On behavioral Arrow Pratt risk process with applications to risk pricing, stochastic cash flows, and risk control

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ON BEHAVIORAL ARROW-PRATT RISK PROCESS WITH APPLICATIONS TO RISK PRICING, STOCHASTIC CASH FLOWS, AND RISK CONTROL

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ON BEHAVIORAL ARROW-PRATT RISK PROCESS WITH APPLICATIONS TO RISK PRICING, STOCHASTIC CASH FLOWS, AND RISK CONTROL

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--- Abstract ---

We introduce a closed form behavioural stochastic Arrow-Pratt risk process, decomposed into discrete asymmetric risk seeking and risk averse components that run on different local times in ϵ-disks centered at risk free states. Additionally, we embed Arrow-Pratt ("AP") risk measure in a simple dynamic system of discounted cash flows with constant volatility, and time varying drift. Signal extraction of Arrow-Pratt risk measure shows that it is highly *nonlinear in constant volatility* for cash flows. Robust identifying restrictions on the system solution confirm that even for small time periods constant volatility is not a measure of AP risk. By contrast, time-varying volatility measures aspects of embedded AP risk. Whereupon maximal AP risk measure is obtained from a convolution of input volatility and idiosyncratic shocks to the system. We provide four applications for our theory. First, we find that Engle, Ng and Rothschild (1990) Factor-ARCH model for risk premia is misspecified because the factor price of risk is time varying and unstable. Our theory predicts that a hyper-ARCH correction factor is required to remove the Factor-ARCH specification. Second, when applied to analysts beliefs about interest rates and volatility, we find that AP risk measure is a feedback control over stochastic cash flows. Whereupon increased risk aversion to negative shocks to earnings increases volatility. Third, we use an oft cited example of Benes, Shepp and Witsenhausen (1980) to characterize a controlled AP diffusion for a conservative investor who wants to minimize the AP risk process for an asset. Fourth, we recover stochastic differential utility functional from the AP risk process and show how it is functionally equivalent to Duffie and Epstein's (1992) parametrization.

Keywords: behavioural Arrow-Pratt risk process; asymmetric risk decomposition; asset pricing; Markov process; local martingale; local time change

JEL Classification: C03, G12, D03, D81**;** **AMS 2010 Classification:** 60

I. INTRODUCTION

This paper contributes to the mammoth literature on stochastic differential equations in financial economics by introducing (1) a closed form stochastic Arrow-Pratt ("AP") risk process that is decomposed into asymmetric risk aversion and risk seeking components, and (2) a linear operator that embeds Arrow-Pratt risk measure in stochastic processes of interest. The AP process is a time changed martingale, and the discrete embedding paves the way for estimation of risk processes via signal extraction methods. Specifically, we use an adaptive infinitesimal generator to embed Arrow-Pratt[1] ("AP") risk measure in a Markov process and then specify a simple theory motivated filtering scheme to estimate it. Robustness of that generator was verified by its recovery with an independent Dambis-Dubins-Schwartz time change theory for local martingales. We impose identifying restrictions on the model to calibrate AP risk measure to volatility. Contrary to Tobin's (1958, pp. 71-72) static model, we find that constant volatility is not a measure of AP risk for any feasible identifying restriction-- even over very small time periods. By contrast, we find time varying volatility captures aspects of Arrow-Pratt risk measure—even though it is unstable. Thereby confirming empirical results by Chou, et al. (1992, pp. 205, 207), who also report that French, et al. (1987, p.20) found no

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[1] Pratt (1964, p. 123 n.2, p. 135, n.4) credited unpublished lecture notes by Kenneth Arrow, and work by Robert Schlaifer with independent "discovery" of the risk aversion measure $r(x) = -\frac{u''(x)}{u'(x)}$ for a regular utility function. Aspects of that formula was subsequently published in Arrow (1965).

statistically significant evidence between expected risk premia and *ex-ante* volatility in a GARCH type model.

Time change for local martingales show that independent risk seeking and risk aversion components of Arrow-Pratt risk process operate on different time clocks. In fact, excursions of each component from a risk free state are asymmetric. Thereby, predicting an asymmetric risk aversion result from von Neuman Morgensterrn utility albeit in a stochastic environment—as opposed to Tversky and Khaneman (1992) static asymmetric loss aversion theories. Additionally, our theory explains empirical results for subordinated processes reported by Clark (1973), and Carr and Wu (2004). Nelson and Foster (1994, pp. 10-11) derive time change results similar to ours—albeit in the context of adaptive data analysis for ARCH models. Our results are obtained from microfoundation theory.

To be sure, variance and or AP risk measure are not the only risk measures proposed in the literature. For example, Ross (1981, p. 625) argued that the AP risk measure is weak because it fails in cases of incomplete insurance, and he proposed a stronger measure of his own. Recently, Artzner, et al (1999) introduced the concept of coherent measures of risk based on 4-axioms, that include the controversial translation variance restriction which resembles the "risk compensation" concept reported in LeRoy and Warner (2000, p. 85 eq. (9.11)). Acerbi (2001, 2008) introduced a purported spectral risk measure based on Artzner, et al (1999). However, closer examination of Acerbi's model shows that it is a version of Tversky and Khaneman's (1992) prospect theory. In fact, Tversky and Khaneman (1992) popularized the concept of loss aversion—a variant of risk aversion—due to asymmetric value functions or skewed preferences observed in laboratory experiments. Even though this paper introduces a behavioural Arrow-Pratt risk process, it does not test whether volatility is a measure of risk for prospect theory specifications[2].

The instant paper proceeds as follows. In Section II.A. we provide a brief introduction to the problem, and in Section II. B we introduce an adaptive infinitesimal generator, in Proposition II.B.1, that embeds Arrow-Pratt risk measure in Markov processes. The latter process is then used to model discounted stochastic [cash] flow with embedded Arrow-Pratt risk measure. Section II.C presents a simple dynamic system for signal extraction of the AP risk measure with a Kalman-Bucy filter. Section II.D imposes identifying restrictions on the estimator to calibrate Arrow-Pratt risk measures to volatility. The feasibility of the restrictions determine whether or not volatility is a measure of Arrow-Pratt risk. Additionally, a maximal Arrow-Pratt risk result is established from which volatility bounds are inferred. Section III.A. recovers the infinitesimal generator for AP risk process by suitable time change of local martingales under the Dambis-Dubins-Schwartz (1965), and Knight's (1971) representation theories. Section III.B. introduces a risk decomposition theory based on AP risk processes in the Ornstein-Uhlenbeck class. We also consider the local time behavior of excursions of AP

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2 Risk measures in prospect theory are asymmetric, and subprobability measures are assumed. See Tversky and Khaneman (1992). Nelson (1991) provides a mechanism for estimating asymmetric volatility. However, implementing the additional complexities of those models here would detract from this paper's focus on Arrow-Pratt risk measure.

risk from an ϵ −disk centered at risk free states. Lemma III.B.5 and Theorem III.B.4 are the main results there.

