conclude that the "theory is not testable unless the exact composition of the true market portfolio is known and used in the tests" (p. 130).

Roll's critique was met by two possible ways forward. First, since validity of the CAPM and mean-variance efficiency of the market return are equivalent, if the proxy is not mean-variance efficient, then any "test" based on this proxy seems besides the point. This stance led to the development of tests for mean-variance efficiency of a proxy, with Gibbons, Ross, and Shanken (1989) serving as the prominent example and MacKinlay and Richardson (1991) providing a useful generalization. Second, if the proxy is invalid (i.e., not mean-variance efficient), then the CAPM prediction based upon this proxy should not be expected to hold exactly, only approximately. From this stance, bounds on the deviations from exact CAPM pricing were developed based upon the relative efficiency of the proxy (i.e., its distance inside the mean-variance frontier). Examples of this approach include Shanken (1987) as well as Kandel and Stambaugh (1987, 1995).

This paper extends the literature on relative efficiency testing by developing a pricing restriction that reflects the way in which (1) a proxy return relates to the market return and (2) individual security returns relate to the proxy. The first relation is addressed in the works of Shanken (1987) and Kandel and Stambaugh (1987). The second is the principal contribution of this paper. A general outline of the approach is as follows. The measure of relative efficiency is the correlation between the proxy return and the market return denoted by ρ . A prior belief on the true value of ρ is denoted by ρ_0 . A value for ρ is computed as the upper bound to a multivariate statistic of

CAPM pricing errors measured against the proxy. If this value is less than ρ_0 , such is interpreted as evidence against the CAPM. This approach is a conditional test of the CAPM based upon a prior belief about the relative efficiency of the proxy. If the proxy is determined to be too far inside the mean-variance frontier, the market return is concluded to be inefficient as well.

Any test of relative efficiency assumes, to some extent, that the proxy return mismeasures the market return. A common starting point for most of these tests is the assumption that the relationship between security returns and the proxy return can be explained by a projection of the former onto the latter. Suppose that mismeasurement of the market return by the proxy return is taken to mean that certain components relevant to the market return are excluded from the proxy. Then, the extent to which these excluded components are correlated with the proxy return will determine the extent to which innovations to the familiar market model will tend to covary with that proxy return, since those innovations will contain the aforementioned omitted components. In other words, mismeasurement renders the proxy return endogenous to the market model. The resulting structural equation will, therefore, differ from a projection equation. This paper shows that relative efficiency tests based upon projections are biased towards rejecting the CAPM because these tests overstate a proxy's distance away from the mean-variance frontier. Mismeasurement of the market return is the source of this bias.

A relative efficiency test based upon the aforementioned structural equation as opposed to the commonly used OLS projection requires a consistent estimator for the former in order to render the test feasible empirically. Towards that end, an estimator for linear equations with an endogenous regressor proposed by Prono (2008) is utilized. This estimator bases identification on exclusionary restrictions within the functional form describing heteroskedasticity in security returns and is a higher-moment analog to common instrumental variables techniques. MacKinlay and Richardson (1991) demonstrate that heteroskedasticity in market model residuals can meaningfully impact the results of mean variance efficiency tests for a given proxy return. This paper extends these authors' findings to tests of relative efficiency, noting that a particular form of heteroskedasticity can be used to describe not only the second moment patterns of market model residuals but also the co-movement of these residuals with the proxy return caused by the latter's mismeasurement of the market return (MacKinlay and Richardson (1991), among others, assume this co-movement to be zero).

The remainder of this paper is organized as follows. Starting from a rather general pricing model, Section 2 develops a pricing restriction that can be used to test the CAPM conditional on a prior belief about the correlation between the market return and the proxy used in the test. This restriction fully encompasses the difference between the market return and an imperfect proxy. Section 3 reviews a conventional test of relative mean-variance efficiency. Section 4 presents an overview of the econometrics used to identify and estimate a structural market model. Section 5 details a method for conducting a test of relative mean-variance efficiency that is based upon the econometric techniques developed in section 4. Section 6 summarizes the results from employing this test, comparing them to the results of the conventional method reviewed in section 3. Section 7 presents the results of a Monte Carlo study of the test proposed in section 5 against conventional alternatives. Section 8 proposes a generalization of the pricing restriction in section 2 that provides a direct link between mismeasurement of the market return and time-variation in beta. Section 9 concludes.

2. Pricing Restriction

Assume there exists an observable risk-free rate. Let r_t be an N -vector of observable excess security returns. Define m_t as a scalar unobservable economic aggregate and $r_{m,t}$ as the excess return on an unobservable portfolio of securities that is efficient with respect to excess security returns inclusive of r_t (i.e., the market return). Let $r_{p,t}$ be a scalar proxy for $r_{m,t}$, and assume that the $N + 1$ components of r_t and $r_{p,t}$ are linearly independent. Consider the following pricing model

$$E[r_t] = Cov[m_t, r_t] \tag{1}$$

that relates expected excess returns to the covariance between excess returns and the economic aggregate. Further suppose

$$m_t = \left(\frac{E[r_{m,t}]}{\sigma^2[r_{m,t}]} \right) r_{m,t} \tag{2}$$

so that the economic aggregate is proportional to the market return. Then (1) and (2) imply

$$E[r_t] = \beta E[r_{m,t}]$$

where

$$\beta = \frac{Cov[r_{m,t}, r_t]}{\sigma^2[r_{m,t}]},$$

which is the familiar CAPM of Sharpe (1964) and Lintner (1965).

The model of (1) and (2) is equivalent to the linear multivariate regression

$$r_t = \alpha + \beta r_{m,t} + e_t, \quad (3)$$

where $E[e_t r_{m,t}] = 0$ by construction, and $\alpha = 0$. If the market return were observable, the zero constraint on alpha would be the single testable restriction of the CAPM. Under the current set-up, however, alpha is immeasurable, and, as a consequence, the theory is untestable. What is lacking is a link between the market return and the observable proxy. In order to provide that link, assume

$$r_{m,t} = r_{p,t} + \phi_t, \quad (4)$$

which casts the relationship between the market return and proxy return as a form of measurement error.³ (4) is a generalization of (17) in Jagannathan and Wang (1996) if $\phi_{vw} = 1$. (4) is also a close counterpart to the decomposition of a proxy return used to develop the CAPM for inefficient portfolios (CAPMI) in Diacogiannis and Feldman (2006), with the differences being (i) ϕ_t is not assumed to be uncorrelated with $r_{m,t}$ and (ii) the expected value of ϕ_t is not, necessarily, zero. The variable ϕ_t reflects components to the market return that are excluded from the proxy return. Examples of these components include returns to nontraded assets and/or the returns to human capital.⁴ Substitution of (4) into (3) produces

$$r_t = \gamma + \delta r_{p,t} + \tilde{e}_t \quad (5)$$

³ ϕ_t is measurement error in a general sense not in a classical sense, since the assumption that $\phi_t \perp r_{m,t}, r_t$ is not made. There is good reason for this omission, since given what the measurement error is intended to reflect, ϕ_t is related to both $r_{m,t}$ and r_t .

⁴Studies by Campbell (1996), Jagannathan and Wang (1996), and Dittmar (2002) note the importance of the returns to human capital in pricing expected returns.

where

$$\begin{aligned}\gamma &= \alpha + \beta E[\phi_t], \quad \delta = \beta \\ \tilde{e}_t &= \beta \tilde{\phi}_t + e_t, \quad \tilde{\phi}_t = \phi_t - E[\phi_t]\end{aligned}\tag{6}$$

(5) is the measurable analog to (3). Notice that $\gamma \neq 0$ even if the CAPM holds, thus complicating a test of the theory if the proxy is not the market. In addition, note that according to Diacogiannis and Feldman (2006), $r_{p,t}$ and ϕ_t must be correlated. From (5),

$$Cov[\tilde{e}_t, r_{p,t}] = \beta Cov[\phi_t, r_{p,t}] = \beta (Cov[\phi_t, r_{m,t}] - \sigma^2[\phi_t]).$$

In general, this expression is not zero, which is to say that $r_{p,t}$ is an endogenous regressor in (5). As a result, (5) is a structural equation that unlike (3) cannot, necessarily, be treated as a linear projection without loss of generality. The fact that the market return is unobservable and any proxy return, by definition, is incomplete affords this distinction. The effects of this distinction on measuring the efficiency of a proxy return relative to the market return is made explicit in the proposition and corollaries to follow.

In the context of (1), empirically-based asset pricing models attempt to tie the unobservable economic aggregate m_t to observable variables. Towards that end, consider a linear projection of m_t onto $r_{p,t}$:

$$m_t = a + br_{p,t} + e_{m,t}.\tag{7}$$

According to Lemma 1 of Shanken (1987),

$$Cov[\tilde{e}_t, e_{m,t}]' \Sigma_{\tilde{e}}^{-1} Cov[\tilde{e}_t, e_{m,t}] \leq \sigma^2(m_t) (1 - \rho^2)\tag{8}$$

where $\Sigma_{\tilde{e}}$ is the covariance matrix of residuals from (5), and ρ is the correlation between m_t and $r_{p,t}$. A proof of (8) along with proofs to all propositions and corollaries is stated in the Appendix.

The right-hand-side of (8) is the variance of $e_{m,t}$. Equality holds if and only if $e_{m,t}$ is an exact linear combination of \tilde{e}_t . Therefore, (8) simply states that the r-squared value from a regression of $e_{m,t}$ on \tilde{e}_t can be at most one. From Shanken (1987), $Cov[\tilde{e}_t, e_{m,t}]$ "may be interpreted as a vector

of deviations from an exact [single] beta expected return relation" (p. 93). The inequality places an upper bound on these deviations and is useful in determining a similar bound for deviations from CAPM pricing measured with respect to a proxy return. Proposition 1 formalizes this result in light of the nonzero covariance between \tilde{e}_t and $r_{p,t}$ in (5).

Proposition 1 *Let the pricing model of (1) hold for all security returns including the proxy return.*

Consider (i) proportionality between the economic aggregate and the market return in (2) and (ii) the measurable analog to the market model in (3) that is given by (5). Define

$$\theta_p = \frac{E[r_{p,t}]}{\sigma[r_{p,t}]} \quad (9)$$

as the Sharpe performance measure for the proxy return, and

$$\eta = \frac{Cov[\tilde{e}_t, r_{p,t}]}{\sigma^2[r_{p,t}]} \quad (10)$$

as a measure of the degree to which unobservable components to the market return covary with the proxy return. Then,

$$d' \Sigma_{\tilde{e}}^{-1} d \leq \theta_p^2 (\rho^{-2} - 1) \quad (11)$$

where

$$d = E[r_t] - (\delta + \eta) E[r_{p,t}].$$

The pricing model of (1) and (2) is not directly testable because $r_{m,t}$ is unobserved. With the exception of ρ , (11) is stated entirely in terms of quantities that are measurable from observed data, provided, of course, that (5) can be identified. Proposition 1, therefore, derives a testable restriction from (1) and (2) conditional on a prior belief about ρ .

The proof of Proposition 1 in Appendix A demonstrates that

$$\rho = \frac{\theta_p}{\sigma[m_t]}.$$

Let $\theta_m = \frac{E[r_{m,t}]}{\sigma[r_{m,t}]}$, the Sharpe performance measure for the market return. Given (2), $\sigma[m_t] = \theta_m$, and ρ is a ratio of Sharpe performance measures. As a consequence, ρ is afforded a geometric

interpretation in mean-standard deviation space as the ratio of the slope of the security market line passing through the excess proxy return to the slope of the security market line tangent to the mean-variance frontier. This ratio links Proposition 1 to a test of relative efficiency for the proxy return.

Let $\beta_p = \frac{Cov[r_{p,t}, r_t]}{\sigma^2[r_{p,t}]}$, the measurable analog to β . Given (5),

$$\beta_p = \beta + \eta. \quad (12)$$

Proposition 1 provides a decomposition of these beta proxies into the true betas and the relation between innovations to the measurable market model and the proxy return caused by measurement error. Affording this decomposition is the structural interpretation of the measurable market model.

