Allocation rules for museum pass programs

Béal, Sylvain and Solal, Philippe

9 December 2009

Allocation rules for museum pass programs*

Sylvain Béal†, Philippe Solal‡

December 9, 2009

Abstract

We consider natural axioms for allocating the income of museum pass programs. Two allocation rules are characterized and are shown to coincide with the Shapley value and the equal division solution of the associated TU-game introduced by Ginsburgh and Zang [1].

Keywords: Museum pass program, fair treatment, Shapley value, equal division solution.
JEL Classification number: C71.

1 Introduction

The museums of many big cities offer a joint entry pass which gives pass buyers unlimited access to several museums during a limited period. As an example the Paris museum pass involves 60 museums and monuments in and around Paris. Three options are available at different prices for 2, 4 or 6 days. For the Louvre museum, 430 000 visits in 2006 have been attributed to the Paris museum pass. This note examines the problem of sharing the income of a museum pass program among the participating museums.

In section 2, we define a museum pass program by considering a set of participating museums, a set of pass buyers and, for each pass buyer, the group of museums visited by the pass buyer. We consider natural axioms for museum pass programs. Some of these axioms have been informally discussed in Ginsburgh and Zang [2]. Combining some of these axioms allows to characterize two allocation rules for sharing the total income of museum pass programs. The first one consists in redistributing the income of each pass equally to the museums visited by the pass buyer. This allocation rule highlights the influence of each coalition of museums in the sharing process. The second one attributes an identical share of the total income to each participating museum. These two characterizations are comparable in the sense that one can switch from the first to the second by dropping one axiom and extending the principle of another axiom. In the last section, we study this problem from a game-theoretic point of view by associating to each museum pass program the TU-game considered by Ginsburgh and Zang [1]. It turns out that the above allocation rules coincide with the Shapley value and the equal division solution of the associated TU-game respectively. As such, our results are in the same spirit as the comparable characterizations of the Shapley value and the equal division solution obtained by van den Brink [4] on the class of all TU-games.

*Financial support by IXXI (Lyon Complex Systems Institute) – research project "Cooperation, allocation and complexity" – and the National Agency for Research (ANR) – research program "Models of Influence and Network Theory" – is gratefully acknowledged.

†Corresponding author. Université de Saint-Etienne, CNRS UMR 5824 GATE Lyon Saint-Etienne, France, sylvain.beal@univ-st-etienne.fr. Tel: (+33)(0)4.77.42.19.68. Fax: (+33)(0)4.77.42.19.50.

‡Université de Saint-Etienne, CNRS UMR 5824 GATE Lyon Saint-Etienne, France, philippe.solal@univ-st-etienne.fr. Tel: (+33)(0)4.77.42.19.61. Fax: (+33)(0)4.77.42.19.50.

2 Axiomatic study of museum pass programs

Let $N = \{1, \ldots, n\}$ be a finite set of museums and $\mathcal{X} \subseteq \mathbb{N}$ be the nonempty universe of pass buyers. For each pass buyer $k \in \mathcal{X}$ the nonempty set of museums visited by k is denoted by $M_k \subseteq N$. The price of each pass is normalized to 1. A *museum pass program* on N is given by a (possibly empty) subset $X \subseteq \mathcal{X}$ of pass buyers. Let $2^{\mathcal{X}}$ denote the set of all museum pass programs on N. A *payoff vector* $y \in \mathbb{R}^n$ is an n-dimensional vector giving a payoff $y_i \in \mathbb{R}$ to each museum $i \in N$. An *allocation rule* on $2^{\mathcal{X}}$ is a function $f : 2^{\mathcal{X}} \longrightarrow \mathbb{R}^n$ that assigns a payoff vector $f(X) \in \mathbb{R}^n$ to each $X \in 2^{\mathcal{X}}$.

We consider five axioms that describe an allocation rule for museum pass programs in a natural way. Full distribution states that the total income of the museum pass program is fully redistributed to the participating museums.

Full distribution. For each $X \in 2^{\mathcal{X}}$, it holds that $\sum_{i \in N} f_i(X) = |X|$.