Section IV of the paper provides four important applications of our theory. First, in Section IV.A we show that hyper-ARCH risk aversion processes leads to specification error in Engle, Ng and Rothschild (1990) Factor-ARCH model for pricing risk premia. Second, in Section IV.B application to analysts beliefs about interest rates, and volatility in discounted cash flows show that risk aversion to negative shocks in earnings leads to increased volatility from feedback effects between risk aversion and volatility. This is the *sui generis* of Barsky and DeLong's (1993) findings. Third, Section IV.C provides heuristics about the time changed AP risk process. Whereupon we apply an oft cited example by Benes, Shepp and Withausen (1980) to the case of a conservative investor who wants to monitor the Arrow-Pratt risk associated to an asset. Fourth, Section IV. D employs an elementary integral to recover stochastic differential utilityfrom AP risk process, and establish functional equivalence with Duffie and Epstein (1992) specification.

II. THE MODEL

A. Introduction to the problem

Given a sample space Ω, i.e., laws of nature, $\mathfrak{F}$ a σ-field of Borel subsets of Ω, a probability measure P on Ω, and a finite time interval $[0,T]$, an analyst wants to estimate the Arrow-Pratt risk aversion process $\{r(t,\omega); 0 \leq t \leq T < \infty\}$--associated with a realization of a sample path for the flow $\{H(t,\omega); 0 \leq t \leq T < \infty\}$ (exponentially discounted over $\mathbb{F}$ with (constant) rate $R = (1+i)^{-1}$ for interest rate i). Technically, all processes are right continuous with left hand limits, and $H(t,\omega)$ is a $\mathbb{F}$-adapted process defined on the measure space $(\Omega, \mathfrak{F}, \mathbb{F}, P)$ where $\mathbb{F}$ is the P completition of the augmented natural filtration of Brownian motions $\mathfrak{F}^U \cup \mathfrak{F}^V$ to be motivated in the sequel. That is, $\mathbb{F} = \{\mathfrak{F}_t\}_{0 \leq t < T < \infty}$ subsumes the natural filtration of Brownian motions $(U(t,\omega); 0 \leq t < \infty)$ and $(V(t,\omega); 0 \leq t < \infty)$. All processes are assumed to have the strong Markov property[3], i.e., present independent of the past.

Pratt (1964, p. 125 eq. (5)) postulated the following relationship for local risk premium $\pi(x,\tilde{z})$ on an asset for some generic risk $\tilde{z}$ with volatility, i.e., variance, σ_z^2

(1) $$\pi(x,\tilde{z}) = \frac{1}{2}\sigma_z^2 r(x) + o(\sigma_z^2)$$

If we assume that $\pi(\cdot)$ is observable and that σ_z^2 and $r(x)$ are not, then AP risk is embedded in $\pi(\cdot)$, and separability of σ_z^2 and $r(x)$ is an artifact of Pratt's theoretical assumptions. Specifically, a generalized risk premium relationship could be rewritten as

(2) $$\pi(x,\tilde{z}) = f(\sigma_z^2, r(x)) + \xi$$

where the functional form of $f(\cdot)$ is unknown, and the random variable ξ depends on σ_z^2. With that in mind, (1) and (2) suggests a signal extraction or filtering procedure for estimating $r(x)$. We proceed as follows.

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[3] See Dynkin (1961, p. 108)

B. Embedding Arrow-Pratt risk measure with infinitesimal generators

By definition, Arrow-Pratt risk measure in choice space X for a utility function $u \in C^2(X)$ is given by

(3) $$r(x) = -\frac{\frac{\partial^2 u(x)}{\partial x^2}}{\frac{\partial u\,(x)}{\partial x}}$$

This can be rewritten as

(4) $$\left(\frac{\partial^2 u}{\partial x^2}\right) + r(x)\left(\frac{\partial u}{\partial x}\right) = 0$$

So that slight modification of (4) leads to a linear operator

(5) $$\mathcal{L} = \frac{\sigma^2(x)}{2}\left(\frac{\partial^2}{\partial x^2}\right) + r(x)\left(\frac{\partial}{\partial x}\right)$$

which is the infinitesimal generator of the following homogenous Markov process[4]

(6) $$dX(t,\omega) = r\big(X(t)\big)dt + \sigma\big(X(t)\big)dW(t,\omega)$$

where $W(t,\omega)$ is a Wiener process, $\sigma(X(t))$ is a coefficient of diffusion, and $r\big(X(t)\big)$ is Arrow-Pratt risk measure embedded in the drift term. In the sequel we assume that $r(\cdot)$ and $\sigma(\cdot)$ are Lipshitz continuous and square integrable, i.e., $\int_0^\infty |r(X(t)|^2 dt + \int_0^\infty |\sigma(X(t,\omega)|^2 < \infty$. These requirement guarantee convergence of the Pickard-Lindelof iteration scheme to a fixed point solution of the AP equation(s). In particular, equation (6) is the stochastic differential analog to Pratt's (1964) risk premium equation generalized in (2) above. Let $\mathfrak{D}(\mathcal{L})$ be the domain of $\mathcal{L}$. The results in (1)-(6) can be summarized by the following

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[4] See Protter (2004, pp. 349-350).

PROPOSITION II.B.1 (Arrow-Pratt embedding). Let $r(x) = -\frac{\frac{\partial^2 u(x)}{\partial x^2}}{\frac{\partial u(x)}{\partial x}}$ be the Arrow-Pratt risk measure on a choice space for a utility function $u \in C^2(X)$; and

$$\mathcal{L} = \frac{\sigma^2(x)}{2}\left(\frac{\partial^2}{\partial x^2}\right) + r(x)\left(\frac{\partial}{\partial x}\right)$$

be an infinitesimal generator, with domain $\mathfrak{D}(\mathcal{L})$, defined on utility functions $u \in C^2(X) \cap \mathfrak{D}(\mathcal{L})$. Then Arrow-Pratt risk measure $r(x)$ is embedded in the Markov process $\{X(t,\omega); 0 \leq t < \infty\}$ defined on the measure space $(\Omega, \mathfrak{F}, \mathbb{F}, P)$ and represented by the Ito process

$$dX(t,\omega) = r\big(X(t)\big)dt + \sigma\big(X(t)\big)dW(t,\omega)$$

So that estimation of Arrow-Pratt risk measure is a signal extraction problem tantamount to estimation of the drift term in a Markov process. $\rrbracket$

C. A simple dynamic system for estimating Arrow-Pratt risk measure

With the foregoing embedding behind us, let $Z(t,\omega)$ be the discounted value of the flow $H(t,\omega)$ over $\mathbb{F}$. Assume that $U(t,\omega)$ and $V(t,\omega)$ are Brownian motions, and that σ_z is a constant volatility associated with $H(t,\omega)$ through $Z(t,\omega)$. We let $C(t)$ be an unobservable deterministic function that controls "jumps" in $U(t,\omega)$ in equation (8) below[5]. In order to estimate $r(Z(t,\omega))$, the analyst proposes the simple solveable dynamic system:

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[5] This assumption implies that the AP risk measure embedded in the Markov process for equation (9) also has its own stochastic process.