Given (12), d is the vector of constants from a linear multivariate regression of r_t on $r_{p,t}$ or the alphas measured with respect to the proxy return.⁵ Term these constants alpha proxies. Proposition 1 places an upper bound on these alpha proxies that is a function of the relative efficiency of the proxy. The left-hand-side of (11) describes observable deviations from CAPM pricing. These deviations need not be zero for the theory to hold because the proxy return not the market return is being used in the restriction. However, these deviations are bounded by the proxy's location in mean-variance space. As ρ approaches one (i.e., the proxy approaches the mean-variance frontier), these deviations approach zero. Let ρ_0 be a prior belief on the true value of ρ (i.e., the true position of the proxy relative to the mean-variance frontier). For a given d , $\Sigma_{\tilde{e}_t}$, and θ_p , let $\bar{\rho}$ be the value of $\rho \in (0, 1]$ that, if it exists, satisfies (11). If $\bar{\rho} < \rho_0$, such is evidence against the pricing model of (1) and (2). The strength of this evidence increases as $\bar{\rho} - \rho_0$ becomes more negative and is, of course, conditional on the correctness of ρ_0 .

Corollary 1 *In (4), suppose that ϕ_t is constant such that $\phi_t = \phi_c$. Then (11) holds as an equality to zero where $d = \alpha$ if and only if $\phi_c = 0$.*

If ϕ_t is constant, the market return is a mean-shift of the proxy return, and (5) is equivalent to an OLS projection. Corollary 1 then relates the pricing restriction of (11) to a statement of mean-variance efficiency similar to either Gibbons, Ross, and Shanken (1989) or MacKinlay and

⁵For a proof of this statement, see (46) and (47) in the Appendix.

Richardson (1991) since each begins with the assertion that $Cov[\tilde{e}_t, r_{p,t}] = 0$. From (44), $\rho = 1$ if and only if the mean-shift is identically zero. In this case

$$d'\Sigma_{\tilde{e}}^{-1}d = 0,$$

which is equivalent to the null hypothesis

$$H_0 : \alpha = 0; \quad \phi_c = 0, \quad (13)$$

since

$$d = E[r_t] - \delta E[r_{p,t}] = \alpha + \beta\phi_c \quad (14)$$

given (5), (6) and the fact that $\eta = 0$.⁶ Failure to reject this null is a failure to reject equivalence between the market and proxy return as well as mean-variance efficiency of the market return. Rejection of this null, on the other hand, is only a rejection of mean-variance efficiency of the proxy return, since either $\phi_c \neq 0$, in which case the proxy return is inefficient because $\rho < 1$, or $\alpha \neq 0$, in which case the proxy and the market return are inefficient, or both. The inability to distinguish between these alternatives illustrates the Roll (1977) critique.

If the market return is a mean-shift of the proxy return, the manner in which Proposition 1 allows for an indirect assessment of the CAPM is parallel to that of Proposition 2 in Shanken (1987). If, on the other hand, $\phi_t \neq \phi_c$, the structural equation in (5) no longer coincides with a projection of r_t onto $r_{p,t}$. Diacogiannis and Feldman (2006) postulate that correlation between $r_{p,t}$ and ϕ_t "might be material when considering the misspecification caused by ignoring, in implementations and tests, the addendum related to $[\phi_t]$ " (p. 20). The following corollary confirms this hypothesis.

Corollary 2 *Let $e_{p,t}$ be the errors from a linear multivariate projection of r_t on $r_{p,t}$, and define*

Σ_{e_p} as the variance-covariance matrix of these errors. Given (4), $\Sigma_{\tilde{e}} - \Sigma_{e_p}$ is positive semi-definite.

From Corollary 2, $\Sigma_{\tilde{e}} \geq \Sigma_{e_p}$. By extension, $\Sigma_{e_p}^{-1} \geq \Sigma_{\tilde{e}}^{-1}$. Compare $d'\Sigma_{e_p}^{-1}d$ to $d'\Sigma_{\tilde{e}}^{-1}d$, noting

⁶From (14), $d = 0$ if $\alpha = -\beta\phi_c$. This latter equality is only satisfied under (13). To see why, note that β cannot be a zero vector. Therefore, if α is nonzero, then ϕ_c needs to be nonzero for the equality to hold. But, given (44) a nonzero ϕ_c means $\rho < 1$, which, in turn, means that $d \neq 0$ given (11).

in both instances that d is a vector of alpha proxies. According to Corollary 2,

$$d' \Sigma_{e_p}^{-1} d \geq d' \Sigma_{\tilde{e}}^{-1} d.$$

From (49), a case where these two quadratic forms equate is when $\phi_t = \phi_c$. In general, however, the degree to which expected returns deviate from the CAPM prediction measured conditional on a proxy return will tend to be overstated if $\Sigma_{e_p}^{-1}$ is used as the weighting matrix as opposed to $\Sigma_{\tilde{e}}^{-1}$. As a consequence, $\bar{\rho}$ will tend to be understated in (11). The end result is that treating the relationship between security returns and the proxy return as a projection equation instead of a structural equation will bias test results of the inequality restriction in Proposition 1 towards rejecting the CAPM theory.

Hansen and Jagannathan (1997) criticize model misspecification tests that depend on the variance-covariance matrix of the pricing errors because these tests grant a "reward for sampling error associated with the sampling mean." In reference to the CAPM, this paper argues that higher sampling error should be accounted for to the extent that it relates to misspecification of the market return. Ignoring this misspecification will bias the test results towards rejecting the theory because of the proxy being used in the test not because of any failing in the theory itself.

Let α_p denote the vector of alpha proxies. If $\phi_t = \phi_c$, then from (14),

$$\alpha_p = \alpha + \beta \phi_c. \tag{15}$$

In this case, differences between the alpha proxies and the true alphas are directly proportional to the mean-shift in the market return relative to the proxy return. These differences are expected to be positive (negative) if β is positive (negative), since a negative mean-shift implies that $\rho > 1$ given (44). If $\phi_t \neq \phi_c$, then

$$\alpha_p = \alpha + \beta E[\phi_t] - \eta E[r_{p,t}] \tag{16}$$

given (6) and (47). In this case, differences between the alpha proxies and the true alphas are ambiguous. Affecting these differences are both the mean of the omitted components as well as the covariance between those components and the proxy return. Provided that the CAPM holds, (15) explains the empirical discovery of "significant" alpha measured with respect to a proxy return

to be the result of the mismeasured portion to the market return. (16) adds to this explanation covariation between this mismeasured portion and the proxy return.

3. Conventional Test

Suppose $\phi_t = \phi_c$ in (4), and assume that $e_t \sim N(0, \Sigma_e)$. Let \hat{d} and $\hat{\Sigma}_e = \frac{1}{T} \sum_{t=1}^T \hat{e}_t \hat{e}_t'$ denote estimates of α_p in (15) and Σ_e , respectively, from N separate OLS regressions of r_{it} on $r_{p,t}$, where r_{it} is the i th element of r_t and $t = 1, \dots, T$. $\hat{\theta}_p$ is an estimate of the proxy performance measure computed from the sample mean and variance of $r_{p,t}$. Consider the following definitions:

$$Q \equiv \frac{T \hat{d}' \hat{\Sigma}_e^{-1} \hat{d}}{1 + \hat{\theta}_p^2}; \quad \lambda \equiv \frac{T d' \Sigma_e^{-1} d}{1 + \hat{\theta}_p^2}.$$

Gibbons, Ross, and Shanken (1989) show that $[N^{-1}(T - N - 1)/T - 2]Q$, conditional on $r_{p,t}$, is distributed as a noncentral F with degrees of freedom N and $T - N - 1$ and non-centrality parameter λ . Multiply both sides of (11) by $T/(1 + \hat{\theta}_p^2)$. Then Proposition 1 is equivalent to

$$H_0 : \lambda \leq \frac{T \theta_p^2 (\rho^{-2} - 1)}{1 + \hat{\theta}_p^2}, \quad (17)$$

which establishes an upper-bound on the non-centrality parameter.

If $\rho = 1$, then $\lambda = 0$, meaning that under Corollary 1, Q follows a central F distribution. In this case, a test of (13) follows immediately because Q is stated entirely in terms of observable quantities. Suppose, instead, that $\rho < 1$. Then, consider conducting a test of $\rho > \bar{\rho}$ conditional on a value for θ_p by evaluating (17) given $\bar{\rho}$ and θ_p to obtain a value for λ which, in turn, can be used in the aforementioned noncentral F test. Shanken (1987) follows this approach. In addition, for a given significance level α , consider finding $\bar{\rho}$ such that the p-value from the non-central F test equals α . Then, $\bar{\rho}$ is the maximum correlation that satisfies Proposition 1 at a significance level of α (for the empirical tests in section 6, $\alpha = 0.05$). Following the discussion in section 2, whether ρ_0 is greater than (less than) $\bar{\rho}$ then determines whether the CAPM is rejected (not rejected).

Violations of the normality assumption for e_t are well documented in the literature.⁷ Numerous studies support Engle's (1982) Autoregressive Conditional Heteroskedasticity (ARCH) and

⁷See Mandelbrot (1963) and Fama (1965) as early examples.

Bollerslev’s (1986) Generalized ARCH (GARCH) in security returns. Common specifications of these models assume e_t to be conditionally normal, which (as demonstrated by Milhoj (1985) or Bollerslev (1986)) results in the unconditional distribution of e_t being leptokurtic; although, the standardized residuals of e_t are still shown to be non-normal empirically. In light of these findings, the potential for mean-variance efficiency tests like those just described to be sensitive to the normality assumption motivated the search for more robust testing methods. From the results of section 2, it is apparent that normality is not necessary for deriving data-dependent restrictions implied by mean-variance efficiency (or relative efficiency). Rather, such a condition is statistically convenient for determining the distributional properties of the resulting test statistics. With this observation in mind, MacKinlay and Richardson (1991) proposed a GMM-based test that, by construction, is distribution free and able to accommodate general forms of heteroskedasticity. These authors uncovered material differences between their approach and that of Gibbons et al. (1989) at conventional levels of significance.

A unifying restriction of both Gibbons et al. (1989) and MacKinlay and Richardson (1991) is that $\phi_t = \phi_c$. Corollary 2 illustrates how a violation of this assumption could impact a test of relative mean-variance efficiency. The testing methodology developed in the next section is robust to ϕ_t and is built upon the premise that \tilde{e}_t follows a GARCH process but one that is not, necessarily, conditionally normal.⁸

4. Econometric Methodology

An empirical investigation into Proposition 1 requires estimation of all quantities, with the exception of ρ , in (11). From the proof of Corollary 2, d is the vector of constant terms from a multivariate projection of r_t onto $r_{p,t}$. As such, the individual elements of d can be estimated following the same approach outlined in section 3. If $\phi_t = \phi_c$, then $\Sigma_{\tilde{e}} = \Sigma_e$ and can also be estimated in the manner described under section 3. If, on the other hand, ϕ_t is stochastic, then $r_{p,t}$ is endogenous to (5). Any method for estimating (5) and, hence, $\Sigma_{\tilde{e}}$ needs to be robust to this endogeneity.

⁸Diebold, Im, and Lee (1989) provide evidence that market model residuals are heteroskedastic.

From (5), the relationship between any given security return and the proxy return can be expressed as

$$r_{1,t} = \gamma_{1,0} + \delta_0 r_{p,t} + \epsilon_{1,t} \quad (18)$$

$$r_{p,t} = \gamma_{2,0} + \epsilon_{2,t}, \quad (19)$$

which is a triangular system without instruments, where r_{1t} is a given security return and $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are unobserved errors or shocks. Let $\epsilon_t = [\epsilon_{1,t} \ \epsilon_{2,t}]'$. The term $\gamma_{1,0}$ refers to the true value of γ_1 , with the same interpretation holding for all other parameter values. The sketch of an identification result together for (18) and (19) with an associated estimator follows. Prono (2008) provides a detailed discussion of both.

Note that (18) makes no explicit use of the error decomposition in (6), meaning that the effect of ϕ_t is considered only at the level of $\epsilon_{1,t}$. No attempt is made to isolate or identify properties unique to ϕ_t . The functional form describing the relationship between $\epsilon_{1,t}$ and $\epsilon_{2,t}$ is sufficiently general to place only minimal constraints on the process governing ϕ_t .