Independence of external visitors states that the payoff of a museum is not affected if one more customer buys a pass but does not visit the museum.

Independence of external visitors. For each $X \in 2^{\mathcal{X}}$, each $k \in X$ and each $i \in N$ such that $i \notin M_k$, it holds that $f_i(X) = f_i(X \backslash \{k\})$.

Fair treatment requires that the change in payoff resulting from the sale of an extra pass is the same for any two museums.

Fair treatment. For each $X \in 2^{\mathcal{X}}$, each $k \in X$ and each $i, j \in N$, it holds that

$$f_i(X) - f_i(X \backslash \{k\}) = f_j(X) - f_j(X \backslash \{k\}).$$

Fair treatment for visited museums consists in applying fair treatment only to museums visited by the extra customer.

Fair treatment for visited museums. For each $X \in 2^{\mathcal{X}}$, each $k \in X$ and each $i, j \in N$ such that $i, j \in M_k$, it holds that

$$f_i(X) - f_i(X \backslash \{k\}) = f_j(X) - f_j(X \backslash \{k\}).$$

Finally, positivity requires that no museum incurs a loss from sharing the total income of the museum pass program.

Positivity. For each $X \in 2^{\mathcal{X}}$ and each $i \in N$, it holds that $f_i(X) \geq 0$.

We are now ready to study the consequences of combining some of these axioms. Note that fair treatment implies fair treatment for visited museums.

Theorem 1 *The unique allocation rule g on $2^{\mathcal{X}}$ that satisfies full distribution, independence of external visitors, fair treatment for visited museums and positivity assigns to each $X \in 2^{\mathcal{X}}$ the payoff vector*

$$g_i(X) = \sum_{S \subseteq N : i \in S} \frac{|\{k \in X : M_k = S\}|}{|S|}, \quad i \in N. \tag{1}$$

Proof. [UNIQUENESS] Consider an allocation rule f on $2^{\mathcal{X}}$ that satisfies the four axioms. The proof that f is uniquely determined is by induction on the number $|X|$ of pass buyers.
INITIAL STEP: assume $|X| = 0$, *i.e.* $X = \emptyset$. Then full distribution implies $\sum_{i \in N} f_i(\emptyset) = 0$. Together with positivity, we obtain $f_i(\emptyset) = 0$ for each $i \in N$. Next, assume that $|X| = 1$, *i.e.* $X = \{k\}$ for some $k \in \mathcal{X}$. Pick any museum $i \in N$ such that $i \notin M_k$ if such a museum exists. By independence of external visitors, we have $f_i(\{k\}) = f_i(\emptyset) = 0$. Full distribution yields $\sum_{i \in M_k} f_i(\{k\}) = 1$. Now, pick any $i, j \in N$ such that $i, j \in M_k$. By fair treatment for visited museums, we get $f_i(\{k\}) - f_i(\emptyset) = f_j(\{k\}) - f_j(\emptyset)$ or equivalently $f_i(\{k\}) = f_j(\{k\})$. Therefore, $f_i(\{k\}) = 1/|M_k|$ for each $i \in M_k$.
INDUCTION HYPOTHESIS: let $x \in \mathbb{N}$ be any number of pass buyers and assume that f is uniquely determined for each $X \in 2^{\mathcal{X}}$ such that $|X| \leq x - 1$.
INDUCTION STEP: consider any $X \in 2^{\mathcal{X}}$ such that $|X| = x$. Pick any $k \in X$. From independence of external visitors, we get $f_i(X) = f_i(X \backslash \{k\})$ for each $i \notin M_k$. Together with full distribution, this implies that

$$\sum_{i \in N} f_i(X) - \sum_{i \in N} f_i(X \backslash \{k\}) = \sum_{i \in M_k} f_i(X) - \sum_{i \in M_k} f_i(X \backslash \{k\}) = 1. \qquad (2)$$

