A complementary approach to transitive rationalizability

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Abstract

In this article, we study the axiomatic foundations of revealed preference theory. We define two revealed relations from the weak and strong revealed preference. The alternative x is preferred to y with respect to U if x, being available in an admissible set implies, the rejecting of y; and x is preferred to y with respect to Q if the rejecting of x implies the rejecting of y. The purpose of the paper is to show that the strong axiom of revealed preference and Hansson's axiom of revealed preference can be given with the help of U and Q and their extension properties.

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1 Introduction

A choice correspondence leads to at least two relations. On the one hand, x is weakly preferred to y if x is chosen, while y could have been selected under some set of alternatives. On the other hand, x is strictly preferred to y if x is chosen, while y is available and rejected. The traditional formulation of the weak (WARP) and the strong (SARP) axioms of revealed preference have been given in terms of the relationship between the weak and the strict revealed preference relations. The strong congruence axiom (SCA) is equivalent to the rationality of the choice correspondence with transitive (and complete) underlying preference (see Richter (1966)). Suzumura (1976) introduced an auxiliary concept – *connected chain*, that provides a uniform framework of WARP, SARP and SCA. Hansson's axiom of revealed preference (HARP) (see Hansson (1968)) can be regarded as an equivalent formulation of SCA in terms of connected chain.

The extension theorem of Szpilrajn was the transfinite tool of Richter's proof; this is why the relationship of revealed preference axioms and extension theorems arose. Recently, Duggan (1999) summarized and gave a unified treatment of Szpilrajn–type extension theorems. He pointed out the possibility of the application of an "intension" theorem in the theory of revealed preferences.

It will be shown that HARP can also be written as a special relationship between the weak and the strict revealed relations, eliminating the distinction between the introduction of Hansson's axiom and the introduction of the weak and strong axioms of revealed preference. It is known that there are equivalent formulations of SCA and SARP (Theorems 1 and 2). This note gives an alternative characterization (Theorem 3) with the help of extensions properties of U and Q.

2 Preliminaries

For a binary relation $S \subseteq X \times X$, let $c_g^S(B)$ denote the *S-greatest elements* of the set B, i.e., $c_g^S(B) \doteq \{x \in B : (x,y) \in S \ \forall y \in B\}$, and let $c_m^S(B)$ denote the *S-maximal elements* of the set B, that is, $c_m^S(B) \doteq \{x \in B : S^x \cap B = \varnothing\}$, where S^x is the upper level set of S at x. Borrowing the terminology from Clark

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(1985), we introduce the *complementarity operator*, defined upon relations as follows: $\Gamma(S) \doteq (S^c)^{-1}$, where $S^c \doteq (X \times X) \setminus S$. It is easy to see that the complementarity operator establishes a relationship between the two concepts of optimality, that is, $c_g^S(B) = c_m^{\Gamma(S)}(B)$ for every $B \subseteq X$. Let $t(S)$ and $nt(S)$ denote the transitive closure and the negative transitive interior of a relation S, respectively.

The triple $(X, \mathcal{B}, c)$ is called a *decision structure*, where $\mathcal{B}$ is a subset of the power set of X and $c : \mathcal{B} \to \mathcal{P}(X)$ is a *choice correspondence* such that $\varnothing \neq c(B) \subseteq B$. There are two kinds of definition for the revealed preference relation. The first one is the *weak*: $(x, y) \in R$ if and only if there exists $B \in \mathcal{B}$ for which $x \in c(B)$ and $y \in B$; the second one is the *strict*: $(x, y) \in P$ if and only if there exists $B \in \mathcal{B}$ for which $x \in c(B)$ and $y \in B \setminus c(B)$.

The traditional formulation of the WARP is the inclusion $P \subseteq \Gamma(R)$, and the SARP is the inclusion $t(P) \subseteq \Gamma(R)$, where P is the strict and R is the weak revealed relations. The SCA introduced by Richter says: $(x, y) \in t(R)$ and $y \in c(B)$, $x \in B$ implies $x \in c(B)$. The choice correspondence is said to be *transitive g-rational* whenever there exists a transitive relation S with $c = c_g^S$.

