Valuing an American Put Option

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13 November 2006

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Abstract: The model presents the valuation of an American Put option by using a duplicating portfolio consisting of riskless security and stock sold short.

Introduction

The valuing of American options has intrigued financial and academic world for a quarter of a century. Due to the possibility of early exercise their valuation is more difficult than the counterparty European. McKean(1965) showed that the optimal stopping problem necessary to solve the value of an American option can be considered as a free boundary problem. Merton(1973) showed that the valuing of an American Put option is more difficult than an European Put option because in every instant there is a positive probability of premature exercise. In fact, the value of an European Put option can be less than its pay off, hence, we can't use it to value the American Put option. The intractability of the optimal stopping problem lead Brennan, Schwartz(1977) and Parkinson(1977) to solve the free boundary problem by using numerical solution. Cox, Ross and Rubinstein(1979) introduced the lattice method for pricing option. This method has an appealing feature because permits to simulate the path of the option by backward iteration. The lattice method can be applied in a Monte Carlo setting but the computational effort increases exponentially as well as the pricing error. Geske, Johnson(1984) solved analytically the free boundary problem by using a series of compound option. Indeed, the value of an American option can be decomposed in the counterparty European and the early exercise premium. This lead Barone, Adesi and Whaley(1987) to decompose the price of American option by approximating the value of the early exercise premium. Along the same line Kim(1990) and Carr, Jarrow and Myneni(1992) obtained the same formulation for the value of an American Put option. Our approach starts from another point of view; we will consider the effective pay off that you will have to replicate the value of an American Put option subject to the free boundary condition. Furthermore, we will assume that the value of an American option in absence of arbitrage opportunity is greater, or equal, than its pay off. Hence, it will not be exercised before of maturity. As result, the value of an American Put option becomes the value of an European Put option on the effective final pay off.

The model and its assumptions

We assume that the dynamic of the stock price S_t is given by the following stochastic continuous process:

$$dS_t \,/\, S_t = \mu \, dt + \sigma_S \, dW_S$$

μ denotes the drift of the process

dW_S denotes a standard Wiener process

σ_S denotes the instantaneous volatility of the stock price S_t

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Since the only parameters which we are assumed variable are the stock price S_t and time, and the stock price follows a stochastic continuous process, Put price changes can be characterized using Itô's lemma. Then, by constructing a self financing, risk free hedge between the Put option, the stock, and a riskless security, the Put equilibrium price path can be described by the familiar Black, Scholes(1973) partial differential equation:

$$P_t + r\,S\,P_S + ½\,\sigma^2 S^2 P_{SS} - r\,P = 0$$

Because the American Put option can be exercised at any instant, the problem is termed as free boundary problem. The free boundary condition that the American Put option must satisfy at every instant is:

$$P(S,t) \geq \text{Max}\,[\,0\,,\,X_t - S_t\,]$$

Many authors have given a numerical solution to the partial differential equation satisfying the free boundary condition. Geske, Johnson(1984) showed that exist an analytic solution to this partial differential equation subject to the free boundary condition. The key of their solution is the assumption that each exercise decision is a discrete event. Thus, the formula derived is a continuous time solution to the partial differential equation subject to the free boundary condition applied at an infinite number of discrete instant. Our approach starts from another point of view, firstly, if the value of an American Put option is greater, or equal, than its pay off it will not be exercised before the maturity, secondly, to replicate the value of an American Put option we must have available the amount of money X_t at every instant due to the possibility that the value of the option goes on the pay off, this amount of money will produce an earnings that will change our final pay off. Thus, the value of an American Put option becomes the value of an European Put option on the following final pay off:

$$P(S,T) = \text{Max}\,[\,0\,,\,X_T - S_T\,]$$

To determine the value of X_T we must follow a logical pattern, to have available means that we don't have to risk any loss. Straightforward, X_t increases at the risk free rate; as such, the value of X_T if the risk free rate is constant is simply:

$$X_T = X_t\, e^{\,r(T-t)}$$

At this point, the value of an American Put option becomes simply the Black, Scholes(1973) formula with strike price equal to X_T . Indeed, we wish to extend the analysis to the case of stochastic interest rate. The solution was given in the literature by using a default free zero coupon bond as forward measure. Heath, Jarrow and Morton(1992) take the observed yield curve as initial condition for the forward rate curve, they assume that the forward rate curve reflects the expectation of the market on the future interest-rates such that to avoid arbitrage opportunity it determines the yield curve. They assume that the yield curve is the mean of the future expected spot rate. Now we assume that the dynamic of the default free zero coupon bond $p(t,T)$ is given by the following stochastic continuous process:

$$dp(t,T)\,/\,p(t,T) = r_t\,dt - \sigma_p(t,T)\,dW_r$$

r_t is the spot rate and denotes the drift of the process

dW_r denotes a standard Wiener process capturing the volatility of the market expectation

$\sigma_{p(t,T)}$ denotes the instantaneous volatility of the default free zero coupon bond

If we put the following interest rate elasticity measure:

$$\eta_{p(t,T)} = -[\partial p(t,T) / \partial r]\,[1 / p(t,T)]$$

We have:

$$\eta_{p(t,T)} = (T - t)$$

Thus, we have:

$$\sigma_{p(t,T)} = \delta_r \, \eta_{p(t,T)}$$

Where:

δ_r denotes the instantaneous volatility of the market expectation

As result, the dynamic of X_t is given by the following stochastic continuous process:

$$dX_t / X_t = r_t \, dt - \sigma_p(t,T)\, dW_r$$

Hence, by using X_t as *numeraire*, we have the following formula for an American Put option:

$$P(S,t) = X_t \, N[h_1] - S_t \, N[h_2]$$

Where:

$$h_1 = \frac{\ln(X_t / S_t) + \tfrac{1}{2}\sigma^2_{(t,T)}(T - t)}{\sigma_{(t,T)}\sqrt{(T - t)}}$$

$$h_2 = \frac{\ln(X_t / S_t) - \tfrac{1}{2}\sigma^2_{(t,T)}(T - t)}{\sigma_{(t,T)}\sqrt{(T - t)}}$$

While:

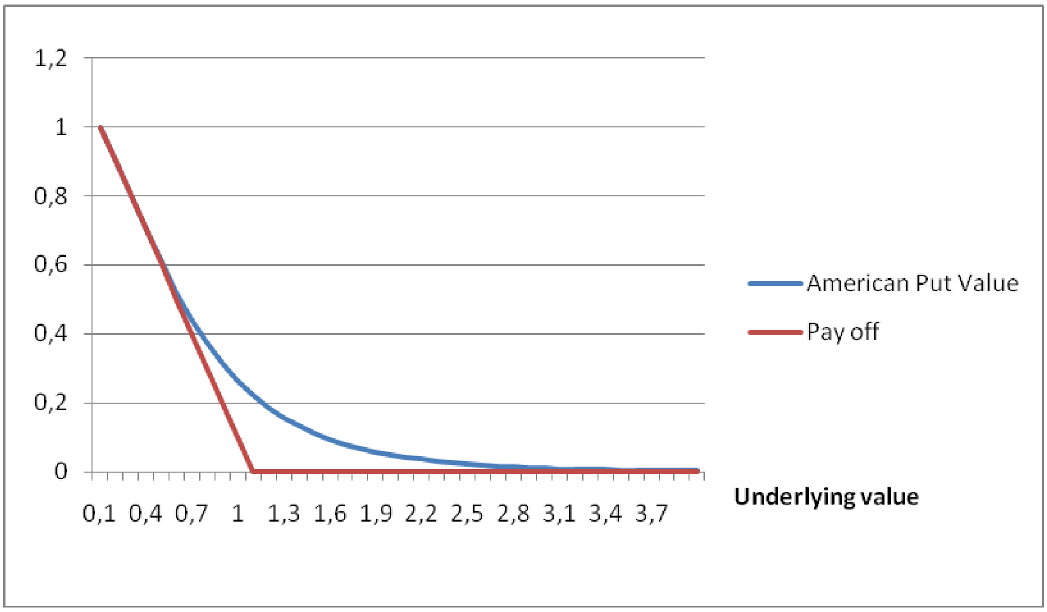
$$\sigma^2_{(t,T)} = [1 / (T - t)] \int_t^T \sigma_S(t)^2 + \sigma_p(t,T)^2 - 2\,\rho\,\sigma_S(t)\,\sigma_{p(t,T)}\, dt$$

ρ represents the correlation between the stock price S_t and the riskless security $p(t,T)$

If we assume that $\sigma_S(t)$ is deterministic we get:

$$\sigma^2_{(t,T)} = \sigma_S^2 + \tfrac{1}{3}\sigma_p(t,T)^2 - \rho\,\sigma_S\,\sigma_p(t,T)$$

For the following value of parameters: $\sigma_S = 0,35$ $\delta_r = 0,02$ $\rho = 0$ $T = 2$ $X_t = 1$, we have:



We can note that the formula satisfies in every instant the free boundary condition imposed to the value of an American Put option.

Numerical result

We have decided to compare our model with the lattice method of Cox, Ross and Rubinstein(1979) because it is the most used by practitioners and converges to the same value of numerical and analytic solution. Instead, the early exercise premium approaches introduce more computational error due to the approximation of the early exercise premium.

σ_S	X_t	r	S_t	T	Lattice 150 nodes	Effective
0,2	1	0,04	1,5	0,25	0,00000	0,00970
0,2	1	0,04	1,25	0,25	0,00037	0,04173
0,2	1	0,04	1	0,25	0,03568	0,11920
0,2	1	0,04	0,75	0,25	0,25000	0,27153
0,2	1	0,04	0,5	0,25	0,50000	0,49876
0,3	1	0,04	1,5	0,5	0,00216	0,03880
0,3	1	0,04	1,25	0,5	0,01491	0,07968
0,3	1	0,04	1	0,5	0,07576	0,15862
0,3	1	0,04	0,75	0,5	0,25055	0,29754
0,3	1	0,04	0,5	0,5	0,50000	0,50471
0,4	1	0,04	1,5	0,75	0,02085	0,07755
0,4	1	0,04	1,25	0,75	0,05207	0,12597
0,4	1	0,04	1	0,75	0,12388	0,20457
0,4	1	0,04	0,75	0,75	0,26867	0,32908
0,4	1	0,04	0,5	0,75	0,50000	0,51394

Conclusion

The advantages of our model state in the fact that it is in line with the Black, Scholes(1973) formula. This means that it is possible to extract and calibrate the implied volatility easily. Furthermore, we take in account the stochastic interest rate by eliminating the parameter from the final formulation and by introducing the volatility of the short rate as new parameter. The computational effort for this formula is very simple and immediate as the Black, Scholes(1973) formula. Indeed, if we use a duplicating portfolio for an American Call option we get the Black, Scholes(1973) formula because we income the dividend that compensate the decrease of the underlying such that we have the same final pay off.

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