In addition, for each $i, j \in M_k$, by fair treatment for visited museums we have

$$f_i(X) - f_i(X \backslash \{k\}) = f_j(X) - f_j(X \backslash \{k\}).$$

Summing both sides on museums j in M_k, we get

$$\sum_{j \in M_k} \Big(f_i(X) - f_i(X \backslash \{k\}) \Big) = \sum_{j \in M_k} \Big(f_j(X) - f_j(X \backslash \{k\}) \Big),$$

which is equivalent to $|M_k|(f_i(X) - f_i(X \backslash \{k\})) = 1$ by (2). We obtain $f_i(X) = f_i(X \backslash \{k\}) + 1/|M_k|$ for each $i \in M_k$. Since $|X \backslash \{k\}| = x - 1$, the induction hypothesis implies that $f_i(X \backslash \{k\})$ is uniquely determined. We conclude that f is uniquely determined for X.

[EXISTENCE] Consider the allocation rule g given by (1). For each $X \in 2^{\mathcal{X}}$, we have

$$\begin{aligned}
\sum_{i \in N} g_i(X) &= \sum_{i \in N} \sum_{S \subseteq N : i \in S} \frac{|\{k \in X : M_k = S\}|}{|S|} \\
&= \sum_{S \subseteq N} |\{k \in X : M_k = S\}| \\
&= |\{k \in X : M_k \subseteq N\}| \\
&= |X|,
\end{aligned}$$

which proves that g satisfies full distribution. Next, pick any $k \in X$ and any $i \in N$ such that $i \notin M_k$. Then

$$g_i(X) = \sum_{S \subseteq N : i \in S} \frac{|\{h \in X : M_h = S\}|}{|S|} = \sum_{S \subseteq N : i \in S} \frac{|\{h \in X \backslash \{k\} : M_h = S\}|}{|S|} = g_i(X \backslash \{k\})$$

so that independence of external visitors holds. Now, for any $k \in X$ and any $i, j \in M_k$, we have

$$\begin{aligned}
g_i(X) - g_i(X \backslash \{k\}) &= \sum_{S \subseteq N : i \in S} \frac{|\{h \in X : M_h = S\}|}{|S|} - \sum_{S \subseteq N : i \in S} \frac{|\{h \in X \backslash \{k\} : M_h = S\}|}{|S|} \\
&= \frac{1}{|M_k|} \\
&= g_j(X) - g_j(X \backslash \{k\})
\end{aligned}$$

which implies that g satisfies fair treatment for visited museums. Finally, g satisfies positivity as a sum of non-negative rational numbers. ■

The allocation rule g consists in redistributing the income of each pass equally to the museums visited by the pass buyer. Replacing fair treatment for visited museums and independence of external visitors by fair treatment yields a characterization of another allocation rule that consists in sharing equally the total income of a museum pass program.

Theorem 2 *The unique allocation rule e on $2^{\mathcal{X}}$ that satisfies full distribution, fair treatment and positivity assigns to each $X \in 2^{\mathcal{X}}$ and each $i \in N$ the payoff $e_i(X) = |X|/n$.*

Proof. [UNIQUENESS] Consider an allocation rule f on $2^{\mathcal{X}}$ that satisfies the three axioms. The proof that f is uniquely determined is by induction on the number $|X|$ of pass buyers.
INITIAL STEP: assume $|X| = 0$, *i.e.* $X = \emptyset$. Then full distribution and positivity imply that $f_i(\emptyset) = 0$ for each $i \in N$.
INDUCTION HYPOTHESIS: let $x \in \mathbb{N}$ be any number of pass buyers and assume that f is uniquely determined for each $X \in 2^{\mathcal{X}}$ such that $|X| \leq x - 1$.
INDUCTION STEP: consider any $X \in 2^{\mathcal{X}}$ such that $|X| = x$. Applying fair treatment for any $k \in X$ and any $i, j \in N$, we have

$$f_i(X) - f_i(X \backslash \{k\}) = f_j(X) - f_j(X \backslash \{k\}).$$

Summing both sides on N, we get

$$\sum_{j \in N} \Big(f_i(X) - f_i(X \backslash \{k\}) \Big) = \sum_{j \in N} \Big(f_j(X) - f_j(X \backslash \{k\}) \Big).$$

By full distribution we have $n\,(f_i(X) - f_i(X \backslash \{k\})) = 1$, which implies that $f_i(X) = f_i(X \backslash \{k\}) + 1/n$ for each $i \in N$. Since $f_i(X \backslash \{k\})$ is uniquely determined by assumption, we conclude that f is uniquely determined for X.