(7) $$Z(t,\omega) = \int_0^t e^{-Rs} H(s,\omega)ds$$

(8) $$dr(Z(t,\omega)) = C(t)dU(t,\omega)$$

(9) $$dZ(t,\omega) = r(Z(t,\omega))dt + \sigma_z dV(t,\omega)$$

These equations[6] describe a filtering problem in which the mean squared error for estimated Arrow-Pratt risk measure, $\hat{r}(Z(t,\omega))$, is

(10) $$S(t,\omega) = E\left[\left(r(Z(t,\omega)) - \hat{r}(Z(t,\omega))\right)^2\right]$$

Whereupon the Ricatti equation[7] is given by

(11) $$\frac{dS(t,\omega)}{dt} = -\sigma_z^2 S^2(t,\omega) + C^2(t)$$

To simplify matters further, assume $C(t) = c$ so that

(12) $$\frac{dS(t,\omega)}{(c^2 - \sigma_z^2 S^2(t,\omega))} = dt$$

The solution to that equation is

(13) $$S(t,\omega) = \left(\frac{c}{\sigma_z}\right)\tanh(\sigma_z ct)$$

According to Kalman-Bucy filter theory, Oksendal (2003, pp. 99-100, Theorem 6.2.8), the dynamics of $\hat{r}(Z(t,\omega))$ is given by

(14) $$d\hat{r}(Z(t,\omega)) = -\sigma_z^2 S(t,\omega)\hat{r}(Z(t,\omega))dt + \sigma_z^2 S(t,\omega)dZ(t,\omega)$$

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[6] In models that posit risk neutrality equation (7) is replaced by

$$Z(t,\omega) = E^{P^*}[\int_0^t e^{-Rs} H(s,\omega)ds \,|\mathfrak{F}_r; 0 \leq r < t]$$

where P^* is an equivalent martingale measure. That is, the probability measure P^* is equivalent to P but it is a function of $Z(t,\omega)$. See Musiela and Rutkowski (2004, pp. 13-14).

[7] See Oksendal (2003, p. 99).

which upon substitution of the value of $S(t,\omega)$ from (13) gives

$$= -\sigma_z^2\left(\frac{c}{\sigma_z}\right)\tanh(\sigma_v ct)\,\hat{r}(t,\omega)dt + \sigma_z^2\left(\frac{c}{\sigma_z}\right)tanh(\sigma_z ct)dZ(t,\omega)$$

$$\Rightarrow d(\cosh(\sigma_z ct))\hat{r}(t,\omega) = \sigma_z c.\sinh(\sigma_z ct)\,dZ(t,\omega)$$

So that

(15) $$\hat{r}(Z(t,\omega)) = \left(\frac{\sigma_z}{\cosh\,(\sigma_z ct)}\right)\int_o^t \sinh(\sigma_z cs)\,dZ(s,\omega)$$

$$= \left(\frac{\sigma_z}{\cosh(\sigma_z ct)}\right)\int_0^t \sinh(\sigma_z cs)\,e^{-Rs}H(s,\omega)ds$$

D. Interpreting the system estimator for Arrow-Pratt risk measure.

This estimate of Arrow-Pratt relative risk aversion plainly shows that it depends on volatility (σ_z), and interest rate (R) for discounted dynamic flows $(H(s,\omega))$. For example, if $H(t,\omega)$ is earnings flow at time t, then equation (7) is discounted earnings $\big(dZ(t,\omega)\big)$ decomposed into unobservable permanent $(r(Z(t,\omega))dt)$ and temporary $\big(\sigma_z dV(t,\omega)\big)$ components. Thus, risk aversion ("don't mess with my money") is a "permanent" part of discounted cash flows in which volatility is also embedded. So the problem here is one of signal extraction[8]. Heuristically, instead of the time separable volatility implied by equation (13), we let $\tilde{\sigma}_z(t) = \sigma_z ct$ be a time varying volatility, and plug it in the solution, for expository purposes to get

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[8] Barsky and DeLong (1993) used fundamental valuation in a rational expectations framework and found that permanent and temporary components of dividend streams explains excess fluctuations in historic stock market index. So our equation (9) could also be interpreted as the present value of a dividend payment stream. In Barsky and De Long's paper investors risk aversion is reflected in their beliefs about dividend growth rates.

(16) $$\hat{r}(Z(t,\omega)) = \left(\frac{\sigma_z}{\cosh\left(\tilde{\sigma}_z(t)\right)}\right)\int_o^t \sinh\left(\tilde{\sigma}_z(s)\right) dZ(s,\omega)$$

$$= \sigma_z \operatorname{sech}\left(\tilde{\sigma}_z(t)\right)\int_0^t \sinh\left(\tilde{\sigma}_z(s)\right) e^{-Rs} H(s,\omega) ds$$

This is equivalent to imposing restrictions on a Heston (1993) type stochastic volatility model

(17) $$d\tilde{\sigma}_z(t,\omega) = \kappa\left(\mu_{\tilde{\sigma}_z} - \tilde{\sigma}_z(t,\omega)\right)dt + \eta\sqrt{\tilde{\sigma}_z(t,\omega)}\, dW(t,\omega)$$

where $\mu_{\tilde{\sigma}_z}$ is long run mean volatility, κ is the rate of mean reversion, η is a constant volatility of $\tilde{\sigma}_z(t,\omega)$, i.e., a secondary source of volatility risk, and $W(t,\omega)$ is a Brownian motion. Typically, $W(t,\omega)$ is assumed to be correlated with $V(t,\omega)$ in equation (9). That is, $E[dW(t,\omega)dV(t,\omega)] = \rho dt$. However in this paper, if we assume that $\tilde{\sigma}_z(t,\omega)$ is deterministic, i.e., $dW(t,\omega) = 0$, then equating the drift term in (17) with the time separable specification implied by (13) shows that

(18) $$\sigma_z c = \kappa\left(\mu_{\tilde{\sigma}_z} - \tilde{\sigma}_z(t,\omega)\right)$$

Which implies that for $\tilde{\sigma}_z(t,\omega) \geq 0$ we get

(18)′ $$\sigma_z c \leq \kappa\mu_{\tilde{\sigma}_z}$$

So that constant volatility σ_z is bounded by long run volatility factors comprised of the rate of volatility reversion (κ) and controls (c)on the jumps of risk aversion. This result is summarized in the following

LEMMA II.D.1 (Volatility bounds). Let $\{r(Z(t,\omega)); 0 \leq t \leq T < \infty\}$ be an Arrow-Pratt risk aversion process, and

$$dZ(t,\omega) = r(Z(t,\omega)dt + \sigma_z dV(t,\omega)$$

be an induced behavioural Markov process defined on a filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0 \le t < T < \infty}, P)$ with constant diffusion coefficient σ_z and background driving Brownian motion $V(t,\omega)$. Let $\mu_{\tilde{\sigma}_z}$ be the long run mean diffusion coefficient for $Z(t,\omega)$; κ be the rate of mean reversion, and c be the constant volatility of the AP risk aversion process. If $\tilde{\sigma}_z(t)$ is a deterministic time separable volatility measure for $Z(t,\omega)$, then

$$\tilde{\sigma}_z(t) = \mu_{\tilde{\sigma}_z} - \left(\frac{\sigma_z c}{\kappa}\right)$$

whereupon $\sigma_z \le \left(\frac{\kappa}{c}\right) \mu_{\tilde{\sigma}_z}$. $\rrbracket$