Define S_{t-1} as the σ -field generated by $\{\epsilon_{t-1}, \epsilon_{t-2}, \dots\}$, and assume

$$E[\epsilon_t | S_{t-1}] = 0, \quad E[\epsilon_t \epsilon_t' | S_{t-1}] = H_t,$$

which is the definition of semi-strong GARCH from Drost and Nijman (1993). An advantage of this definition is that no particular conditional distribution for ϵ_t needs to be assigned. Let

$$vech(H_t) = h_t, \quad vech(\epsilon_t \epsilon_t') = e_t,$$

and parameterize h_t as

$$h_t = C_0 + A_0 e_{t-1} + B_0 h_{t-1}, \quad (20)$$

where C_0 is a 3×1 column vector and A_0 and B_0 are both 3×3 diagonal matrices. Throughout this section, $vech(\cdot)$ denotes the matrix operator that stacks the lower triangle, including the diagonal, of a symmetric matrix into a column vector, while $vec(\cdot)$ is the matrix operator that stacks the columns of a matrix into a column vector. In addition, $A = [a_{jk}]$ denotes any matrix A .

(20) is a bivariate version of the familiar GARCH(p, q) model introduced by Bollerslev (1986).

The special case of $p = q = 1$ is popular because of its parsimony and forecasting power. Examples of its use in the modeling of multivariate financial time series include Bollerslev, Engle, and Wooldridge (1988) and Bollerslev (1990), where the former investigates a conditional CAPM. For the purposes of identifying the triangular system, the GARCH model of (20) is the key identifying assumption; in particular, the diagonal specification of the parameter matrices A_0 and B_0 .

H_t needs to be positive definite almost surely. This requirement translates into restrictions on the parameters $c_{j1,0}$, $a_{jk,0}$, and $b_{jk,0}$ in (20). One way to satisfy this requirement (and the one relied on in the empirical work of this paper) is to specify h_t according to a bivariate diagonal BEKK(1,1) model. The BEKK model is for general multivariate GARCH processes and is developed by Engle and Kroner (1995).

Consider only the conditional covariance between $\epsilon_{1,t}$ and $\epsilon_{2,t}$ as well as the conditional variance of $\epsilon_{2,t}$. In doing so, let $\bar{e}_t = \begin{bmatrix} \epsilon_{1,t}\epsilon_{2,t} & \epsilon_{2,t}^2 \end{bmatrix}'$, and define $Z_{t-2} = [\bar{e}'_{t-2} \cdots \bar{e}'_{t-L}]'$ for a finite $L \geq 2$. If \bar{e}_t is covariance stationary (which requires $\epsilon_{2,t}$ to be fourth moment stationary), then the conditional moment restrictions from (20) imply for the unconditional autocovariances of \bar{e}_t that

$$Cov[\bar{e}_t, Z_{t-2}] = (\bar{A}_0 + \bar{B}_0) Cov[\bar{e}_t, Z_{t-1}], \quad (21)$$

where \bar{A}_0 is a 2×2 diagonal matrix formed from the elements $a_{22,0}$ and $a_{33,0}$ in A_0 (a parallel definition holds for \bar{B}_0), and $Cov[\bar{e}_t, Z_{t-i}] \equiv E[(\bar{e}_t - E[\bar{e}_t])(Z_{t-i} - E[Z_{t-i}])']$ for $i \geq 1$ (the Lemma in Prono (2008) demonstrates this result).

For ARCH models (i.e., where $B_0 = 0$), Rich, Raymond, and Butler (1991) realized that since (20) implies

$$e_t = h_t + \omega_t,$$

where $E[\omega_t | S_{t-1}] = 0$, then $E[\omega_t e'_{t-1}] = 0$, thus enabling A_0 to be identified by unconditional moment restrictions familiar to the OLS and IV estimators. (21) is an extension of this result to the GARCH(1,1) model.

Let $R_t = \begin{bmatrix} R_{1,t} & R_{2,t} \end{bmatrix}'$ be the vector of reduced form errors from (18) and (19). The structural errors ϵ_t are related to R_t by

$$\epsilon_t = \Delta_0^{-1} R_t, \quad (22)$$

where $\Delta_0 = \begin{bmatrix} 1 & \delta_0 \\ 0 & 1 \end{bmatrix}$. Given (22), substituting the reduced form expressions for $\epsilon_{1,t}$, $\epsilon_{2,t}$ and $\epsilon_{2,t}^2$ into (21) produces

$$Cov [\bar{r}_t, Z_{r,t-2}] = (\bar{A}_{0,r} + \bar{B}_{0,r}) Cov [\bar{r}_t, Z_{r,t-1}],$$

where \bar{r}_t and $Z_{r,t-i}$ (for $i = 1, 2$) are the reduced forms of \bar{e}_t and Z_{t-i} , respectively, and $\bar{A}_{0,r}$ and $\bar{B}_{0,r}$ are reduced form parameter matrices. While \bar{A}_0 and \bar{B}_0 are diagonal, $\bar{A}_{0,r}$ and $\bar{B}_{0,r}$ are upper triangular. Furthermore, since $dg(\bar{A}_{0,r} + \bar{B}_{0,r}) = \bar{A}_0 + \bar{B}_0$, where $dg(\cdot)$ is the matrix operator that forms a diagonal matrix from the principal diagonal of any matrix, the off-diagonal element of $\bar{A}_{0,r} + \bar{B}_{0,r}$ is a function of the diagonal elements and of δ_0 . Therefore, if $\bar{A}_{0,r} + \bar{B}_{0,r}$ is identified (which it is if $Cov[\bar{r}_t, Z_{r,t-1}]$ or, equivalently, $Cov[\bar{e}_t, Z_{t-1}]$ has a row rank of two), then this parameter matrix defines a system of 3 reduced form equations in 3 structural unknowns (those unknowns being $a_{22,0}$, $a_{33,0}$, and δ_0) with a unique solution for δ_0 .

Iglesias and Phillips (2004) demonstrate that if the structural errors from a triangular system follow a diagonal GARCH process, the reduced form errors, while still GARCH, are no longer diagonal GARCH. The reduced form parameter matrices $\bar{A}_{0,r}$ and $\bar{B}_{0,r}$ both illustrate this point and evidence how departures from diagonality permit identification. In particular, since $dg(\bar{A}_{0,r} + \bar{B}_{0,r}) = \bar{A}_0 + \bar{B}_0$, the element in the upper triangle of $\bar{A}_{0,r} + \bar{B}_{0,r}$ is restricted by the diagonal terms. It is from this restriction that identification follows. In discussing how the relationship between structural and reduced form GARCH models can identify simultaneous systems, Rigobon (2002) states "the model of heteroskedasticity of the structural residuals impose[s] important constraints on how the reduced form heteroskedasticity can evolve" (p.433). The relevant constraint here is the exclusion of all off-diagonal terms in the formulation of h_t in (20).

The traditional method for identifying the triangular system is to impose exclusion restrictions on some of the exogenous variables affecting the conditional mean, which is equivalent to assuming the existence of outside instruments. For the triangular system of (18) and (19), no such exclusions are available. However, diagonality of the parameter matrices A_0 and B_0 in (20) is the extension of exclusion restrictions onto the second moments. Without these restrictions, the triangular system would remain unidentified, which is to say that the existence of conditional heteroskedasticity alone is not sufficient for identification. For instance, suppose h_t follows a fully general GARCH

model, which requires A_0 and B_0 to be composed entirely of nonzero terms. Then the structural form GARCH model imposes no constraints on how the reduced form can evolve. In this case, the reduced form parameter matrix $\bar{A}_{0,r} + \bar{B}_{0,r}$ defines a system of 4 equations in 5 unknowns.⁹

In the sketch of the identification result above, the conditional variance of $\epsilon_{1,t}$ plays no role. Therefore, the parameterization of $h_{11,t}$ in (20) need not imply a finite fourth moment for $\epsilon_{1,t}$ as is the requirement for $h_{22,t}$ in regards to $\epsilon_{2,t}$.

Given the identification result, an estimator for (18) and (19) can be constructed from (21) and the fact that $E[\epsilon_t] = 0$. Towards that end, let $\epsilon_t = [\epsilon_{1,t} \quad \epsilon_{2,t}]'$, and $\bar{e}_t = [\epsilon_{1,t}\epsilon_{2,t} \quad \epsilon_{2,t}^2]'$, where

$$\begin{aligned}\epsilon_{1,t} &= r_{1,t} - \gamma_1 - \delta r_{p,t} \\ \epsilon_{2,t} &= r_{p,t} - \gamma_2\end{aligned}$$

Define $\psi = \{\gamma_1, \gamma_2, \delta, \bar{C}, \bar{A} + \bar{B}\}$, where $\bar{C} = [c_{21} \quad c_{31}]'$ and Ψ as the set of all possible values for ψ . Let $\bar{\sigma} = [I - (\bar{A} + \bar{B})]^{-1} \bar{C}$, where I is the identity matrix, and $z_{t-2} = [(\bar{e}_{t-2} - \bar{\sigma})' \cdots (\bar{e}_{t-L} - \bar{\sigma})']'$. Construct the sample moments

$$\begin{aligned}g_1 &= \widehat{E}[\epsilon_t], & g_2 &= \widehat{E}[\bar{e}_t] - \bar{\sigma}, \\ g_3 &= \widehat{Cov}(\bar{e}_t, z_{t-2}) - (\bar{A} + \bar{B}) \widehat{Cov}(\bar{e}_t, z_{t-1}),\end{aligned}$$

where \widehat{E} and \widehat{Cov} are estimates of the expectation and covariance operators, respectively, and stack these moments into a single vector $g = [g_1 \quad g_2 \quad vec(g_3)]'$. Given certain regulatory conditions, the standard Hansen (1982) GMM estimator

$$\widehat{\psi} = \arg \min_{\psi \in \Psi} g' W_T g \tag{23}$$

is weakly consistent for some sequence of positive definite W_T . The choice of W_T follows Prono (2008) and is constructed such that the autocovariances in g_3 are transformed into autocorrelations. This transformation requires preliminary estimates of the variance to both $\epsilon_{1,t}\epsilon_{2,t}$ and $\epsilon_{2,t}^2$.

Application of (23) to the N separate structural equations in the form of (18) that are implied

⁹Rigobon (2002) formalizes this result in an appendix.

by (5) produces consistent estimates of the N elements in \tilde{e}_t , which can then be used to estimate $\Sigma_{\tilde{e}}$. To close this section, $\hat{\theta}_p = \frac{\hat{E}[r_{p,t}]}{\hat{\sigma}_{22}}$.

5. Test Methodology

The inequality restriction in (11) is equivalent to

$$H_0 : \rho \leq \sqrt{\frac{1}{1 + \theta_p^2 d' \Sigma_{\tilde{e}}^{-1} d}} \quad (24)$$

which identifies an upper bound for ρ , since ρ is strictly positive. Define $\xi \equiv \sqrt{\frac{1}{1 + \theta_p^2 d' \Sigma_{\tilde{e}}^{-1} d}}$. Section 4 outlines a methodology for obtaining $\hat{\xi}$. An analogous approach to Shanken (1987) would be to determine the distribution (either asymptotic or exact) of ξ so that a test of $\xi > \bar{\xi}$ could be conducted. For a given significance level α , the value of $\bar{\xi}$ that produces a p-value from that distribution equal to α is then the maximum correlation supporting Proposition 1. A comparison of $\bar{\xi}$ to ρ_0 determines whether the CAPM is rejected (not rejected) depending on whether the inequality is $<$ ($>$). Determining a distribution for ξ , however, would be difficult, owing, in no small part, to the heteroskedastic properties assumed for \tilde{e}_t that permit its identification. An alternative approach would be to bootstrap a standard error for $\hat{\xi}$ and use this standard error to determine $\bar{\xi}$. This paper adopts the alternative methodology.