[EXISTENCE] It is obvious that e satisfies full distribution and positivity. Next, consider any $X \in 2^{\mathcal{X}}$, $X \neq \emptyset$, any $k \in X$ and any $i, j \in N$. We have

$$e_i(X) - e_i(X \backslash \{k\}) = \frac{|X|}{n} - \frac{|X| - 1}{n} = e_j(X) - e_j(X \backslash \{k\})$$

which completes the proof. ■

Logical independence of the axioms in Theorems 1 and 2 is shown by the following allocation rules on $2^{\mathcal{X}}$.

- The null allocation rule that assigns to each $X \in 2^{\mathcal{X}}$ and each $i \in N$ a null payoff satisfies independence of external visitors, fair treatment for visited museums, fair treatment and positivity. It does not satisfy full distribution.

- The allocation rule e satisfies full distribution, fair treatment for visited museums and positivity. It does not satisfy independence of external visitors.

- The allocation rule f^g that assigns to each $X \in 2^{\mathcal{X}}$, the payoffs $f_i^g(X) = 1-n+g_i(X)$ for some $i \in N$ and $f_j^g(X) = 1 + g_j(X)$ if $j \in N\backslash\{i\}$ satisfies full distribution, independence of external visitors and fair treatment for visited museums. It does not satisfy positivity.

- The allocation rule $f^{\max}$ that assigns to each $X \in 2^{\mathcal{X}}$ the payoff vector

$$f_i^{\max}(X) = \sum_{S \subseteq N : i = \max_{j \in S} j} |\{k \in X : M_k = S\}|, \quad i \in N$$

  satisfies full distribution, independence of external visitors and positivity. It does not satisfy fair treatment for visited museums and fair treatment.

- The allocation rule f^e that assigns to each $X \in 2^{\mathcal{X}}$, the payoffs $f_i^e(X) = 1-n+e_i(X)$ for some $i \in N$ and $f_j^e(X) = 1 + e_j(x)$ if $j \in N\backslash\{i\}$ satisfies full distribution and fair treatment. It does not satisfy positivity.

3 Concluding remarks

In this section, we associate to each museum pass program $X \in 2^{\mathcal{X}}$ the TU-game v_X on N such that for each coalition of museums $S \subseteq N$, $v_X(S)$ is the total income of passes that would have been sold if only admissions for the subgroup of museums S were possible, *i.e.*

$$v_X(S) = |\{k \in X : M_k \subseteq S\}|.$$

The game v_X was introduced by Ginsburgh and Zang [1]. It is easy to check that for each $X \in 2^{\mathcal{X}}$, the payoff vector $e(X)$ coincides with the equal division solution of the museum TU-game v_X. Furthermore, the Proposition in Ginsburgh and Zang [1] establishes that for each museum pass program $X \in 2^{\mathcal{X}}$, the payoff vector $g(X)$ given by (1) is the Shapley value (Shapley [3]) of the TU-game v_X.

References

[1] V. Ginsburgh and I. Zang – "The Museum Pass Game and its Value", *Games and Economic Behavior* **43** (2003), pp. 322–325.

[2] — , "Sharing the Income of a Museum Pass Program", *Museum Management and Curatorship* **19** (2004), pp. 371–383.

[3] L. S. Shapley – "A Value for n-person Games", Contribution to the Theory of Games vol. II (H.W. Kuhn and A.W. Tucker eds), *Annals of Mathematics Studies* **28**, Princeton University Press, Princeton, 1953.

[4] R. van den Brink – "Null or Nullifying Players: The Difference Between the Shapley Value and Equal Division Solutions", *Journal of Economic Theory* **136** (2007), pp. 767–775.