1. Identifying restrictions

In order to extend Arrow-Pratt risk measure $\hat{r}(Z(t,\omega))$ to constant volatility σ_z it must satisfy the identifying restriction

(19) $$\hat{r}(Z(t,\omega)) = \sigma_z$$

This is tantamount to asserting that *the drift term for discounted flow variables in equation (9) is constant*. Whereupon we have the identifying restriction[9]

(20) $$\operatorname{sech}\big(\tilde{\sigma}_z(t)\big) \int_0^t \sinh\big(\tilde{\sigma}_z(s)\big)\, dZ(s,\omega) = 1$$

But the law of the iterated logarithm for Brownian motion (Brieman (1968, p. 263)) states that

[9] This restriction depends on the zero drift specification of equation (8) because if $c = 0$, then $\tilde{\sigma}(t) = 0$ and we get "$0 = 1$" which is a *non sequitur*. Therefore, we must have $c \ne 0$ for a plausible restriction and for equation (19) to hold. Additionally, there is no loss of generality in specifying (8) without a constant drift term. Because Girsanov's Theorem would remove it and we would have a functionally equivalent specification.

(21) $$P - \overline{\lim}_{t\downarrow 0}\left\{\frac{V(t,\omega)}{\sqrt{2t\log\log\frac{1}{t}}} = 1\right\} = 1 \text{ a.s}$$

So that for small t, equation (16) is approximately equal to

(22) $$\tanh\tilde{\sigma}_z(t)\,.\,\Delta Z(t,\omega) = 1$$

where $\Delta Z(t,\omega) = r(Z(t,\omega))\Delta t + \sigma_z\Delta V(t,\omega)$. Thus, $\tanh\tilde{\sigma}_z(t)\,\sigma_z\Delta V(t,\omega) \leq 1$, where $\Delta V(t,\omega)$ is the modulus of Brownian motion. The modulus, $\Delta V(t,\omega)$ in equation (22) is characterized as follows

PROPOSITION II.D.1.1 (Modulus of Continuity of Brownian motion). Let $\{V(t); t \geq 0\}$ be Brownian motion, and define the following random subsets on the space of continuous functions $C[0,1]$:

(23) $$\mathfrak{F}(h) = \{V(s+h) - V(s); 0 \leq s \leq 1-h\}$$

Then

(24) $$P - \lim_{\mathrm{h}\downarrow 0}\sup\{\left(2\mathrm{h}\log(\mathrm{h}^{-1})\right)^{-\left(\frac{1}{2}\right)}\sup_{0\leq \mathrm{h}\leq 1}|V(s+h) - V(s)| = 1\} = 1$$

Further,

(25) $$P - \lim_{\mathrm{h}\downarrow 0}\left\{\left(\mathrm{h}^{-1}\log(\mathrm{h}^{-1})\right)^{\left(\frac{1}{2}\right)}\inf_{0\leq \mathrm{s}\leq 1-\mathrm{h}}\sup_{0\leq \mathrm{t}\leq \mathrm{h}}|V(t+s) - V(s)| = \frac{\pi}{\sqrt{8}}\right\} = 1$$

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<u>Proof</u>. See de Acosta (1985).

Because $\tanh\tilde{\sigma}_z(t) \approx \tilde{\sigma}_z(t) + o_p(\tilde{\sigma}^2(t))$ is small, and according to Proposition II.D.1.1, $\Delta V(t,\omega) \in \mathfrak{F}(h)$ is also a Brownian motion, see Gikhman and Skorokhod (1969), equation (22) is further reduced to

(26) $$\tilde{\sigma}_z(t)\Delta V(t,\omega) \leq 1$$

Treating $\Delta V(t,\omega)$ as a Brownian motion, and substituting the incipient time varying relation $\tilde{\sigma}_z(t) = \sigma_z ct$ in (26) reveals that the restriction is satisfied only if

(27) $$\sigma_z \leq \left(\frac{1}{ct}\right)\left(\frac{1}{\sqrt{2t \log \log \frac{1}{t}}}\right)$$

This is contrary to our initial hypothesis of constant variance because the upper bound above fluctuates with time. Therefore, constant volatility is not a measure of Arrow-Pratt risk almost surely because according to (27) the identifying restriction in (26) fails for even small values of t.

Without the identifying restrictions for constant volatility, small values of t in equation (20) and (22) implies

(28) $$\hat{r}(Z(t,\omega)) = \sigma_z\tilde{\sigma}_z(t)r(Z(t,\omega))\Delta t + \ \sigma_z\tilde{\sigma}_z(t)\Delta V(t,\omega) + o_p(\tilde{\sigma}^2(t))$$

So that for time separable $\tilde{\sigma}_z(t) = \sigma_z ct$, we have

(29) $$\hat{r}(Z(t,\omega)) = \sigma_z^2 ct\Delta Z(t,\omega) + o_p\big(\tilde{\sigma}^2(t)\big)$$

$$= \sigma_z^2 ct\, r(Z(t,\omega))\Delta t + \sigma_z^2 \Delta V(c^2t^2,\omega) + o_p\big(\tilde{\sigma}^2(t)\big)$$

where $\Delta V(t,\omega)$ is the modulus of Brownian motion. In which case Arrow-Pratt risk measure extends to volatility as indicated by the term

(29)′ $$var(\hat{r}(t,\omega) = \hat{\sigma}_z^2(t,\omega) = \sigma_z^4 var(\Delta V(c^2t^2,\omega)) = \sigma_z^4 c^2 t^2$$

2. Volatility-shock convolution and maximal Arrow-Pratt risk measure

Equation (28) suggests the existence of a convolution between time-varying volatility input and idiosyncratic shocks, i.e. the impulse from a Brownian motion.

Assume a time interval $0 \leq t \leq 1$[10], and partition the interval in equal increments Δu. Specifically we rewrite (29)′ as follows:

(30) $$\hat{\sigma}_r(t,\omega) = \tilde{\sigma}_z^2(t) * \Delta V(\sigma_z^2 t, \omega) = \int_0^1 \tilde{\sigma}_z^2(t-u)\Delta V(\sigma_z^2 u, \omega)\, du$$

Thus, by Cauchy-Schwartz inequality

(31) $$\sup_{t\in[0,T]}|\hat{\sigma}_r(t,\omega)|^2 \leq \left\{\int_0^1 \left|\tilde{\sigma}_z^2(t-u)\right|^2 du\right\}\left\{\int_0^1 \left|\Delta V(\sigma_z^2 u,\omega)\right|^2 du\right\}$$

If we use a "matched filter"[11] so that $A\tilde{\sigma}_z^2(t-u) = \Delta V(\sigma_z^2 u, \omega)$, where A is a constant, then

(32) $$\sup_{t\in[0,T]}|\hat{\sigma}_r(t,\omega)|^2 = A^{-2}\max_{\Delta u}\left\{\int_0^1 (\sigma_z^2 \Delta u)du\right\}^2$$

$$= A^{-2}\max_{0<\Delta u\leq t<1}(\sigma_z^2\Delta u)^2 = A^{-2}\sigma_z^4 t^2$$

Since the total variation $\max \Delta u = t$. Thus, depending on whether $A \lesseqgtr 1$ from (18), the maximal volatility of Arrow-Pratt risk

(33) $$\sup_{t\in[0,T]}|\hat{\sigma}_r(t,\omega)| = A^{-1}\sigma_z^2 t \leq A^{-1}\left(\frac{\mu_{\tilde{\sigma}_z}\theta}{c}\right)^2 t$$

is bounded by long run volatility factors.