Bootstrapping a standard error for $\hat{\xi}$ requires resampling from the N excess security returns and the excess proxy return used to form the quantities $\hat{\theta}_p$, \hat{d} , and $\hat{\Sigma}_{\tilde{e}}$. Such is a nontrivial exercise since these returns are not iid and, in fact, their departure from independence (both within and across return series) is a key assumption underlying the estimator that generates $\hat{\Sigma}_{\tilde{e}}$. Define

$$\epsilon_t^{(i)} = \begin{bmatrix} \epsilon_{i,t} & \epsilon_{\tilde{2},t} \end{bmatrix}', \quad i = 1, \dots, N,$$

where $\epsilon_{i,t} = r_{i,t} - \gamma_{i,0} - \delta_{i,0} r_{p,t}$, the errors from the structural equation for the i th security return, and $\epsilon_{\tilde{2},t}$ is the demeaned proxy return. Suppose

$$\epsilon_t^{(i)} = \left(H_t^{(i)} \right)^{1/2} V_t^{(i)}, \quad (25)$$

where $H_t^{(i)}$ is the conditional variance-covariance matrix for the i th security return and the proxy return parameterized according to (20), and $V_t^{(i)} = \begin{bmatrix} V_{i,t} & V_{2,t} \end{bmatrix}'$. The vector $V_t^{(i)}$ is assumed to be iid with mean zero and identity variance-covariance matrix. (25) defines a strong GARCH process. Unlike most applications of strong GARCH, however, no particular distribution is assumed for $V_t^{(i)}$. The estimator in (23) supplies $\widehat{\epsilon}_t^{(i)}$. Conditional on this estimate, one can obtain $\widehat{H}_t^{(i)}$. As a result, $\widehat{V}_t^{(i)} = \left(\widehat{H}_t^{(i)}\right)^{-1/2} \widehat{\epsilon}_t^{(i)}$. Bootstrap samples are drawn from $\widehat{V}_t^{(i)}$. Let $\widehat{V}_t^{(i)*}$ be a bootstrap sample. Then $\widehat{\epsilon}_t^{(i)*} = \left(\widehat{H}_t^{(i)*}\right)^{1/2} \widehat{V}_t^{(i)*}$, where $\widehat{H}_t^{(i)*}$ is based upon parameter estimates from the original sample, and

$$\widehat{r}_{p,t}^* = \widehat{\gamma}_2 + \widehat{\epsilon}_{2,t}^*, \quad (26)$$

$$\widehat{r}_{i,t}^* = \widehat{\gamma}_i + \widehat{\delta}_i \widehat{r}_{p,t}^* + \widehat{\epsilon}_{i,t}^*, \quad i = 1, \dots, N, \quad (27)$$

where $\widehat{\gamma}_2$, $\widehat{\gamma}_i$, and $\widehat{\delta}_i$ are also obtained from the original sample. The resulting bootstrap series is then used to estimate $\widehat{\xi}^*$ given the estimation method described in section 4.

Define E^* as the expectation operator relative to the distribution of the bootstrap sample conditional on the original sample, and let

$$g = \frac{1}{T} \sum_{t=1}^T g_t.$$

Following Hall and Horowitz (1996), the bootstrap version of g_t is

$$g_t^* = g_t - E^* \widehat{g}_t, \quad (28)$$

where \widehat{g}_t is g_t evaluated at $\widehat{\psi}$, the parameter estimates from the original data sample. (28) recenters the bootstrap moment conditions such that $E^* g_t^* = 0$. In general, $E^* g_t \neq 0$ when the number of moment conditions exceeds the number of parameters in ψ . If g_t is used instead of g_t^* , then $\widehat{\psi}^*$ will have different asymptotic properties than $\widehat{\psi}$. In order to avoid this discrepancy,

$$\widehat{\psi}^* = \arg \min_{\psi \in \Psi} g^{*'} W_T^* g^*. \quad (29)$$

The bootstrap standard error of $\widehat{\xi}$ is based on $\widehat{\psi}^*$.

Given a standard error for $\hat{\xi}$, the asymptotic t statistic

$$\hat{\tau} = \frac{\hat{\xi} - \bar{\xi}}{\widehat{se}(\hat{\xi})} \quad (30)$$

can be constructed to test $\hat{\xi} > \bar{\xi}$. This statistic is asymptotically pivotal with an asymptotic distribution assumed to be well approximated by a standard normal.¹⁰ As a result, for $\alpha = 0.05$, the value of $\bar{\xi}$ can be determined such that $\Phi(\hat{\tau}) = \alpha$.

According to MacKinnon (2007), bootstrapping (30) will generally lead to an asymptotic refinement. Such a practice is referred to as the double or iterated bootstrap. Implementing the double bootstrap, however, is very computationally expensive. For example, define B_1 as the number of bootstrap iterations used to generate $\widehat{se}(\hat{\xi})$ and B_2 as the number of iterations used to generate the bootstrap distribution of (30). If $B_1 = B_2 = 1000$, then the total number of iterations required for the double bootstrap is approximately 1 million. Given the size of the data samples used to construct $\hat{\xi}$ (see section 6), the standard normal will likely provide a descent approximation to the asymptotic distribution of (30). A Monte Carlo study (see section 7) verifies this claim. As a result, this approximation is used as opposed to the double bootstrap alternative.

6. Test Results

All tests are conducted using size, B/M, and momentum portfolios. These portfolios are studied because they reflect the size, value, and momentum "premiums" that empirical applications of the CAPM struggle to explain. The returns are measured weekly (in percentage terms) from 10/6/67 through 9/28/07. Test results consider 20- and 10-year subperiods of this overall date range. The daily 25 size-B/M and 25 size-momentum return files (each 5×5 sorts with breakpoints determined by NYSE quintiles) formed from all securities traded on the NYSE, AMEX, and NASDAQ exchanges are used to construct the weekly return series.¹¹ Monte Carlo studies of (23) reveal sizable benefits in terms of reduced finite sample bias and increased efficiency from using large sample sizes due to the fact that higher moments are being estimated. In light of this finding, weekly

¹⁰From MacKinnon (2007), "a test statistic is asymptotically pivotal if its asymptotic distribution does not depend on anything that is unknown" (p.5).

¹¹These return files are available on Kenneth French's website.

returns are utilized. Further supporting this frequency choice is the fact that weekly returns reduce day-of-the-week and weekend effects as well as the effects of nonsynchronous trading and bid-ask bounce. The size portfolios considered are "Small," "Mid," and "Large." "Small" is the average of the five low-market-cap portfolios, "Mid" the average of the five medium-market-cap portfolios, and "Big" the average of the five large-market-cap portfolios. The B/M portfolios considered are "Value," "Neutral," and "Growth." "Value" is the average of the five high-B/M portfolios, "Neutral" the average of the five middle-B/M portfolios, and "Growth" the average of the five low-B/M portfolios.¹² Finally, the momentum portfolios considered are "Losers," "Draws," and "Winners." "Losers" is the average of the five low-return-sorted portfolios, "Neutral" the average of the five middle-return sorted portfolios, and "Winners" the average of the five high-return-sorted portfolios. The proxy return is the CRSP value-weighted index return formed from all securities traded on the NYSE, AMEX, and NASDAQ exchanges. The risk-free rate is the one-month Treasury bill rate from Ibbotson Associates.

The tests focus on (17) and (24). The former is conducted following the approach developed in Gibbons, Ross, and Shanken (1989), referred to hereafter as GRS, that is implemented in Shanken (1987) described in section 3. The latter is conducted following the approach of section 5 under two cases: (1) $\phi_t = \phi_c$, (2) ϕ_t is stochastic. Case 1 will be referred to as Bootstrap Proposition 1 constant (BPC), while case 2 will be referred to as Bootstrap Proposition 1 stochastic (BPS). The only difference in implementation between BPC and BPS is that under the former, OLS regressions estimate the measurable market model while, under the latter, this model is estimated using (23) and its bootstrap analog in (29). A comparison of BPC to GRS evidences the effects of conditional heteroskedasticity on a test of relative mean-variance efficiency. A comparison of BPC to BPS evidences the effects of treating the market model as a projection as opposed to a structural equation. When implementing BPS, the number of lags used in (23) is set to $L = 4$. The choice of this lag length is motivated by the frequency of returns as well as the finding in Prono (2008) that higher lag lengths, while successful at reducing the variability of $\hat{\psi}$ also increases the finite sample bias. Finally, all bootstrap routines are conducted over 1000 trials.

Table 1 (A and B) and 2 (A and B) summarizes results from two 20-year subperiods: (1) 10/6/67 - 9/25/87, (2) 11/6/87 - 9/28/07. Tables 1A and 2A provide summary statistics of the returns used

¹²Definitions for the "Small," "Large," "Value," and "Growth" portfolios are taken from Lewellen and Nagel (2006).

in the tests as well as the alpha proxies (accompanied with heteroskedasticity-corrected standard errors) from individual OLS regressions of those returns on the proxy. Tables 1B and 2B describe the maximum correlation between the proxy return and the market return that still supports the CAPM result at a 5% significance level according to the GRS, BPC, and BPS tests. Recall that all three tests are based on the inequality restriction in (11). If $\rho < 1$, then a test of this restriction requires a prior belief on the true value of the correlation ρ_0 . From Roll (1977), $\rho_0 = 0.90$ or above. This value will be used throughout the discussions of the test results.

For the GRS test, if $\rho < 1$, then a test of (17) also requires the true value of θ_p . Possible values for θ_p are taken from Shanken (1987). $\theta_p = 0.52$ is the most likely (or expected) value. $\theta_p = 0.22$ and $\theta_p = 0.86$ are - 1 standard deviations and + 2 standard deviations away from this expected value, respectively. All values of θ_p are annualized for presentation but expressed in weekly terms when used in the tests. Assuming an annual standard deviation of 20% for the proxy return, $\theta_p = 0.22$ corresponds to a "market" premium of 4.4%, $\theta_p = 0.52$ a "market" premium of 10.4%, and $\theta_p = 0.86$ a "market" premium of 17.2%.¹³ This range for θ_p is sufficiently wide to encompass the point estimates for θ_p implied by the different subperiods considered. Finally, $\theta_p = 1.00$ is also reported as a value for the proxy Sharpe ratio that is greater than any conceivable true value.

Since (17) requires θ_p to be known, for comparative purposes θ_p is treated as known in (24) for the BPC and BPS tests. In addition, however, θ_p is also treated as unknown in the latter two tests, meaning that its value is bootstrapped along with every other random quantity in $\hat{\xi}$. In the tables, the heading "unknown" under Panel F: BPC and Panel G: BPS details the results of this more general treatment.

Under Tables 1B and 2B, note that (1) the projection errors appear to be non-normal, characterized by (at times) significant skewness and (often times) excess kurtosis, and (2) there exist apparent differences between the projection and structural errors. These two findings foreshadow differences between the GRS, BPC, and BPS tests. Also under Tables 1B and 2B, a comparison of the maximum correlations for known values of θ_p between the GRS and BPC tests reveals the general tendency of higher correlations implied by the former. Such a tendency implies that the former test will tend to under-reject the CAPM relative to the latter. MacKinlay and Richardson

¹³By "market" premium, what is meant is the market premium implied by the proxy.

(1991) document a similar finding in their empirical work. As an example, for the period 10/6/67 - 9/25/87 at $\theta_p = 0.52$, $\bar{\rho} = 0.86$ according to GRS but $\bar{\rho} = 0.74$ according to BPC for the size portfolios. For the period 11/6/87 - 9/28/07, the same comparison yields $\bar{\rho} = 1.00$ according to GRS as opposed to $\bar{\rho} = 0.93$ according to BPC. This latter comparison possesses economic significance since the former cannot reject mean-variance efficiency of the proxy (see Corollary 1), while the latter can. When comparing GRS and BPC across the two 20-year time periods, the largest differences in relative efficiency occur for the size portfolios. The B/M portfolios show a similar directional difference, though on a more muted scale. For both 20-year time periods, the maximum correlations measured relative to the momentum portfolios are higher for BPC than for GRS. The difference between these correlations, however, is small.

