Criteria for Monotonicity of Demand Functions

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In consumer's choice theory (see, for example, [1, p. 313]), it is often postulated that the consumer's behavior is described by the extremal problem

$$u(c) \to \max, \qquad pc \le \beta,\ c \ge 0, \tag{1}$$

where $u(c)$ is the utility function (goal function), c is a consumption goods vector, p is a price vector, β is the consumer's income (in the sequel, β is assumed to be a constant).

Denote by R^n_+ the set of nonnegative column-vectors of dimension n; T stands for the transpose operation. We consider c to be a column-vector and p to be a row-vector (thus, $p^T \in R^n_+$). Let $C(p)$ be the set of solutions of (1) for fixed p and let G be a subset of R^{2n}_+. The correspondence $C(p)$ is called a demand function. We say that $C(p)$ is monotone on G if

$$(p-q)(c_p - c_q) \le 0 \tag{2}$$

for any $c_p \in C(p)$, $c_q \in C(q)$, $(p^T, c_p) \in G$, and $(q^T, c_q) \in G$.

For a number of economic equilibrium models (see, for example, [2, 3, 4]), the requirement of the monotonicity of the demand function (or some modification of such a requirement) is necessary to ensure the uniqueness of equilibrium, turnpike theorems validity, and the convergence of the computational algorithms [3, 5, 6].

Our goal is the following: To describe the class of preference indicators $u(c)$ such that the problem (1) generates a monotone demand function. Moreover, a generalization of the condition (2) is considered.

Assume that $G = P \times S$, where $S \subset R^n_+$ and

$$P = \{p \mid p \ge 0, C(p) \cap S \ne \emptyset\}. \tag{3}$$

Suppose that

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1. S is a convex set and its interior S_0 is not empty;
2. the function $u(c)$ is defined on R^n_+, is concave, and has continuous second derivatives on S;[1]
3. the gradient $v(c)$ of the function $u(c)$ is nonnegative and satisfies the inequality $v(c)c > 0$ for any $c \in S$;
4. the income β is positive.

In the problem (1), a maximum is reached at the point c for a given prices vector vector p if and only if

$$v(c) = \lambda p - w, \qquad \lambda(pc - \beta) = 0, \qquad wc = 0, \tag{4}$$

where $\lambda \in R^1_+$, $w^T \in R^n_+$.

Our assumptions imply that $pc = \beta$. Hence, using (4), we get

$$\lambda = \frac{1}{\beta} v(c)c. \tag{5}$$

Let us denote $\frac{1}{v(c)c} v(c)$ by $f(c)$. From 1–3, we see that the vector function $f(c)$ is defined and has continuous derivatives on S. Let c_p and c_q be the solutions of the problem (1) for the prices p and q respectively. From (4) and (5), we obtain

$$(p - q)(c_p - c_q) \leq \beta(f(c_p) - f(c_q))(c_p - c_q). \tag{6}$$

Lemma 1. *The function $C(p)$ is monotone on G if and only if the Jacobian matrix $F(c)$ of the function $f(c)$ satisfies the condition*

$$z^T F(c) z \leq 0 \tag{7}$$

for any $c \in S$ and $z \in R^n$.

P r o o f. From [7, p. 262], we see that the inequality (7) holds on S_0 if and only if $f(c)$ is a monotone function. The continuity argument shows that (7) is equivalent to the condition

$$(f(c_p) - f(c_q))(c_p - c_q) \leq 0 \qquad \text{for any } c_p, c_q \in S. \tag{8}$$

[1]The last requirement implies that there exists a smooth extension of $u(c)$ onto some open set containing S (this set is not required to be a subset of R^n_+).

The inequalities (8) and (6) imply (2). If (8) is false, then there exist $c_p, c_q \in S$ such that

$$(f(c_p) - f(c_q))(c_p - c_q) > 0. \tag{9}$$

Let $p = \beta f(c_p)$, $q = \beta f(c_q)$. Taking into account (4) and (9), we conclude that $C(p)$ is nonmonotone. Thus, Lemma 1 is proved.

Differentiating $f(c)$, we get instead of (7) an equivalent relation

$$(vc)z^T Uz - (vz)z^T Uc - (vz)^2 \leq 0, \tag{10}$$

where $v = v(c)$; $U = U(c)$ is the Hessian matrix of $u(c)$.