III. AN INVARIANT ARROW-PRATT RISK PROCESS

In this section we introduce an invariant Arrow-Pratt risk process based on the relation posited for Arrow-Pratt risk measure in equation (4). First we begin by stating

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[10] Normalizing the interval $[0, T]$ by dividing it by T, and scaling Brownian motion accordingly permits this without loss of generality.

[11] See Gershenfeld (1999, p. 187).

the Dambis-Dubins-Schwartz time change theorem for local martingales after we provide some basic definitions.

Definition III.1 (Stopping time functional) (Revuz and Yor (1999)). Let $\mathbb{F} = \{\mathfrak{F}_t\}_{t\geq 0}$ be a right continuous filtration, and $\{A(t,\omega); 0 \leq t < \infty\}$ be an increasing right continuous process adapted to $\mathbb{F}$, i.e. $A(\cdot)$ can be derived from information in $\mathbb{F}$. Let

(34) $$T(s) = \inf\{t; A(t,\omega) > s\}$$

where $\inf\{\phi\} = +\infty$. Then $T(s)$ is a stopping time.

Definition II.2 (Time change) A time change $T$is a family $T(s)$, $s \geq 0$ of stopping times such that $s \longmapsto T(s)$ are almost surely increasing and right continuous.

LEMMA III.3. $A(t,\omega)$ is an $\mathfrak{F}_{T(s)}$ stopping time.]

Proof. See Revuz and Yor (1999, p. 180)

A. Time changed Arrow-Pratt risk processes

Now we are ready to state the following. For notational convenience we write $r(t,\omega)$ instead of $r(Z(t,\omega))$.

THEOREM III.A.1 (Dambis-Dubins-Schwartz). Let $\{r(t,\omega), \mathfrak{F}_t; 0 \leq t < \infty\}$ be a nowhere constant right continuous local martingale, and $< r >_t$ be its variation such that $\lim_{t\to\infty} < r >_t = \infty$. Let $T(s)$ be a stopping time defined by

(35) $$T(s) = \inf\{t \geq 0; < r >_t \ > s\}$$

and $\{V(s,\omega), \mathcal{G}_s; 0 \leq s < \infty\}$ be a Brownian motion defined over a filtration $\{\mathcal{G}_s\}_{s\geq 0}$. Then the time changed Brownian motion process

(36) $$V(s,\omega) = r(T(s),\omega)$$

over the filtration $\mathcal{G}_s \triangleq \mathfrak{F}_{T(s)}$ is a standard Brownian motion. Specifically, we have

(37) $$r(t,\omega) = V(<r>_t,\omega)$$ $\rrbracket$

Proof. See Karatzas and Shreve (1991, p. 174).

Specifically, to apply the Dambis-Dubins-Schwartz theorem, let

(38) $$\mathfrak{L} = \left(\frac{\partial^2}{\partial x^2}\right) + r(x)\left(\frac{\partial}{\partial x}\right) =$$

be the infinitesimal generator of an Arrow-Pratt process as indicated by equation (4). The following steps are adapted from Pedersen (2003, pp. 208-209). Let

(39) $$T(t) = 1 - e^{-2t}$$

be a time change, and

(40) $$r(t,\omega) = \frac{V(T(t),\omega)}{\sqrt{1-T(t)}} = e^t V(1-e^{-2t},\omega)$$

Further, let

(41) $$W(t,\omega) = \frac{1}{\sqrt{2}}\int_0^{T(t)} \frac{1}{\sqrt{1-u}} dV(u,\omega)$$

$$= \frac{1}{\sqrt{2}}\int_0^{t} \frac{1}{\sqrt{1-T(u)}} dV(T(u),\omega)$$

Let $f \in C^2(X)$ and consider the Ito formula for a generic Markov process

(42) $$dX(t,\omega) = \mu(X)dt + \sigma(X)dW(t,\omega)$$

starting at $X(0) = x$. Then

(43) $$f\big(X(t,\omega)\big) = f(x) + f'\big(X(t,\omega)\big)dX(t,\omega) + \frac{1}{2}f''\big(X(t,\omega)\big)\big(d\big(X(t,\omega)\big)^2$$

From which we get the infinitesimal generator

(44) $$Af = \lim_{t\downarrow 0}\frac{f\big(X(t,\omega)\big)-f(x)}{t} \equiv \mathfrak{L}f$$

assuming $f \in \mathcal{D}(A) \cap \mathcal{D}(\mathfrak{L}) \cap C^2(X)$ where $\mathcal{D}(\cdot)$ is the domain of $(\cdot)$. Thus computation shows that

(45) $$Af = \mu(X)f^{'}(x) + \frac{1}{2}f^{''}(x)(dW)^2$$

But from (39) and (41) we get

(46) $$\frac{1}{2}(dW)^2 = \frac{1}{2}\left\{\frac{1}{\sqrt{1-T(t)}}dV(T(t),\omega)\right\}^2 = \frac{1}{2}\left(\frac{dT(t)}{1-T(t)}\right) = 1$$

which upon evaluation from the relation for $Af = \mathfrak{L}f$ in (44) implies that $\mu(X) = r(x)$ and $\sigma(x) = \sqrt{2}$. Whereupon we get the Arrow-Pratt risk process

(47) $$r(Z(t,\omega)) = r(Z(t,\omega))dt + \sqrt{2}\,dW(t,\omega)$$

PROPOSITION III.A.2 (Time Changed Arrow-Pratt Processes). Let $T(t) = 1 - e^{-2t}$ be a time change process, and $\{V(t,\omega), \mathfrak{F}_t; 0 \leq t < \infty\}$ be a Brownian motion. Let $\{r(t,\omega), \mathfrak{F}_t; 0 \leq t < \infty\}$ be a nowhere constant local martingale Arrow-Pratt risk process given by the time change transformation

(48) $$r(t,\omega) = \frac{V(T(t),\omega)}{\sqrt{1-T(t)}} = e^t V(1 - e^{-2t}, \omega)$$

Then the infinitesimal generator for Arrow-Pratt risk process is

(49) $$\mathfrak{L} = \left(\frac{\partial^2}{\partial x^2}\right) + r(x)\left(\frac{\partial}{\partial x}\right)$$

in the class of Ornstein-Uhelenbeck processes.