Duggan (1999) defined the notion of compatible extension and compatible "intension" of a binary relation. Given relations D and D', D' is a *compatible extension* of D if $D \subseteq D'$ and $A(D) \subseteq A(D')$; here $A(D) \doteq D \cap \Gamma(D)$ is the asymmetric part of a relation D. We denote by $\mathcal{E}(D)$ the set of compatible extensions of D. Similarly D' is a *compatible "intension"* of D if $D' \subseteq D$ and $A(D) \subseteq A(D')$. We denote by $\mathcal{I}(D)$ the set of compatible "intensions" of a relation D. It is not too hard to see that $\mathcal{I}(D) = \{\Gamma(D') : D' \in \mathcal{E}(\Gamma(D))\}$, that is, the operation Γ is a one-to-one correspondence between $\mathcal{I}(D)$ and $\mathcal{E}(\Gamma(D))$.

Suzumura (1976a) showed that the relation D has complete, transitive, compatible extensions if and only if $A(D) \circ t(D) \subseteq \Gamma(D)$, which is equivalent to $t(D) \in \mathcal{E}(D)$ as Duggan clarified. Of course the statement that D has asymmetric, negative transitive compatible "intension" if and only if $nt(D) \in \mathcal{I}(D)$, is an easy consequence of the previous one.

3 Results

Put $U \doteq \Gamma(R)$, where R is the weak revealed relation of a decision structure. The $(x, y) \in U$ can be interpreted as follows. By the definition of Γ it is equivalent to $(y, x) \notin R$, which is equivalent to the fact that $y \in c(B)$ and $x \in B$ can not both hold for any $B \in \mathcal{B}$, that is, for every $B \in \mathcal{B}$, $x \in B$ implies $y \notin c(B)$. This can be phrased as *the availability of x implies the rejecting of y*. Since SARP is formulated by $t(P) \subseteq U$, we can give a characterization of SARP as a congruence axiom as follows: *if $(x, y) \in t(P)$, then the selection of y is prohibited if x is available.*

Set $Q \doteq \Gamma(P)$, where P is the strong revealed relation. As before, we investigate the meaning of $(x, y) \in Q$. The inclusion $(x, y) \in Q$ is equivalent to $(y, x) \notin P$, which is equivalent to $y \notin c(B)$ or $x \notin B$ or $x \in c(B)$ for any $B \in \mathcal{B}$. Therefore $(x, y) \in Q$ if and only if $y \in c(B)$ and $x \in B$ together imply that $x \in c(B)$ for all $B \in \mathcal{B}$. Comparing this with the original notion of SCA we get the equivalence of SCA and the inclusion $t(R) \subseteq Q$. This equivalence illuminates the distinction between the introduction of Hansson's axiom and the weak and strong axioms of revealed preferences.

The inclusion $(x, y) \in Q$ can also be expressed in the following way: $x \in B$ and $x \notin c(B)$ together imply that $y \notin c(B)$ for any $B \in \mathcal{B}$, or $x \in B \setminus c(B)$ implies that $y \notin c(B)$. Thus *the rejecting of x implies the rejecting of y*. Since SCA is equivalent to the assumption $t(R) \subseteq Q$, the strong congruence axiom can be expressed as follows: *if $(x, y) \in t(R)$, then the selection of y is prohibited if x is rejected.*

The equivalence of $t(R) \subseteq \Gamma(P)$ and SCA leads to the summarization of the transitive rationalizability in terms of revealed preferences R, P, U, and operations of transitive hull and negative transitive interior.

Theorem 3.1 *Let $(X, \mathcal{B}, c)$ be a decision structure. All of the following assumptions are equivalent to SCA:*
1. $t(R) \subseteq \Gamma(P)$;
2. $P \subseteq nt(U)$;
3. $c = c_g^{t(R)}$, that is, c is transitive g-rational;
4. $c = c_m^{nt(U)}$;
5. for any $B \in \mathcal{B}$ and for all $x \in B \setminus c(B)$, there exists $y \in B$ such that

$(y,x) \in nt(U)$;
6. for any $B \in \mathcal{B}$ and for all $x \in B \setminus c(B)$ and for all $y \in c(B)$, the inclusion $(y,x) \in nt(U)$ *holds.*
Here R is the weak, and P the strict revealed, relation of c and $U = \Gamma(R)$.