Let us fix c. Note that if $vz = 0$, then the inequality (10) holds (since the matrix U is negative semidefinite). To make sure that this inequality is valid on the whole space, it is sufficiently to check it only for the vectors z that comply with the condition

$$vz = vc. \tag{11}$$

We now may rewrite the inequality (10) as follows:

$$z^T Uz - z^T Uc \leq vc, \tag{12}$$

or

$$\left(z - \frac{1}{2}c\right)^T U \left(z - \frac{1}{2}c\right) - \frac{1}{4}c^T Uc \leq vc, \tag{13}$$

where z is an arbitrary vector satisfying equation (11). Using a substitution

$$x = \frac{1}{vc}(2z - c),$$

we finally get the following condition that is equivalent to (13):

$$\frac{1}{4}\alpha(vc)^2 - \frac{1}{4}c^T Uc \leq vc, \tag{14}$$

where

$$\alpha = \sup_{vx=1} x^T Ux. \tag{15}$$

If U is negative definite, then it is easy to see that $\alpha = \frac{1}{vU^{-1}v^T}$. Thus, we have proved the following statement.

Theorem 1. *Suppose that the conditions 1–4 are valid. Then the demand function $C(p)$ is monotone on G if and only if for any $c \in S$ the relations* (14), (15) *hold. If, moreover, the Hessian matrix $U(c)$ is negative definite, then $C(p)$ is monotone if and only if the inequality*

$$\frac{vc}{vU^{-1}v^T} - \frac{c^T U c}{vc} \leq 4 \tag{16}$$

holds for any $c \in S$.

Remark 1. One can show that any of the relations (7), (14), and (16) also implies the monotonicity of $C(p)$ even if S has no interior points.

The first summand in (14) is nonpositive. If we delete it, then we obtain the sufficient condition of monotonicity of $C(p)$. We can check this condition without inverting of the matrix U:

$$-c^T U c \leq 4vc. \tag{17}$$

The following sufficient (but weaker) condition is even more convenient to verify:

$$-Uc \leq 4v^T \tag{18}$$

(after multiplying (18) by c^T, we obtain (17)).

Let $c = (c_1, \cdots, c_n)$, $v = (u_1, \cdots, u_n)$, $U = (u_{ij})$. If $v > 0$, then the inequality (18) may be rewriteen as follows:

$$\xi_i = -\frac{1}{u_i} \sum_j u_{ij} c_j \leq 4, \qquad i = 1, \cdots, n. \tag{19}$$

It is obvious that $\xi_i = \sum_j \xi_{ij}$, where $\xi_{ij} = -\frac{\partial}{\partial c_j}(\ln u_i) / \frac{\partial}{\partial c_j}(\ln c_i)$ is the elasticity of the marginal utility of the product i with respect to j. Thus, it is naturally to say that ξ_i is the total elasticity of the marginal utility of the product i.

The criterion (17) (as well as (18)) is additive: if it holds for each function $u^{(k)}(c)$, then it is also valid for $u(c) = \sum_k u^{(k)}(c)$. This property is often useful.

Corollary 1. Suppose conditions 1–4 hold and, moreover, $u(c) = \sum_k u^{(k)}(c)$, each function $u^{(k)}$ is nonnegative and positive homogeneous of degree $\alpha_k \geq 0$. Then the map $C(p)$ is monotone.

Indeed, from Euler's formula we have

$$\alpha_k u^{(k)}(c) = v^{(k)}(c) c, \tag{20}$$

where $v^{(k)}$ is the gradient of $u^{(k)}(c)$. Differentiating (20), then multiplying the result by the vector c and summing over k, we get the inequality (17).

Suppose now that the preference indicator has the form

$$u(c) = \frac{1}{2} c^T U c + a^T c, \tag{21}$$

where $a \in R^n_+$, $a \neq 0$, and the matrix U is negative semidefinite and independent of c. In this case, we can rewrite the inequality (18) as follows:

$$-Uc \leq \frac{4}{5} a. \tag{22}$$

It is interesting to compare (22) with the natural condition of nondecrease of the function $u(c)$:

$$-Uc \leq a. \tag{23}$$

Thus, for the case (21), the function $C(p)$ is monotone if S is a polyhedron defined by the inequality (22) and the additional requirement $a^T c > 0$ (this requirement ensures that $v(c)c$ is positive). One can easily find examples showing that after an extension of S to the set (23), monotonicity may be violated.