$\rrbracket$

We will state below a theorem by Frank B. Knight (1971) as presented in Karatzas and Shreve (1991, p. 179), and then provide an application in Section IV. Before doing so, we define the following local Brownian functional for $0 \leq t < \infty$

(50) $$V^{(1)}(t,\omega) = \int_0^t \mathbb{I}_{\{W(s,\omega)\geq 0\}}\, dV(s,\omega)$$

(51) $$V^{(2)}(t,\omega) = \int_0^t \mathbb{I}_{\{W(s,\omega)< 0\}}\, dV(s,\omega)$$

Where $V^{(1)}$ and $V^{(2)}$ correspond to positive and negative shocks respectively, so that $|V| = V^{(1)} - V^{(2)}$ is the variation of V. In the sequel we will suppress the ω sample point unless otherwise needed.

THEOREM III.A.3 (Frank B. Knight (1971)). Let $r(t) = (r^{(1)}(t), r^{(2)})$ where $r^{(i)}(t)$ is a local martingale, $i = 1,2$ and $r = \{r(t), \mathfrak{F}_t; 0 \leq t < \infty\}$ be a continuous process adapted to $\mathbb{F}$ such that $\lim_{t\to\infty} < r^{(i)} >_t = \infty$ w.r.t $(\Omega, \mathfrak{F}_t, \mathbb{F}, P)$, and the cross-variation

(52) $$< r^{(1)}, r^{(2)} >_t = 0$$

Define

(53) $$T_i(s) = \inf\{t \geq 0; < r^{(i)} >_t > s\} \quad 0 \leq s < \infty$$

So that $T_i(s)$ is a stopping time for $\mathbb{F}$. Then the processes

(54) $$V^{(i)}(s,\omega) \triangleq r^{(i)}(T_i(s),\omega)$$

are independent standard Brownian motions. Additionallly

(55) $$r^{(i)}(t,\omega) = V^{(i)}\big(< r^{(i)} >_t, \omega\big)$$

for $0 \leq t < \infty$. ⟧

Proof. See Karatzas and Shreve (1991, p. 179)

This theorem implies that our Arrow-Pratt risk process can be decomposed into a process for positive and negative excursions of $W(t,\omega)$ which we address below.

B. Risk decomposition with asymmetric Ornstein-Uhlenbeck processes.

The Arrow-Pratt risk process in Proposition III.A.2, obtained from time changed transformation of Brownian motion in (48), is an Ornstein-Uhlenbeck process. See Karatzas and Shreve (1991, p. 354). However, according to Theorem III.A.1 that process is decomposed according to whether the background driving Brownian motion is in the positive or negative quadrant. Additionally, negative Arrow-Pratt risk measure is associated with risk seeking behavior while risk aversion is associated with a positive measure. So that starting from a risk free position $r\big(Z(t,\omega)\big) = 0$ positive excursions represent risk aversion, and negative excursions represent risk seeking. Thus

(56) $$\mathit{risk\ averse}\text{:}\ r^{(1)}(t,\omega) = r^{(1)}(t,\omega)dt + \sqrt{2}V^{(1)}(t,\omega)$$

Where $r^{(1)}(t) \geq 0, V^{(1)}(t,\omega) \geq 0$, and

(57) $$\mathit{risk\ seeking}\text{:}\ r^{(2)}(t,\omega) = r^{(2)}(t,\omega)dt + \sqrt{2}V^{(2)}(t,\omega)$$

and $r^{(2)}(t) \leq 0,\ V^{(2)}(t,\omega) \leq 0$

Asuming the strong Markov property, for $0 \leq s \leq t$ the solution for each of those Ornstein-Uhlenbeck processes, starting afresh at some initial point $r(t,\omega) = r_0$ are:

(58) $$r^{(1)}(t,\omega) = r_0^{(1)}e^{t} + \sqrt{2}\int_0^t e^{(t-s)}\,dV^{(1)}(s,\omega)$$

(59) $$r^{(2)}(t,\omega) = r_0^{(2)}e^{-t} + \sqrt{2}\int_0^t e^{-(t-s)}\,dV^{(2)}(s,\omega)$$

Arrow-Pratt risk process is recovered from superposition of these two asymmetric equations. So that for $r_0^{(1)} = r_0^{(2)} = r_0$ upon superposition

(60) $$r(t,\omega) = r^{(1)}(t,\omega) + r^{(2)}(t,\omega)$$

$$= r_0^{(1)} e^t + r_0^{(2)} e^{-t} + \sqrt{2} \int_0^t \left(e^{(t-s)} dV^{(1)}(s,\omega) + e^{-(t-s)} dV^{(2)}(s,\omega) \right)$$

$$= r_0 \sinh t + \sqrt{2} \int_0^t \left(\left(dV^{(1)}(s,\omega) - dV^{(2)}(s,\omega) \right) \sinh(t-s) \right.$$

$$\left. + \left(dV^{(1)}(s,\omega) + dV^{(2)}(s,\omega) \right) \cosh(t-s) \right) ds$$

$$= \quad \sinh t$$

$$+ \sqrt{2} \int_0^t \left(\sinh(t-s) \left(dV^{(1)}(s,\omega) - dV^{(2)}(s,\omega) \right) + \cosh(t-s) dV(s,\omega) \right) ds$$

$$= \quad r_0 \sinh t + \sqrt{2} \int_0^t \sinh(t-s) \left(dV^{(1)}(s,\omega) - dV^{(2)}(s,\omega) \right) ds$$

$$+ r_0 \cosh t + \sqrt{2} \int_0^t \cosh(t-s) dV(s,\omega)$$

By an abuse of notation, for AP risk process moving away from the origin in a disk B_ϵ centered at the origin, i.e., risk free state, with radius $\epsilon > 0$ let

(60) $$\tilde{r}_{B_\epsilon, \epsilon \uparrow 0}(t,\omega) = \quad r_0 \sinh t$$

$$+ \sqrt{2} \int_0^t \sinh(t-s) \left(dV^{(1)}(s,\omega) - dV^{(2)}(s,\omega) \right) ds$$

It should be noted that $|dV| = \; dV^{(1)} - dV^{(2)}$ is the variation of dV. So we can rewite (60) as

(60)′ $$\tilde{r}_{B_\epsilon, \epsilon \uparrow 0}(t,\omega) = \quad r_0 \sinh t \; + \sqrt{2} \int_0^t \sinh(t-s) \, |dV| \, ds$$

For AP risk near the origin, heading towards the origin, let

(61) $$\tilde{r}_{B_\epsilon, \epsilon \downarrow 0}(t,\omega) = r_0 \cosh t + \sqrt{2} \int_0^t \cosh(t-s) dV(s,\omega)$$

So that for AP processes $\tilde{r}_{B_\epsilon}(t,\omega)$ in a disk $B_\epsilon$centered at the origin, we have