Proof. The equivalence of assumption (1) and SCA has just been proved. Since $\Gamma(t(S)) = nt(\Gamma(S))$ for any relation S, condition (2) is equivalent to condition (1). The equivalence of condition (3) and SCA is easy and well-known. The identities $c_g^{t(R)} = c_m^{\Gamma(t(R))} = c_m^{nt(U)}$ hold because $c_g^S = c_m^{\Gamma(S)}$ for any relation S, which shows the equivalence of (3) and (4). The identity $c = c_g^{t(R)}$ is equivalent to the inclusion $c_g^{t(R)}(B) \subseteq c(B)$ for any $B \in \mathcal{B}$. This inclusion implies that, when $x \in B$ and $x \notin c(B)$, then $x \notin c_g^{t(R)}(B)$, that is, if $x \in B \setminus c(B)$, then there exists $y \in B$ such that $(x,y) \in t(R)$ does not hold. Hence for all $x \in B \setminus c(B)$ there exists $y \in B$ such that $(y,x) \in nt(U)$, which is assumption (5). The equivalence of assumption (2) and (6) is an easy consequence of the definition of the strict revealed relation P. ❑

The original definition of SARP is the inclusion $t(P) \subseteq \Gamma(R)$; hence a parallel theorem can be stated in terms of revealed preferences R, P, Q, and operations of negative transitive interior and transitive hull. The equivalence of (1) and (3) is due to Clark (1985) and the equivalence of the other conditions are obvious.

Theorem 3.2 *Let $(X, \mathcal{B}, c)$ is a decision structure. All of following assumptions are equivalent to SARP:*
1. $t(P) \subseteq \Gamma(R)$;
2. $R \subseteq nt(Q)$;
3. $c = c_m^{t(P)}$;
4. $c = c_g^{nt(Q)}$;
5. for any $B \in \mathcal{B}$ and for every $x \in c(B)$ and for all $y \in B$, the inclusion
$(x,y) \in nt(Q)$ holds,
where R is the weak, and P the strict revealed, relation and $Q = \Gamma(P)$.

The difference between HARP and SARP is one of the most interesting problems in revealed preference theory. We are going to show that this difference can be restated in terms of compatible extension and "intension".

It can be easily proved that D' is a compatible extension of D if and only if one of the following conditions holds:
· $S(D) \subseteq S(D')$ and $A(D) \subseteq A(D')$;
· $D \subseteq D' \subseteq W(D)$,
where $S(D) \doteq D \cap D^{-1}$ is the symmetric part of relation D and $W(D) = D \cup \Gamma(D)$ is the weak extension of relation D. It is also easy to verify that either of the following conditions:
· $S_c(D) \subseteq S_c(D')$ and $A(D) \subseteq A(D')$;
· $A(D) \subseteq D' \subseteq D$
is equivalent to D' being a compatible "intension" of D, where $S_c(D) \doteq S(D^c)$ is the symmetric complement of the relation D.

Theorem 3.3 *Suppose that, the decision structure $(X, \mathcal{B}, c)$ has the property WARP. Let R and P denote the weak and the strict revealed relations respectively, and let $U = \Gamma(R)$, $Q = \Gamma(P)$.*
The following assumptions are equivalent to SCA (and therefore to HARP as well):
1. $R \subseteq t(R) \subseteq Q$;
2. $t(R) \in \mathcal{E}(R)$;
3. $nt(U) \in \mathcal{I}(U)$.
The following assumptions are equivalent to SARP:
1. $R \subseteq nt(Q) \subseteq Q$;
2. $nt(Q) \in \mathcal{E}(R)$;
3. $t(P) \in \mathcal{I}(U)$.

Proof. We have seen that SCA is equivalent to $t(R) \subseteq Q$. It is well-known that WARP is equivalent to the equation $A(R) = P$, whence $W(R) = \Gamma(A(R)) = \Gamma(P) = Q$. Thus $\mathcal{E}(R) = \{D' : R \subseteq D' \subseteq Q\}$ on the assumption that WARP holds. This proves the equivalence of assumptions (1) and (2). The equivalence of (3) and (2) is an easy consequence of the involution property of Γ.

The definition of SARP is $t(P) \subseteq \Gamma(R)$, that is, $R \subseteq \Gamma(t(P)) = nt(Q)$. This shows the equivalence of SARP and the inclusions $R \subseteq nt(Q) \subseteq Q$. Therefore, as we have seen, $nt(Q) \in \mathcal{E}(R)$, and the involution property of Γ implies that $nt(Q) \in \mathcal{E}(R)$ and $t(P) \in \mathcal{I}(U)$ are equivalent conditions. ❑

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