Remark 2. If U is a negative definite matrix and the inequality (16) is strict, then the inequality (2) becomes also strict whenever $c_p \neq c_q$. If S belongs to the interior of R^n_+, then different prices lead to different demand. In this case, $C(p)$ is a strictly monotone map.

Remark 3. The results stated above remain valid, if we

- replace the non-negativity requirement in (1) by a more general one: $c \in K$, where K is a convex cone in R^n_+ with interior points, and
- let S be a subset of K.

If there exist only one consumer of the form (1) in an equilibrium model, then this model reduces to an extremal problem and is rather easy to study. Monotonicity criteria are useful in those cases, where there are many consumers and their total demand isn't generated by a single goal function. The following example shows that exactly this is a quite typical situation.

Let the number of products be equal to 3 and each of the three participants has unit income. The utility functions are as follows:

$$\begin{aligned} u^{(1)}(c_1, c_2, c_3) &= c_1^{1/2} + \sqrt{2} c_2^{1/2} + \frac{2\sqrt{2}}{3} c_3^{3/4}, \\ u^{(2)}(c_1, c_2, c_3) &= u^{(1)}(c_2, c_3, c_1), \\ u^{(3)}(c_1, c_2, c_3) &= u^{(1)}(c_3, c_1, c_2). \end{aligned}$$

By corollary 1, these functions generate monotone demand. Let us show that the total demand $C(p)$ couldn't be described by the problem of the form (1), since there exists a point p such that, at this point, $C(p)$ doesn't satisfy well known Slutsky's conditions [8, p. 258]

$$\frac{\partial C_i}{\partial p_k} + C_k c_i^0 = \frac{\partial C_k}{\partial p_i} + C_i c_k^0. \tag{24}$$

Here, $C(p) = C = (C_1, C_2, C_3)$ and c_i^0 are some constants.

Let $p = (1, 1, 1)$. Then it is easy to check that the demand of the first consumer is equal to $(1/4, 1/2, 1/4)^T$ and the demands of others may be got by a cyclic permutation. Thus, $C = (1, 1, 1)^T$. Using formulas from [8, p. 262], one can find the Jacobian matrix of the total demand

$$\left(\frac{\partial C_i}{\partial p_k}\right) = \begin{pmatrix} -1.95 & 0.5 & 0.45 \\ 0.45 & -1.95 & 0.5 \\ 0.5 & 0.45 & -1.95 \end{pmatrix}.$$

It is easy to see that for such $\frac{\partial C_i}{\partial p_k}$ and C_i, the system of linear equations (24) in variables c_i^0 is unsolvable.

Note that if the utility functions are positive homogeneous, then the total demand is generated by a single goal function [3].

To conclude, let us consider a natural generalization of the monotonicity concept. We say that the function $C(p)$ is quasimonotone on G if

$$\min\{p(c_p - c_q), q(c_q - c_p)\} \leq 0 \tag{25}$$

for any $(p^T, c_p) \in G$, $(q^T, c_q) \in G$, $c_p \in C(p)$, and $c_q \in C(q)$. It is obvious that (25) follows from (2). If $u(c)$ is concave and doesn't reach its supremum, then the demand function generated by $u(c)$ satisfies the inequality (25). Contrary to (2), the inequality (25) is nonadditive, so the criterion of quasimonotonicity of the total demand must include some characteristcs of the utility functions of different participants. We don't know any satisfactory formulation of such a criterion. Below, we point out a necessary condition that the Jacobian matrix of $C(p)$ must meet in the case, where (25) holds. That condition is also close to sufficient.

We use the concept of the single-valued function $C(p)$ monotonicity on the set P of the independent variable values, meaning that G is the graph of $C(p)$ on the set P. Let us introduce the concept of strict quasimonotonicity:

$$\min\{p(C(p) - C(q)), q(C(q) - C(p))\} < 0 \tag{26}$$

for all $p, q \in P$, $p \neq q$. For the equilibrium models from [2, 3], the inequality (26) ensures the uniqueness of the equilibrium prices and the inequality (25) together with the strict convexity of the technology set entails the uniqueness of the equilibrium outputs[2].