(62) $$\tilde{r}_{B_\epsilon}(t,\omega) = \tilde{r}_{B_\epsilon,\epsilon\uparrow 0}(t,\omega) + \tilde{r}_{B_\epsilon,\epsilon\downarrow 0}(t,\omega)$$

1. Local time behavior of AP risk processes and random time change of Clark's (1973) subordinate process

It is clear that there are two different high frequency dynamics at work here by virtue of the pseudo sinusoidal representations for the components of AP risk process. One dynamic ($\tilde{r}_{B_\epsilon,\epsilon\downarrow 0}(t,\omega)$) is pulling towards the "origin"[12], and the other is pulling away ($\tilde{r}_{B_\epsilon,\epsilon\uparrow 0}(t,\omega)$). Thus, we have just constructed the local time behavior of AP risk processes in an ϵ-disk, and shown that risk aversion and risk seeking run on different time clocks near the origin. In equation (60), the quantity $r_0 \sinh t$ is increasing in time away from the origin, and it is zero at the origin. It approaches 0 from the left. By contrast, in equation (61) the process converges to r_0 when approached from the right since it is right continuous. This asymmetric result, derived from Arrow-Pratt risk measure for von Neuman Morgenstern utility functions, predicts Tversky and Khaneman's (1992) asymmetric value function loss aversion result--relative to a reference point, i.e. "origin". It should be noted in passing that for $t = \infty$ risk aversion is infinite while risk seeking is zero. That is, in the long run people are risk averse.

An important paper by Clark (1973) opined:

> [T]he hypothesis presented and tested in this paper is that the distribution of price change is subordinate to a normal distribution. The price series for cotton futures evolves at different rates during identical rates of time. The number of individual effects added together to give price change

[12] Under the strong Markov property the "origin" is the starting point of the risk process, and does not necessarily mean the coordinate point 0.

during a day is variable and in fact random, making the Central Limit Theorem inapplicable. *Id* at 137.

In the context of our theory, it appears that depending on whether a 'critical mass" of futures traders were risk averse (bearish) or risk seeking (bullish), the weighted average "added effects" in (60) and (61) also renders the Central Limit Theory inapplicable. That is, if there are a total of $N = N_{ra} + N_{rs}$ futures traders, N_{ra} of whom are bearish and N_{rs} bullish, then the AP risk process generated by them is $N_{ra} r^{(1)}(t,\omega) + N_{rs} r^{(2)}(t,\omega)$. If the N_{ra} and N_{rs} are modeled after an Ehrenfest urn scheme in which traders go in and out of bull or bear positions, with given probabilities, and there are no new entrants or exits from the market, then one can see Clark's empirical results at work. Because the limiting value of this set-up, for traders moving in and out of bull or bear positions with given probabilities, is a stochastic process. See e.g., Kac (1947, 374-375). Similarly, Carr and Wu (2004) used a time changed Levy process to address the subordinated processes problem identified by Clark (1973). Daniel and Foster (1994, pp. 10-11) provide excellent heuristics on the efficacy of change in time scales. In fact, they also derive Ornstein-Uhlenbeck process representation for unobservable variance in a filtering scheme for ARCH models. Our theory is distinguished because it employs microfoundations of utility theory in lieu of adaptive data analysis for its results--in the context of AP risk processes.

We summarize the foregoing with the following propositions and theorems.

DEFINITION III.B.1 (Local time). Let $B_\epsilon(a)$ be a disk of radius ϵ centered at a point $a \in [0,\infty)$, and

(63) $$\Lambda(t,a,\omega) = \lim_{2\epsilon\downarrow 0} meas\{0 \le s \le t; |r(s,\omega) - a| \le \epsilon\}$$

where "meas" is the measure of the time spent in the vicinity of a. Then $\Lambda(t,a,\omega)$ is the local time of $r(s,\omega)$ at a.

THEOREM III.B.2 (Karatzas and Shreve 1991, p. 218). Let $r(t,\omega)$ be an AP risk process on $(\Omega, \mathfrak{F}, \mathbb{F}, \mathrm{P})$. There exist a $\mathfrak{F}_t$-measureable local time for AP risk measure, i.e., a random field

$$\Lambda = \{\Lambda(t,a,\omega); (t,a) \in [0,\infty) \times \mathbb{R}, \qquad \omega \in \Omega$$

such that

1. $\int_0^\infty \mathbb{I}_{\{a\}^c}\big(r(t,\omega)\big) d\Lambda(t,a,\omega) = 0$
2. For each Borel function $k: \mathbb{R} \to [0,\infty)$, and local continuous martingale component $M(s,\omega)$ of $r(s,\omega)$

$$\int_0^t k\big(r(s,\omega)\big) d <M>_s = 2\int_{-\infty}^{\infty} k(a)\Lambda(t,a,\omega)\, da$$

3. Λ is jointly continuous in t and right continuous left hand limit in a so that

$$\lim_{\substack{\tau\to t \\ b\downarrow a}} \Lambda(t,b,\omega) = \Lambda(t,a,\omega)$$

$$\lim_{\substack{\tau\to t \\ b\uparrow a}} \Lambda(t,b,\omega) = \Lambda(t,a^-,\omega)$$

$\rrbracket$

Proof. See Karatzas and Shreve (1991, p. 218).

LEMMA III.B.3 The local time representation of AP risk process is given by

$$\lim_{\epsilon\to 0} \tilde{r}_{B_\epsilon}(t,\omega) = \lim_{\epsilon\uparrow 0} \tilde{r}_{B_\epsilon}(t,\omega) + \lim_{\epsilon\downarrow 0} \tilde{r}_{B_\epsilon}(t,\omega)$$

$$\tilde{r}(\Lambda,\omega) = \tilde{r}(\Lambda^-,\omega) + \tilde{r}(\Lambda^+,\omega)$$

Proof. See equations (60)-(62) and Theorem III.B.2

THEOREM III.B.4. (Asymmetric AP representation theorem). Let $\{V(t,\omega); 0 \leq t < \infty\}$ be a background driving Brownian motion, and $|V|$ be the total variation of V. Let $r^{(1)}(t,\omega)$ and $r^{(2)}(t,\omega)$ be risk aversion and risk seeking components of an AP risk process $\{r(t,\omega), \mathfrak{F}_t; 0 \leq t < \infty\}$ defined on a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{F}, P)$. Let B_ϵ be an ϵ -disk centered at the origin. Then the equations of motion for risk seeking, and risk averting in B_ϵ are given by, respectively

$$r^{(1)}(t,\omega) = r_0 \sinh t + \sqrt{2}\int_0^t \sinh(t-s)\,|dV|\,ds$$

$$r^{(2)}(t,\omega) = r_0 \cosh t + \sqrt{2}\int_0^t \cosh(t-s) dV(s,\omega)$$

⟧

IV. APPLICATIONS

A. Hyper-ARCH risk aversion and specification error in Engle, Ng and Rothschild (1990) Factor-ARCH

The heteroskedasticity correction in (29) implies

(64) $$\left(ct\sqrt{\Delta t}\right)^{-1} r(t,\omega) = \sigma_z^2 B(1,\omega) + o_p\left(\left(ct\sqrt{\Delta t}\right)^{-1}\tilde{\sigma}(t)\right)$$