A comparison of the maximum correlations for constant values of θ_p between the BPC and BPS tests supports the results of Corollary 2. These correlations are generally higher under the latter. The largest correlation difference between BPC and BPS occurs for the size portfolios in the second 20-year time period for $\theta_p = 0.22$. In this case, $\bar{\rho} = 0.64$ under BPC, while $\bar{\rho} = 0.86$ under BPS. Also in the second 20-year period, positive correlation differences between the BPS and BPC tests are apparent across all values of θ_p for the B/M and momentum portfolios.

Treating θ_p as unknown under BPS offers the most general test considered and is the principal contribution of this paper to relative efficiency testing. In contrast, the GRS method requires θ_p to be known. The most natural means of comparison between GRS and BPS with an unknown θ_p is to assume that $\theta_p = \hat{\theta}_p$ (the sample-specific estimate of θ_p) for the former, since $\hat{\theta}_p$ is the point estimate around which bootstrap samples are generated. For the first 20-year time period, $\hat{\theta}_p = 0.22$ while for the second, $\hat{\theta}_p \approx 0.52$. In the second 20-year time period, therefore, $\bar{\rho} = 0.54$ under GRS as compared to $\bar{\rho} = 0.75$ under BPS for the momentum portfolios. The former result implies that the proxy return explains less than 30% of the variation in the market return ($0.54^2 = 0.29$), while the latter implies that the proxy return accounts for over 55% of the variation in the market return. In this case, GRS rather significantly understates the relative efficiency of the proxy return when compared to BPS. An inference from this result is that the GRS test will tend to over-reject the CAPM prediction. Understatement of the relative efficiency of the proxy return by GRS when compared to BPS (with an unknown θ_p) is consistent across both time periods and for all portfolios considered. The difference in the implied mean-variance location of the proxy

portfolio can be striking. For instance, $\bar{\rho}$ is 59% and 64% higher according to BPS when compared to GRS for the B/M and momentum portfolios, respectively, in the first 20-year time period.

The BPS test with an unknown θ_p cannot reject the null hypothesis that the proxy return is mean-variance efficient for size portfolios in the most-recent 20-year period. Otherwise, the test results do not speak favorably for the CAPM. If $\rho_0 = 0.90$, then the result of Proposition 1 is rejected for all remaining time periods and portfolios. The CAPM fares decidedly worse on B/M and momentum portfolios relative to size portfolios and performs the poorest on momentum portfolios. A potential bright-spot emerges when comparing results between the two time periods. The size of the CAPM errors for all the portfolios considered is greatly reduced in the most-recent period, since the implied correlations very nearly double. This paper, therefore, documents a significant increase in the ability of the CAPM to explain the size, value, and momentum "premiums" post the 1987 market crash.

As a robustness check, the GRS, BPC, and BPS tests are also applied to three 10-year subperiods: (1) 10/7/77 - 9/25/87, (2) 11/6/87 - 9/26/97, (3) 10/3/97 - 9/28/07.¹⁴ Tables 3 (A and B) through 5 (A and B) summarize the results. These results are largely consistent with those for the two 20-year subperiods discussed above. Namely, for constant values of θ_p , GRS tends to imply higher correlations than BPC, and BPC tends to imply lower correlations than BPS. In addition, GRS when evaluated at $\theta_p = \hat{\theta}_p$ tends to imply lower correlations than BPS evaluated with an unknown θ_p . Moreover, significant differences between the tests continue to be evidenced. For example, during the period 11/6/87 - 9/26/97, the GRS test rejects the CAPM prediction at all levels of θ_p for the B/M portfolios. The BPS test with an unknown θ_p , on the other hand, does not. During the period 10/3/97 - 9/28/07, also for the B/M portfolios, the GRS test fails to reject the CAPM at all levels of θ_p , while the BPS test with an unknown θ_p offers a sound rejection.

7. Monte Carlo

The previous section establishes that inferences on the relative efficiency of a given proxy return can be sensitive to the test considered. This section investigates the source of these differences. For instance, do these differences signal the inappropriateness of a normality assumption

¹⁴The period 10/6/67 - 9/30/77 is not considered because the mean of the proxy return is negative.

for innovations to the market model? Do they signal the inappropriateness of treating the market model as a projection? Alternatively, do they reflect poor finite sample performance of the GMM estimator in (23) or the fact that the asymptotic distribution of the test statistic in (30) is not well approximated by a standard normal?

In order to assess these possibilities, consider a Monte Carlo study of the following design. From (25), with $N = 3$ let the individual components of $V_t^{(i)}$ be distributed as standardized Gamma(2,1) random variables, and parameterize $H_t^{(i)}$ using the estimates obtained for the B/M portfolios over the period 11/6/87 - 9/28/07 that do not assume $h_{i\tilde{2},t} = 0$.¹⁵ Further, let $\hat{\gamma}_{\tilde{2}}$ and $\hat{\delta}_i$ from (26) and (27), respectively, be obtained from (23) applied to the same data set mentioned above. Consider the pricing restriction of (11) stated in terms of estimated quantities. Conditional on $\hat{H}_t^{(i)}$, $\hat{\gamma}_{\tilde{2}}$, and $\hat{\delta}_i$, $\hat{\gamma}_i$ in (27) is set so that the individual components of \hat{d} are equal and support $\rho = 0.90$.¹⁶ Therefore, the data generating process (DGP) considered in this study supports the CAPM.

Given the DGP described above, this study examines the rejection rates of the GRS, BPC, and BPS tests at 10%, 5%, and 1% significance levels when either $\bar{\rho}$ or $\bar{\xi}$ is set equal to 0.90. For the GRS test, θ_p is assumed to be known and is set equal to the estimate from the original sample. For the BPC and BPS tests, θ_p is treated as unknown. For all three test statistics, simulations are conducted across 500 trials generating excess return series of 1000 observations each. When constructing the individual excess return series for each trial, the first 200 observations are dropped to avoid initialization effects. For the BPC and BPS statistics, within each simulation trial a bootstrap of $\hat{\xi}$ is conducted over 250 repetitions.¹⁷ Parameter estimates for implementing these routines do not vary by simulation trial. These parameter estimates are generated from the original data sample

¹⁵The Gamma(2,1) distribution is chosen because, when combined with $\hat{H}_t^{(i)}$, this distribution produces errors with unconditional skewness and kurtosis measures comparable to those described under Panel D for the B/M portfolios of Table 2B.

¹⁶From (47),

$$\hat{\gamma}_i = \hat{d}_i + \hat{\eta}_i \hat{E} [r_{p,t}].$$

Let $\hat{d}_i = \hat{d}_j \forall i, j = 1, 2, 3$. From the B/M portfolios and proxy return measured over the period 11/6/87 - 9/28/07, \hat{d}_i is calibrated such that $\rho = 0.90$ if (11) is treated as an equality. $\hat{\eta}_i$ is the slope parameter from an OLS regression of $\hat{\epsilon}_{i,t}$ on $r_{p,t}$.

¹⁷The test results bootstrap $\hat{\xi}$ for 1000 repetitions. Only 250 repetitions are considered here in order to keep the simulation time feasible. This truncated number of repetitions should still produce a decent estimate of the standard error.

in the manner described above, although in the case of the BPC statistic, the additional constraint of $h_{i\bar{2},t} = 0$ is imposed.

Table 6 reports the simulation results. Both the GRS and BPC statistics over-reject the null hypothesis that ρ is at least 0.90 or, equivalently, that the CAPM holds. Across the size levels considered, the GRS statistic over-rejects more than does the BPC statistic. Simulation studies of Zhou (1993) and Chou (1996) show that the GRS statistic tends to over-reject the null hypothesis of an efficient proxy return (i.e., that $\rho = 1$) when the distribution of the errors to the market model is non-elliptical.¹⁸ The results presented here compliment those of Zhou (1993) and Chou (1996) by showing that the tendency for GRS to over-reject extends to tests of relative efficiency where temporal dependence is a factor governing the non-elliptical nature of the error distributions.

In general, the BPS statistic is appropriately sized. A tendency for under-rejecting the null hypothesis is evident at the 10% significance level, but this tendency is modest and is no larger in absolute value than the size distortions evidenced by the other two tests.¹⁹ No such tendency (in either direction) is meaningfully evidenced at either the 5% or 1% significance levels. As a result, it does not appear that poor finite sample performance of the GMM estimator and/or poor approximation of the asymptotic distribution of the test statistic by a standard normal meaningfully distorts the size of the BPS test. In addition, results from the BPC and BPS tests support the finding of Corollary 2 that treating the market model as a linear projection leads to an over-rejection of the CAPM in cases where $\phi_t \neq \phi_c$. Finally, from Table 2B, note that the maximum correlations determined by the GRS, BPC, and BPS statistics are 0.73, 0.824, and 0.841, respectively.²⁰ The rank order of these maximum correlations is supported by the simulation results.

8. Extension

This section generalizes Proposition 1 in terms of conditional moment restrictions and, in doing so, links mismeasurement of the market return to time-variation in "beta." In order to develop this generalization, define I_{t-1} as the information set containing past histories of r_t , $r_{p,t}$, and x_t , a vector

¹⁸The method proposed by Zhou (1993) requires the error distributions to be specified, while Chou (1996) utilizes a bootstrap approach, but one where temporal independence is assumed.

¹⁹Chou (1996) reports similar size distortions at a 10% level for bootstrap tests of mean-variance efficiency.

²⁰For the GRS test, $\theta_p = \hat{\theta}_p = 0.52$. For both the BPC and BPS tests, θ_p is unknown.

of forecasting instruments. Moments for period t conditional on I_{t-1} are labeled with a t subscript as are parameters conditional on I_{t-1} . Consider a conditional version of the pricing model in (1)

$$E_t[r_t] = Cov_t[m_t, r_t], \quad (31)$$

where

$$m_t = \left(\frac{E_t[r_{m,t}]}{\sigma_t^2[r_{m,t}]} \right) r_{m,t}. \quad (32)$$

(31) and (32) imply a conditional CAPM.²¹ In addition, assume

$$\beta = \frac{Cov_t[r_{m,t}, r_t]}{\sigma_t^2[r_{m,t}]} \quad (33)$$

so that the true betas are constant parameters and time variation in expected security returns is driven by changes in the market risk premium. Ferson (1990) asserts that the specification of constant betas "is an important assumption in the context of models with conditional expectations" (p.399).²²

Consider the following generalization of the measurable market model in (5)

$$r_t = \gamma_t + \delta r_{p,t} + \tilde{e}_t, \quad (34)$$

where

$$\gamma_t = \alpha + \beta E_t[\phi_t], \quad \tilde{\phi}_t = \phi_t - E_t[\phi_t].$$

and δ as well as \tilde{e}_t retain their definitions from (6). A case for $Cov[\tilde{e}_t, r_{p,t}] \neq 0$ follows the same logic outlined in section 2. (34) affords a general specification for the time-varying mean of security returns.²³ This time variation is linked to time variation in the expected proxy return as

²¹Harvey (1989), Bodurtha and Mark (1991), Jagannathan and Wang (1996), Lettau and Ludvigson (2001), Adrian and Franzoni (2004), and Ang and Chen (2007) all consider versions of the CAPM in this form.

²²In nearly all cases, a conditionally mean-variance efficient portfolio will exist, implying that so too will a single beta model for expected returns. In general, the beta from this model will be time-varying.

²³There is a consensus in the literature that expected returns are time-varying conditional on a set of forecasting instruments. Potential instruments include (i) lagged values of the proxy return to capture reversion as evidenced in Keim and Stambaugh (1986) and Fama and French (1989) among others, (ii) the term spread as measured by the difference between the 10-year and 3-month yields and advocated by Fama and French (1989), (iii) Moody's BAA - AAA credit spread (see, e.g, Campbell (1996), and (iv) the value spread as measured by the return difference between

well as the expected value of the components omitted from that proxy return. In the special case where the market return is a mean-shift of the proxy return, the single source of time variation in expected returns is time variation in the expected proxy return.