In the sequel, we assume that $C(p)$ is defined on an open convex set $P \subset R^n_+$, its values belong to R^n_+, it is differentiable on P, and satisfies the budget identity

$$pC(p) = \beta, \tag{27}$$

where β is a constant.

Lemma 2. *The function $C(p)$ is strictly quasimonotone if and only if for any $p_0 \in P$ there exists a neighborhood $P_0 \ni p_0$ such that for $p = p_0$ and any $q \in P_0 \setminus \{p_0\}$, the inequality* (26) *holds.*

P r o o f. The necessity is obvious. To check the sufficiency, let us suppose that (26) is false for some $p, q \in P$, $p \neq q$. By (27), this means that

$$\max\{pC(q), qC(p)\} \leq \beta. \tag{28}$$

Let $r = \frac{1}{2}(p + q)$. From (28) and (27), we get the following inequality:

$$\max\{qC(r), pC(r)\} \leq \beta. \tag{29}$$

Using the identity (27) for the point r, we see that

$$\min\{qC(r), pC(r)\} \leq \beta. \tag{30}$$

By the inequalities (29) and (30), we conclude that at least one of the pairs (p, r) and (r, q) satisfies the inequality of the form (28), so we can carry out a similar construction. Thus, we obtain a sequence of nested intervals that converge to their common point r_0. It is obvious that no neighborhood of the point p_0 satisfies the condition of Lemma 2. This contradiction concludes the proof.

Note that for quasimonotone functions, quite analogous statement is valid.

Let us denote

$$Z(p) = \{z \mid z^T \in R^n, zC(p) = 0, z \neq 0\} \tag{31}$$

[2]In [4], the inequality (26) is identified with the weak axiom of revealed preference. This identification isn't quite correct. Following the general treatment, one can see that this axiom requires that (26) holds not for $p \neq q$, but just for $C(p) \neq C(q)$ (see, for example, [1, p. 317]).

and let $H(p)$ be the Jacobian matrix of the function $C(p)$.

Theorem 2. *If for any $p \in P$ and $z \in Z(p)$ the inequality $zH(p)z^T < 0$ holds, then the function $C(p)$ is strictly quasimonotone. If $C(p)$ is quasimonotone and $C(p) \neq 0$ for any $p \in P$, then $zH(p)z^T \leq 0$ for any $p \in P$, $z \in Z(p)$.*

P r o o f. Using (27), we can rewrite (26) as follows:

$$\min\{(p-q)C(p),(q-p)C(q)\} < 0. \tag{32}$$

Let $q = p + z$. By the inequality (32) and Lemma 2, for proving the strict quasimonotonicity of $C(p)$, it is sufficiently to check that for any p, the inequality

$$\min\{-zC(p), zC(p+z)\} < 0 \tag{33}$$

holds for small values of z.

Let us fix p and suppose that $zH(p)z^T < 0$ for $z \in Z(p)$. Then there exist constants $\alpha, \delta > 0$ such that $|z(C(p)| \leq \alpha||z||$ implies that $zH(p)z^T \leq -\delta||z||^2$. Here, $||\ ||$ stands for the Euclidian norm. Thus, in the case, where $-\alpha||z|| \leq zC(p) \leq 0$, we obtain

$$zC(p+z) = zC(p) + zH(p)z^T + o(||z||^2) \leq -\delta||z||^2 + o(||z||^2),$$

so the inequality (33) holds for small values of z. If $zC(p) < -\alpha||z||$, then $zC(p+z) < -\alpha||z|| + o(||z||)$ and hence (33) is valid for all values of z from some neighborhood of zero. Finally, in the case, where $zC(p) > 0$, the inequality (33) is obviously valid. The first statement is proved.

Suppose now that $C(p)$ is quasimonotone, but $zH(p)z^T > 0$ for some $p \in P$ and $z \in Z(p)$. Then there exists $t > 0$ such that

$$zC(p+tz) = tzH(p)z^T + o(t) > 0,$$

hence there exists a vector z_0 (close to tz) such that $z_0C(p) < 0$ and $z_0C(p+z_0) > 0$. Assume that $q = p + z_0$, then

$$\min\{(p-q)C(p),(q-p)C(q)\} > 0.$$

By (27), the last inequality contradicts the quasimonotonicity of $C(p)$. This concludes the proof of Theorem 2.

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