In order to relate the economic aggregate to observable variables, consider the linear projection of m_t onto $r_{p,t}$ conditional on I_{t-1}

$$m_t = a + b_t r_{p,t} + e_{m,t}, \quad (35)$$

where

$$b_t = \frac{Cov_t [r_{p,t}, m_t]}{\sigma_t^2 [r_{p,t}]}.$$

A straightforward adaptation of the Lemma in the Appendix to conditional moments grants

$$Cov_t [\tilde{e}_t, e_{m,t}]' \Sigma_{\tilde{e}_t}^{-1} Cov_t [\tilde{e}_t, e_{m,t}] \leq \sigma_t^2 [m_t] (1 - \rho_t^2), \quad (36)$$

where $\Sigma_{\tilde{e}_t}^{-1}$ is the variance-covariance matrix of residuals from (34) conditional on I_{t-1} , and ρ_t is the conditional correlation between m_t and $r_{p,t}$. Given (36), a conditional analog to Proposition 1 may be stated as

Proposition 2 *Let the pricing model of (31) hold for all security returns including the proxy return.*

Consider (i) proportionality between the economic aggregate and the market return in (32) and (ii) the generalization of the measurable market model given by (34). Define

$$\theta_{p,t} = \frac{E_t [r_{p,t}]}{\sigma_t [r_{p,t}]}$$

as the conditional Sharpe performance measure for the proxy return, and

$$\eta_t = \frac{Cov_t [\tilde{e}_t, r_{p,t}]}{\sigma_t^2 [r_{p,t}]}.$$

as a conditional measure of the degree to which unobservable components to the market return covary with the proxy return. Assume that the relative efficiency of the proxy return is

value and growth stocks (see Campbell and Vuolteenaho (2004)).

constant. Then,

$$d' \Sigma_{\tilde{e}_t}^{-1} d \leq \theta_{p,t}^2 (\rho^{-2} - 1) \quad (37)$$

where

$$d = E_t [r_t] - (\delta + \eta_t) E_t [r_{p,t}].$$

Proof. See the proof of Proposition 1 in the Appendix, and condition the moments contained therein on I_{t-1} . Let $\theta_{m,t} = \frac{E_t [r_{m,t}]}{\sigma_t [r_{m,t}]}$. Given (32) and the fact that (31) also holds for $r_{p,t}$,

$$\rho_t = \frac{Cov_t [m_t, r_{p,t}]}{\sigma_t [m_t] \sigma_t [r_{p,t}]} = \frac{\theta_{p,t}}{\theta_{m,t}},$$

which is constant by assumption. ■

The deviation vector d from Proposition 2 is a N -vector of constant terms from the following model for r_t :

$$r_t = d + (\delta + \eta_t) E_t [r_{p,t}] + u_t \quad (38)$$

where $E_t [u_t] = 0$.²⁴ Let $\beta_{p,t} = \frac{Cov_t [r_t, r_{p,t}]}{\sigma_t^2 [r_{p,t}]}$. Given (34),

$$\beta_{p,t} = \beta + \eta_t. \quad (39)$$

As a result, d is a vector of deviations from conditional CAPM pricing measured with respect to the proxy return, where both the beta proxies and the expected "market" premium are time-varying. From (39), time-variation in the beta proxies results from mismeasuring the market return, since in the true model, betas are constant. Proposition 2 establishes an upper bound for these pricing errors based on the location of the proxy return relative to the conditional mean-variance frontier. Like its unconditional counterpart, a principal strength of Proposition 2 is that except for ρ , it is stated entirely in terms of quantities that can be directly estimated from observable data. This proposition, therefore, sets up a test of the conditional CAPM based upon a prior belief of the proxy's relative efficiency.

Suppose the market return is a mean-shift of the proxy return. Then (34) is a projection of r_t

²⁴(38) is a vector statement of (4) in Bodurtha and Mark (1991).

onto $r_{p,t}$, and the beta proxies are not time varying since $\eta_t = 0$. In this case, time variation in expected security returns is the result of a time-varying "market" premium.

Works by Harvey (1989), Bodurtha and Mark (1991), and more recently Adrian and Franzoni (2004) and Ang and Chen (2007) consider time-varying betas for the CAPM. Adrian and Franzoni (2004) and Ang and Chen (2007) stress time-varying betas as meaningful contributors to the improved performance of conditional specifications of the CAPM relative to their unconditional counterparts. All of these works measure time-variation in beta proxies. The findings of these authors, therefore, may be interpreted as evidence supporting the significance of measurement error on the outcomes of proxy-based investigations into the CAPM prediction.

Given (37), the conditional analog to (24) is

$$H_0 : \rho \leq \xi_t,$$

where $\xi_t \equiv \sqrt{\frac{1}{1+\theta_{p,t}^2 d' \Sigma_{\bar{e}_t}^{-1} d}}$. Taking expectations of both sides produces a null hypothesis testable following the same methodology outlined in section 5, with an asymptotic t statistic

$$\hat{\tau}_c = \frac{\hat{E}[\xi_t] - \bar{\xi}}{\hat{se}(\hat{E}[\xi_t])}.$$

To render this test feasible, the vector of forecasting instruments x_t needs to be specified as does the manner in which I_{t-1} conditions security returns and the "market" premium. In addition, the estimator employed for (34) needs to separately treat the conditional covariance and conditional variance operators that define η_t , so that the time series of this measure can be recovered.

9. Conclusion

This paper develops a new test of the CAPM that accounts for a proxy's mismeasurement of the market return both in terms of the former's relation to the latter as well as the former's relation to the assets it is assumed to price. For a given collection of test assets, conventional investigations of the CAPM prediction based upon the relative mean-variance efficiency of a given proxy estimate the market model by a linear projection. This paper demonstrates that estimating this projection

is not without loss of generality. The returns to nontraded assets and the returns to human capital are omitted from common "market"-based proxies. The extent to which these returns correlate with a given proxy will determine the extent to which innovations to the measurable market model covary with the proxy return. The resulting structural equation will necessarily differ from the projection equation. A novel estimator for this structural equation is reviewed that does not require outside instruments. This estimator is then used to show that the proposed test of relative mean-variance efficiency built upon the structural equation differs in economically significant ways from competing tests based upon the projection equation. In particular, the competing tests over-reject the CAPM prediction because these tests ignore the effects of omitted components from the market return on the measurable market model.

An extension of the pricing restriction implied by a mismeasured market return to conditional moments separates the beta measured with respect to a proxy return into a constant and a time-varying component. Time-variation in the second component is sourced to mismeasurement of the market return. Estimating this time-variation is central to evaluating the performance of a conditional CAPM where movements in beta significant to the pricing of expected returns are caused by measurement error. The estimator described in section 4 treats the parameters governing the conditional covariance matrix as nuisance parameters and only estimates composite functions of these parameters. Given the specification of η_t in Proposition 2, a complete treatment of the conditional covariance between the market model residuals and the proxy return as well as the conditional variance of the proxy return is necessary to render the result of Proposition 2 testable. Future research could develop such an estimator. The performance of the conditional pricing restriction in Proposition 2 could then be compared against the unconditional pricing restriction of Proposition 1 and alternative pricing models like the three-factor model of Fama and French (1993).

Appendix

Lemma Consider the structural model in (5) and the linear projection in (7). Then

$$Cov [\tilde{e}_t, e_{m,t}]' \Sigma_{\tilde{e}}^{-1} Cov [\tilde{e}_t, e_{m,t}] \leq \sigma^2 [m_t] (1 - \rho^2)$$

where $\Sigma_{\tilde{e}}$ is the $N \times N$ covariance matrix of \tilde{e}_t , and ρ is the correlation between m_t and $r_{p,t}$.

Proof. Since (7) describes a linear projection of m_t onto $r_{p,t}$, $b = \frac{Cov[r_{p,t}, m_t]}{\sigma^2[r_{p,t}]}$ and $\sigma^2 [e_{m,t}] = \sigma^2 [m_t] (1 - \rho^2)$. Consider regressing $e_{m,t}$ on \tilde{e}_t . The explained variance from that regression is $Cov [\tilde{e}_t, e_{m,t}]' \Sigma_{\tilde{e}}^{-1} Cov [\tilde{e}_t, e_{m,t}]$, which cannot be greater than $\sigma^2 [m_t] (1 - \rho^2)$, the total variance of $e_{m,t}$. ■

Proof of Proposition 1 Substitution of (5) into the right-hand-side of (1) produces

$$Cov [r_t, m_t] = \delta Cov [r_{p,t}, m_t] + Cov [\tilde{e}_t, m_t]. \quad (40)$$

Given (7),

$$Cov [\tilde{e}_t, m_t] = \left(\frac{Cov [\tilde{e}_t, r_{p,t}]}{\sigma^2 [r_{p,t}]} \right) Cov [r_{p,t}, m_t] + Cov [\tilde{e}_t, e_{m,t}]. \quad (41)$$

Combining (40) and (41) produces

$$Cov [r_t, m_t] = (\delta + \eta) Cov [r_{p,t}, m_t] + Cov [\tilde{e}_t, e_{m,t}],$$

where η is defined in (10). Given (1) and (8), it follows that

$$d' \Sigma_{\tilde{e}}^{-1} d \leq \sigma^2 [m_t] (1 - \rho^2) \quad (42)$$

where

$$d = E [r_t] - (\delta + \eta) E [r_{p,t}],$$

since (1) holds for both r_t and $r_{p,t}$. Also since (1) holds for $r_{p,t}$,

$$\rho = \frac{Cov [m_t, r_{p,t}]}{\sigma [m_t] \sigma [r_{p,t}]} = \frac{\theta_p}{\sigma [m_t]}, \quad (43)$$

which equates (11) with (42). ■

Proof of Corollary 1 Given (2) and (43), $\rho = \frac{\theta_p}{\theta_m}$. Substituting (4) into this result with the constraint that $\phi_t = \phi_c$ produces

$$\rho = \frac{E [r_{p,t}]}{\phi_c + E [r_{p,t}]}, \quad (44)$$

from which follows the statement that $\rho = 1$ if and only if $\phi_c = 0$. If $\rho = 1$, then $d' \Sigma_{\tilde{e}_t}^{-1} d = 0$ in (11). Since $\phi_c = 0$, $\eta = 0$ in (10), and $d = E_t [r_t] - \delta E_t [r_{p,t}]$. From (5) then follows that $d = \alpha$. ■

Proof of Corollary 2 Let

$$r_t = \alpha_p + \beta_p r_{p,t} + e_{p,t} \quad (45)$$

be a multivariate linear projection of r_t onto $r_{p,t}$, where α_p is a vector of alpha proxies, β_p is a vector of beta proxies, and $e_{p,t}$ is a vector of projection errors. Then

$$\begin{aligned} \alpha_p &= E [r_t] - \beta_p E [r_{p,t}] \\ \beta_p &= \frac{Cov [r_t, r_{p,t}]}{\sigma^2 [r_{p,t}]} \end{aligned} \quad (46)$$

Substitution of (5) into the expression for β_p yields the following relationships between the parameters in (45) and the structural parameters in (5):

$$\begin{aligned} \alpha_p &= \gamma - \eta E [r_{p,t}] \\ \beta_p &= \delta + \eta \end{aligned} \quad (47)$$

where η is defined by (10). Given these relationships,

$$e_{p,t} = r_t - \alpha_p - \beta_p r_{p,t} = \tilde{e}_t - \eta \tilde{r}_{p,t}$$

where $\tilde{r}_{p,t} = r_{p,t} - E[r_{p,t}]$. It then follows that

$$\Sigma_{e_p} = \Sigma_{\tilde{e}} - \frac{Cov[\tilde{e}_t, \tilde{r}_{p,t}] Cov[\tilde{e}_t, \tilde{r}_{p,t}]'}{\sigma^2[\tilde{r}_{p,t}]}, \quad (48)$$

since given the definition of $\tilde{r}_{p,t}$, $Cov[\tilde{e}_t, r_{p,t}] = Cov[\tilde{e}_t, \tilde{r}_{p,t}]$ and $\sigma^2[r_{p,t}] = \sigma^2[\tilde{r}_{p,t}]$.

Substitution of the expression for \tilde{e}_t in (6) into (48) produces

$$\Sigma_{\tilde{e}} - \Sigma_{e_p} = \left(\frac{Cov[\tilde{\phi}_t, \tilde{r}_{p,t}]}{\sigma[\tilde{r}_{p,t}]} \right)^2 \beta \beta'. \quad (49)$$

In general, there exists an x such that $\beta'x = 0$. Let $y = \beta'x$. Then $y'y \geq 0$. ■

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Table 1A

Summary statistics for size, B/M, and momentum portfolios, 10/6/67 - 9/25/87. The portfolio return series are measured weekly (in percentage terms) so that relatively high frequency data is utilized (to estimate higher moments) that reduces day-of-the-week and weekend effects as well as the effects of nonsynchronous trading and bid-ask bounce. The proxy return is the CRSP market-value-weighted index of all securities on the NYSE, AMEX, and NASDAQ exchanges. Security returns are constructed from the 25 size-B/M portfolios and the 25 size-momentum portfolios (each 5×5 sorts with breakpoints determined by NYSE quintiles). "Small" is the average of the five low-market-cap portfolios, "Mid" the average of the five medium-market-cap portfolios, and "Big" the average of the five large-market-cap portfolios. "Value" is the average of the five high-B/M portfolios, "Neutral" the average of the five middle-B/M portfolios, and "Growth" the average of the five low-B/M portfolios. Finally, "Losers" is the average of the five low-return-sorted portfolios, "Neutral" the average of the five middle-return-sorted portfolios, and "Winners" the average of the five high-return-sorted portfolios.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Growth
Panel A: Excess returns									
mean	0.103	0.120	0.075	0.169	0.108	0.023	-0.034	0.101	0.198
stdev	2.31	2.14	2.03	2.08	1.96	2.64	2.72	1.99	2.44
skew	-0.33	-0.20	0.11	-0.21	-0.18	-0.19	0.32	-0.10	-0.58
kurt	5.12	4.80	4.83	5.18	4.70	4.71	5.84	5.22	4.95
Panel B: Alpha Proxy									
est	0.045	0.060	0.016	0.113	0.053	-0.053	-0.106	0.045	0.132
std error ^a	0.039	0.024	0.012	0.028	0.021	0.025	0.038	0.020	0.031

Notes:

^aHeteroskedasticity consistent

Table 1B

Test results for size, B/M, and momentum portfolios, 10/6/67 - 9/25/87. Projection errors are the residuals from OLS regressions of security returns on the proxy return. Structural errors are the residuals from linear equations relating security returns to the proxy return, where the residual from each equation and the proxy return is allowed to covary. Gibbons, Ross, and Shanken (1989), or GRS, Bootstrap Proposition 1 constant covariance (BPC), and Bootstrap Proposition 1 stochastic (BPS), are alternative ways of determining the maximum correlation between the CRSP value-weighted proxy return and the market return that supports the CAPM at a 5% significance level. GRS is based on the assumption that the projection errors are normally distributed. BPC is also based on the projection errors, but assumes those errors to follow strong, univariate GARCH(1,1) processes with unknown distributions. BPS is based on the assumption that the structural errors follow strong, univariate GARCH (1,1) processes with unknown distributions.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Winners
Panel C: Projection errors									
skew	0.23	0.22	0.09	0.39	0.48	0.11	0.98	0.39	-0.70
kurt	5.08	5.85	4.15	5.76	8.19	4.45	7.95	5.32	6.83
Panel D: Structural errors									
skew	0.17	0.23	0.00	0.40	0.47	0.10	0.59	0.30	-0.85
kurt	5.67	5.81	5.68	5.99	8.15	4.41	5.97	4.99	6.89
Panel E: GRS ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.565			0.277			0.236	
0.52		0.855			0.570			0.504	
0.86		0.939			0.753			0.694	
1.00		0.953			0.800			0.746	
Panel F: BPC ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.410			0.272			0.260	
0.52		0.737			0.551			0.530	
0.86		0.892			0.737			0.715	
1.00		0.924			0.787			0.766	
unknown		0.581			0.429			0.398	
Panel G: BPS ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.409			0.271			0.242	
0.52		0.741			0.552			0.511	
0.86		0.896			0.738			0.706	
1.00		0.927			0.787			0.761	
unknown		0.590			0.440			0.388	

Notes:

^bMaximum correlations are reported that support the CAPM prediction.

^cValues for the proxy Sharpe ratio are taken from Shanken (1987). These values are annualized. 0.52 is the expected value; 0.22 and 0.86 are - 1 and + 2 standard deviations away from this expected value, respectively. 1.00 is a value for the proxy Sharpe ratio that is greater than any conceivable true value. Unknown means that the proxy Sharpe ratio is bootstrapped along with every other estimated quantity in the expression determining an upper bound for the correlation between the proxy and the market return.

Table 2A

Summary statistics for size, B/M, and momentum portfolios, 11/6/87 - 9/28/07. The portfolio return series are measured weekly (in percentage terms) so that relatively high frequency data is utilized (to estimate higher moments) that reduces day-of-the-week and weekend effects as well as the effects of nonsynchronous trading and bid-ask bounce. The proxy return is the CRSP market-value-weighted index of all securities on the NYSE, AMEX, and NASDAQ exchanges. Security returns are constructed from the 25 size-B/M portfolios and the 25 size-momentum portfolios (each 5×5 sorts with breakpoints determined by NYSE quintiles). "Small" is the average of the five low-market-cap portfolios, "Mid" the average of the five medium-market-cap portfolios, and "Big" the average of the five large-market-cap portfolios. "Value" is the average of the five high-B/M portfolios, "Neutral" the average of the five middle-B/M portfolios, and "Growth" the average of the five low-B/M portfolios. Finally, "Losers" is the average of the five low-return-sorted portfolios, "Neutral" the average of the five middle-return-sorted portfolios, and "Winners" the average of the five high-return-sorted portfolios.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Growth
Panel A: Excess returns									
mean	0.169	0.176	0.157	0.209	0.183	0.103	0.041	0.172	0.315
stdev	2.16	2.09	1.92	1.88	1.83	2.61	2.94	1.74	2.60
skew	-1.04	-0.64	-0.29	-0.91	-0.75	-0.75	0.03	-0.60	-0.89
kurt	11.76	6.64	4.97	9.24	6.66	9.05	7.00	6.72	10.96
Panel B: Alpha Proxy									
est	0.049	0.040	0.027	0.094	0.063	-0.070	-0.129	0.059	0.148
std error ^a	0.044	0.029	0.023	0.032	0.025	0.034	0.056	0.025	0.039

Notes:

^aHeteroskedasticity consistent

Table 2B

Test results for size, B/M, and momentum portfolios, 11/6/87 - 9/28/07. Projection errors are the residuals from OLS regressions of security returns on the proxy return. Structural errors are the residuals from linear equations relating security returns to the proxy return, where the residual from each equation and the proxy return is allowed to covary. Gibbons, Ross, and Shanken (1989), or GRS, Bootstrap Proposition 1 constant covariance (BPC), and Bootstrap Proposition 1 stochastic (BPS), are alternative ways of determining the maximum correlation between the CRSP value-weighted proxy return and the market return that supports the CAPM at a 5% significance level. GRS is based on the assumption that the projection errors are normally distributed. BPC is also based on the projection errors, but assumes those errors to follow strong, univariate GARCH(1,1) processes with unknown distributions. BPS is based on the assumption that the structural errors follow strong, univariate GARCH (1,1) processes with unknown distributions.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Winners
Panel C: Projection errors									
skew	-0.04	0.14	1.41	-0.08	0.17	-0.07	1.13	0.89	-0.25
kurt	7.56	6.06	19.54	6.68	6.70	7.71	9.22	11.49	6.83
Panel D: Structural errors									
skew	-0.43	-0.33	-0.07	-0.75	-0.63	-0.28	1.07	-0.36	-0.72
kurt	9.72	6.68	5.05	10.20	6.59	8.36	9.24	7.39	10.96
Panel E: GRS ^{b, c}									
Proxy Sharpe ratio:									
0.22		1.000			0.406			0.258	
0.52		1.000			0.730			0.540	
0.86		1.000			0.870			0.727	
1.00		1.000			0.899			0.776	
Panel F: BPC ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.639			0.397			0.277	
0.52		0.926			0.698			0.554	
0.86		0.997			0.852			0.745	
1.00		1.000			0.888			0.796	
unknown		1.000			0.824			0.679	
Panel G: BPS ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.857			0.442			0.319	
0.52		1.000			0.739			0.622	
0.86		1.000			0.878			0.805	
1.00		1.000			0.908			0.851	
unknown		1.000			0.841			0.745	

Notes:

^bMaximum correlations are reported that support the CAPM prediction.

^cValues for the proxy Sharpe ratio are taken from Shanken (1987). These values are annualized. 0.52 is the expected value; 0.22 and 0.86 are - 1 and + 2 standard deviations away from this expected value, respectively. 1.00 is a value for the proxy Sharpe ratio that is greater than any conceivable true value. Unknown means that the proxy Sharpe ratio is bootstrapped along with every other estimated quantity in the expression determining an upper bound for the correlation between the proxy and the market return.

Table 3A

Summary statistics for size, B/M, and momentum portfolios, 10/7/77 - 9/25/87. The portfolio return series are measured weekly (in percentage terms) so that relatively high frequency data is utilized (to estimate higher moments) that reduces day-of-the-week and weekend effects as well as the effects of nonsynchronous trading and bid-ask bounce. The proxy return is the CRSP market-value-weighted index of all securities on the NYSE, AMEX, and NASDAQ exchanges. Security returns are constructed from the 25 size-B/M portfolios and the 25 size-momentum portfolios (each 5×5 sorts with breakpoints determined by NYSE quintiles). "Small" is the average of the five low-market-cap portfolios, "Mid" the average of the five medium-market-cap portfolios, and "Big" the average of the five large-market-cap portfolios. "Value" is the average of the five high-B/M portfolios, "Neutral" the average of the five middle-B/M portfolios, and "Growth" the average of the five low-B/M portfolios. Finally, "Losers" is the average of the five low-return-sorted portfolios, "Neutral" the average of the five middle-return-sorted portfolios, and "Winners" the average of the five high-return-sorted portfolios.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Growth
Panel A: Excess returns									
mean	0.221	0.226	0.151	0.257	0.205	0.140	0.077	0.184	0.316
stdev	1.96	1.98	1.99	1.79	1.83	2.49	2.31	1.82	2.43
skew	-1.08	-0.52	0.04	-0.91	-0.54	-0.30	0.36	-0.39	-0.87
kurt	7.96	5.99	4.51	7.81	5.59	4.91	5.72	5.61	6.59
Panel B: Alpha Proxy									
est	0.091	0.081	-0.002	0.130	0.068	-0.045	-0.081	0.048	0.141
std error ^a	0.048	0.031	0.016	0.035	0.024	0.034	0.048	0.025	0.044

Notes:

^aHeteroskedasticity consistent

Table 3B

Test results for size, B/M, and momentum portfolios, 10/7/77 - 9/25/87. Projection errors are the residuals from OLS regressions of security returns on the proxy return. Structural errors are the residuals from linear equations relating security returns to the proxy return, where the residual from each equation and the proxy return is allowed to covary. Gibbons, Ross, and Shanken (1989), or GRS, Bootstrap Proposition 1 constant covariance (BPC), and Bootstrap Proposition 1 stochastic (BPS), are alternative ways of determining the maximum correlation between the CRSP value-weighted proxy return and the market return that supports the CAPM at a 5% significance level. GRS is based on the assumption that the projection errors are normally distributed. BPC is also based on the projection errors, but assumes those errors to follow strong, univariate GARCH(1,1) processes with unknown distributions. BPS is based on the assumption that the structural errors follow strong, univariate GARCH (1,1) processes with unknown distributions.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Winners
Panel C: Projection errors									
skew	-0.71	-0.35	0.20	-0.77	-0.63	-0.13	0.95	-0.15	-0.97
kurt	7.00	4.90	3.72	7.64	5.57	3.88	8.35	5.06	7.10
Panel D: Structural errors									
skew	-0.77	-0.29	0.20	-1.06	-0.76	-0.15	1.08	-0.11	-0.55
kurt	7.39	4.58	3.69	9.29	6.23	4.03	9.26	4.91	5.50
Panel E: GRS ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.437			0.244			0.352	
0.52		0.759			0.518			0.671	
0.86		0.888			0.707			0.831	
1.00		0.913			0.758			0.867	
Panel F: BPC ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.291			0.220			0.286	
0.52		0.590			0.465			0.577	
0.86		0.785			0.660			0.770	
1.00		0.835			0.719			0.821	
unknown		0.772			0.614			0.733	
Panel G: BPS ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.353			0.231			0.316	
0.52		0.645			0.483			0.644	
0.86		0.815			0.672			0.849	
1.00		0.872			0.728			0.899	
unknown		0.810			0.644			0.795	

Notes:

^bMaximum correlations are reported that support the CAPM prediction.

^cValues for the proxy Sharpe ratio are taken from Shanken (1987). These values are annualized. 0.52 is the expected value; 0.22 and 0.86 are - 1 and + 2 standard deviations away from this expected value, respectively. 1.00 is a value for the proxy Sharpe ratio that is greater than any conceivable true value. Unknown means that the proxy Sharpe ratio is bootstrapped along with every other estimated quantity in the expression determining an upper bound for the correlation between the proxy and the market return.

Table 4A

Summary statistics for size, B/M, and momentum portfolios, 11/6/87 - 9/26/97. The portfolio return series are measured weekly (in percentage terms) so that relatively high frequency data is utilized (to estimate higher moments) that reduces day-of-the-week and weekend effects as well as the effects of nonsynchronous trading and bid-ask bounce. The proxy return is the CRSP market-value-weighted index of all securities on the NYSE, AMEX, and NASDAQ exchanges. Security returns are constructed from the 25 size-B/M portfolios and the 25 size-momentum portfolios (each 5×5 sorts with breakpoints determined by NYSE quintiles). "Small" is the average of the five low-market-cap portfolios, "Mid" the average of the five medium-market-cap portfolios, and "Big" the average of the five large-market-cap portfolios. "Value" is the average of the five high-B/M portfolios, "Neutral" the average of the five middle-B/M portfolios, and "Growth" the average of the five low-B/M portfolios. Finally, "Losers" is the average of the five low-return-sorted portfolios, "Neutral" the average of the five middle-return-sorted portfolios, and "Winners" the average of the five high-return-sorted portfolios.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Growth
Panel A: Excess returns									
mean	0.172	0.220	0.221	0.242	0.217	0.149	0.075	0.207	0.336
stdev	1.54	1.58	1.71	1.47	1.43	1.95	2.05	1.37	1.95
skew	-0.51	-0.44	-0.10	-0.30	-0.50	-0.35	0.15	-0.48	-0.51
kurt	6.51	5.36	4.14	5.26	5.58	4.91	5.95	5.91	4.39
Panel B: Alpha Proxy									
est	0.029	0.039	0.008	0.079	0.047	-0.074	-0.134	0.044	0.112
std error ^a	0.048	0.032	0.017	0.033	0.023	0.040	0.054	0.023	0.038

Notes:

^aHeteroskedasticity consistent

Table 4B

Test results for size, B/M, and momentum portfolios, 11/6/87 - 9/26/97. Projection errors are the residuals from OLS regressions of security returns on the proxy return. Structural errors are the residuals from linear equations relating security returns to the proxy return, where the residual from each equation and the proxy return is allowed to covary. Gibbons, Ross, and Shanken (1989), or GRS, Bootstrap Proposition 1 constant covariance (BPC), and Bootstrap Proposition 1 stochastic (BPS), are alternative ways of determining the maximum correlation between the CRSP value-weighted proxy return and the market return that supports the CAPM at a 5% significance level. GRS is based on the assumption that the projection errors are normally distributed. BPC is also based on the projection errors, but assumes those errors to follow strong, univariate GARCH(1,1) processes with unknown distributions. BPS is based on the assumption that the structural errors follow strong, univariate GARCH (1,1) processes with unknown distributions.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Winners
Panel C: Projection errors									
skew	0.24	0.09	0.14	0.88	-0.02	-0.16	0.74	0.29	-0.15
kurt	5.07	3.95	3.17	5.99	5.32	3.83	5.64	5.29	3.81
Panel D: Structural errors									
skew	-0.20	-0.08	0.11	0.86	-0.48	-0.16	0.60	-0.27	-0.18
kurt	6.50	4.74	3.25	6.36	6.13	3.69	6.55	7.02	3.95
Panel E: GRS ^{b, c}									
Proxy Sharpe ratio:									
0.22		1.000			0.388			0.165	
0.52		1.000			0.712			0.372	
0.86		1.000			0.858			0.553	
1.00		1.000			0.889			0.611	
Panel F: BPC ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.554			0.310			0.174	
0.52		0.892			0.597			0.385	
0.86		1.000			0.780			0.562	
1.00		1.000			0.828			0.619	
unknown		1.000			0.941			0.714	
Panel G: BPS ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.636			0.366			0.172	
0.52		0.971			0.661			0.383	
0.86		1.000			0.826			0.562	
1.00		1.000			0.866			0.619	
unknown		1.000			0.931			0.690	

Notes:

^bMaximum correlations are reported that support the CAPM prediction.

^cValues for the proxy Sharpe ratio are taken from Shanken (1987). These values are annualized. 0.52 is the expected value; 0.22 and 0.86 are - 1 and + 2 standard deviations away from this expected value, respectively. 1.00 is a value for the proxy Sharpe ratio that is greater than any conceivable true value. Unknown means that the proxy Sharpe ratio is bootstrapped along with every other estimated quantity in the expression determining an upper bound for the correlation between the proxy and the market return.

Table 5A

Summary statistics for size, B/M, and momentum portfolios, 10/3/97 - 9/28/07. The portfolio return series are measured weekly (in percentage terms) so that relatively high frequency data is utilized (to estimate higher moments) that reduces day-of-the-week and weekend effects as well as the effects of nonsynchronous trading and bid-ask bounce. The proxy return is the CRSP market-value-weighted index of all securities on the NYSE, AMEX, and NASDAQ exchanges. Security returns are constructed from the 25 size-B/M portfolios and the 25 size-momentum portfolios (each 5×5 sorts with breakpoints determined by NYSE quintiles). "Small" is the average of the five low-market-cap portfolios, "Mid" the average of the five medium-market-cap portfolios, and "Big" the average of the five large-market-cap portfolios. "Value" is the average of the five high-B/M portfolios, "Neutral" the average of the five middle-B/M portfolios, and "Growth" the average of the five low-B/M portfolios. Finally, "Losers" is the average of the five low-return-sorted portfolios, "Neutral" the average of the five middle-return-sorted portfolios, and "Winners" the average of the five high-return-sorted portfolios.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Growth
Panel A: Excess returns									
mean	0.166	0.133	0.094	0.177	0.150	0.057	0.007	0.138	0.293
stdev	2.63	2.49	2.11	2.22	2.16	3.12	3.61	2.04	3.12
skew	-1.04	-0.60	-0.36	-1.00	-0.74	-0.76	0.03	-0.57	-0.90
kurt	9.85	5.62	5.00	8.53	5.78	7.96	5.51	5.85	9.91
Panel B: Alpha Proxy									
est	0.087	0.048	0.021	0.107	0.077	-0.052	-0.102	0.070	0.190
std error ^a	0.072	0.048	0.038	0.054	0.043	0.054	0.096	0.043	0.067

Notes:

^aHeteroskedasticity consistent

Table 5B

Test results for size, B/M, and momentum portfolios, 10/3/97 - 9/28/07. Projection errors are the residuals from OLS regressions of security returns on the proxy return. Structural errors are the residuals from linear equations relating security returns to the proxy return, where the residual from each equation and the proxy return is allowed to covary. Gibbons, Ross, and Shanken (1989), or GRS, Bootstrap Proposition 1 constant covariance (BPC), and Bootstrap Proposition 1 stochastic (BPS), are alternative ways of determining the maximum correlation between the CRSP value-weighted proxy return and the market return that supports the CAPM at a 5% significance level. GRS is based on the assumption that the projection errors are normally distributed. BPC is also based on the projection errors, but assumes those errors to follow strong, univariate GARCH(1,1) processes with unknown distributions. BPS is based on the assumption that the structural errors follow strong, univariate GARCH (1,1) processes with unknown distributions.

	Size			B/M			Momentum		
	Small	Mid	Large	Value	Neutral	Growth	Losers	Draws	Winners
Panel C: Projection errors									
skew	-0.11	0.16	1.15	-0.32	0.15	-0.02	1.09	0.81	-0.29
kurt	6.75	5.51	12.23	5.51	5.16	8.02	7.69	8.72	5.49
Panel D: Structural errors									
skew	-0.82	-0.44	-0.18	-1.01	-0.70	-0.46	0.89	-0.46	-0.84
kurt	9.33	5.62	5.29	9.22	5.59	8.06	7.13	6.11	10.07
Panel E: GRS ^{b, c}									
Proxy Sharpe ratio:									
0.22		1.000			1.000			0.469	
0.52		1.000			1.000			0.788	
0.86		1.000			1.000			0.904	
1.00		1.000			1.000			0.926	
Panel F: BPC ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.648			0.427			0.337	
0.52		0.955			0.766			0.645	
0.86		1.000			0.928			0.836	
1.00		1.000			0.960			0.883	
unknown		0.910			0.714			0.579	
Panel G: BPS ^{b, c}									
Proxy Sharpe ratio:									
0.22		0.771			0.434			0.391	
0.52		1.000			0.798			0.741	
0.86		1.000			0.975			0.933	
1.00		1.000			1.000			0.974	
unknown		0.966			0.655			0.597	

Notes:

^bMaximum correlations are reported that support the CAPM prediction.

^cValues for the proxy Sharpe ratio are taken from Shanken (1987). These values are annualized. 0.52 is the expected value; 0.22 and 0.86 are - 1 and + 2 standard deviations away from this expected value, respectively. 1.00 is a value for the proxy Sharpe ratio that is greater than any conceivable true value. Unknown means that the proxy Sharpe ratio is bootstrapped along with every other estimated quantity in the expression determining an upper bound for the correlation between the proxy and the market return.

TABLE 6

Simulation evidence for the size of the three relative efficiency tests considered under the null hypothesis that the correlation of the proxy return with the true market return is at least 90%. Errors to the excess security returns and excess proxy return follow semi-strong GARCH(1,1) processes with standardized Gamma(2,1) innovations. Parameters for the GARCH processes are the sample estimates obtained from the B/M portfolios measured over the weekly period 11/6/87 - 9/28/07 that are robust to endogeneity of the proxy return. These parameter estimates are termed the "true" values. The Gamma(2,1) distribution is chosen because, when combined with these GARCH parameters, this distribution produces errors with unconditional skewness and kurtosis measures comparable to those described under Panel D for the B/M portfolios of Table 2B. Betas for the excess security returns are the sample estimates from the same time period. Alpha proxies for each of the excess security returns are calibrated from the sample returns so that (1) they are all equal and (2) they imply a 90% correlation between the proxy and the market return. For the GRS test, the Sharpe Performance Measure for the proxy return is assumed to be known and is set equal to the estimate from the original sample. For the BPC and BPS tests, the Sharpe Performance Measure is treated as unknown. For all three test statistics, the simulations are conducted across 500 trials with excess return series of 1000 observations each. When constructing the individual excess return series for each trial, the first 200 observations are dropped to avoid initialization effects. For the BPC and BPS statistics, within each simulation trial is a bootstrap of the maximum correlation between the proxy and market return conducted over 250 repetitions. In each case, the bootstrap routines use parameter estimates from the original sample along with constant terms implied by the calibrated alpha proxies. Parameter estimates used in the BPC test assume that innovations to the excess security returns are uncorrelated with the proxy return. Parameter estimates used in the BPS test are the "true" values described above. The table reports rejection rates for the test statistics at 10%, 5%, and 1% significance levels.

Statistic	Size =		
	0.10	0.05	0.01
GRS	0.180	0.112	0.038
BPC	0.114	0.082	0.026
BPS	0.084	0.050	0.016