This suggests that Arrow-Pratt risk measure has a hyper-ARCH representation because even after applying the heteroskedasticity correction displayed on the left hand side of (64), the right hand side is still a random variable $\sigma_z^2 B(1,\omega)$ Thus, AP risk measure is unstable and it has an ARCH process of its own. To see this, since Brownian motion $B(\sigma_z^2, \omega) \sim N(0, \sigma_z^2)$ can be scaled as $\sigma_z B(1, \omega)$ let $\epsilon_t(\omega) \sim N(0, \sigma_z^2)$, then

(65) $$\left(ct\sqrt{\Delta t}\right)^{-1} r(t, \omega) = \sigma_z \epsilon_t(\omega) + o_p\left(\left(ct\sqrt{\Delta t}\right)^{-1} \tilde{\sigma}(t)\right)$$

and the term $\sigma_z \epsilon_t(\omega)$ is Engle's ARCH specification. Chou, et al (1992, p. 207) confirmed hyper-ARCH risk aversion in excess stock index returns, and proposed an ARCH-M system equation with Kalman filter approach to address the problem. Therefore, the theoretical results presented here are robust.

More on point, Engle, Ng and Rothschild (1990) ("ENR-90") posited a k-Factor-ARCH relationship for risk premium with the generic cross-sectional equation

(66) $$\pi(t, \omega) = \alpha + \sum_k \beta_k \sigma_k^2(t)$$

In that case the $\beta_k's$ are estimates of Arrow-Pratt risk measure in the context of Pratt's risk premium equation (1). There, in ENG, a signal extraction procedure, based on factor analysis methodology popularized by psychometricans, was used to obtain estimates for constant $\beta's$. But according to (29) and (65) the $\beta_k's$ are time varying and unstable. So the ENG-90 model in equation (66) is misspecified. Thus, an auxillary regression is required to find a more accurate "price of risk premium" per unit of volatility. In fact, equation (57) suggests that a heteroskedasticity correction factor should be applied to the ENG cross-sectional equation in order to get the "true" price of risk for risk premia.

B. Arrow-Pratt risk aversion as the cause of volatility induced by analysts beliefs about interest rates and shocks to cash flows streams

Persistence in earnings growth in equation (3) depends on changes in interest rates (R), volatility (σ_z) and idiosyncratic shocks $V(t, \omega)$ to earnings. If analysts believe that interest rates will increase, then, other things equal, they become more risk averse. For instant, $H(t, \omega)$ may be the cash flow stream from a bond—in the absence of default risk here. Similarly, with high market volatility σ_z, other things equal, analysts become more uncertain and hence more risk averse. Other things constant, negative shocks in equation (9) tend to increase AP risk measure, and consequently volatility in equation (27). For instance, it is clear that Arrow-Pratt risk measure $r(t, \omega)$ is adapted to the augmented filtration of cash flows $\sigma\{H(t, \omega); 0 \leq t < \infty\} \cup \mathbb{F}$, and that it is a feedback control for the sample path $\omega(t)$ realized in a coordinate mapping with $H(t, \omega)$. Hence increased risk aversion $r(t, \omega) \uparrow$ due to negative shocks to $V(t, \omega)$ causes more volatility in $H(t, \omega)$ through $Z(t, \omega)$, and vice versa. If our model is properly specified, then $\hat{r}(t, \omega) \in \Lambda_{\mathfrak{L}} = \{\lambda_1(r(x)), \lambda_2(r(x)), \ldots\}$. That is, the distribution of Arrow-Pratt risk aversion coincides with the spectral distribution on $\Lambda_{\mathfrak{L}}$--the spectrum of the infinitesimal generator

(67) $$\mathfrak{L} = \left(\frac{\partial^2}{\partial x^2}\right) + r(x)\left(\frac{\partial}{\partial x}\right)$$

It should be noted in passing that $\mathfrak{L}$ is compact because the risk free measure $0 \in \Lambda_{\mathfrak{L}}$ is a limit point for recovering utility from $r(x)$. Thus $\{u_n\}_{n=1}^{\infty}$ is a sequence of approximating utility functions whose limit is u, where $\mathfrak{L}u_n = \lambda_n u_n$.

This spectral relationship coincides with the one posited by Engle, Ng and Rothschild (1990) only if $\mathfrak{L}$ can be represented by a function of their covariance matrix H_t. The system of equations (7)-(9) can be easily modified to accomodate, inter alia, interest rate and volatility dynamics. For example, as indicated by equation (17) σ_z can be parametrized by a Heston (1993) type stochastic volatility model with mean reversion. Additionally, term structure equations[13] could be added when the cash flow stream pertains to bond payments. However, formal inclusion of those equations in this paper would complicate our erstwhile simple system and detract from our narrow focus on the interface between volatility and embedded Arrow-Pratt risk. For a non-technical overview of risk management in financial markets the interested reader is directed to "New Directions for Understanding Risk," A Report of a Conference Cosponsored by the Federal Reserve Bank of New York and the National Academy of Sciences, May 18-19, 2006 in Economic Policy Review, 13(2) (November 2007)

C. Controlled Arrow-Pratt risk processes for negative and positive shocks

On the basis of Theorem III.5 let $r(t,\omega) = r^{(1)}(t,\omega) + r^{(2)}(t,\omega)$ be a decomposition of Arrow-Pratt risk process where $r^{(1)}(t)$ is the response to positive shocks and $r^{(2)}(t)$ the commensurate response to negative shocks on $(\Omega, \mathfrak{F}_t, \mathbb{F}, P)$. Then we can write

(68) $$dr^{(i)}(t,\omega) = r^{(i)}(t,\omega)dt + \sqrt{2}\, dV^{(i)}(t,\omega)$$

for $i = 1,2$. By definition $< V^{(1)} >_t + < V^{(2)} >_t = t$, so that

(69) $$< V^{(2)} >_t = t - < V^{(1)} >_t$$

and $r^{(i)}(t) = V^{(i)}(< r^{(i)} >_t)$. Then for negative shocks, the Arrow-Pratt risk aversion process runs on a different clock. That is,

[13] The seminal papers of Vasicek (1977), Cox, Ingersoll, and Ross (1985); and Heath, Jarrow and Morton (1992) provide ample evidence of the complexity in modelling interest rate behavior.

(70) $$r^{(2)}(t) = V^{(2)}\big(t - < V^{(1)} >_t\big) = V^{(2)}(< V^{(2)} >_t)$$

So that $< r^{(2)}(t) >_t = t - < V^{(1)} >_t)$ is the variation of Arrow-Pratt risk process for negative shocks. Intuitively, this means that notwithstanding symmetry in Brownian motion, the positive and negative excursions of AP risk process are asymmetric. This result is consistent with Tversky and Khaneman (1992) loss aversion results.

More on point, heuristically, we implement the oft cited example of Benes, Shepp and Witsenhausen (1980)[14] reported in Karatzas and Shreve (1991, pp. 438). In this case, we define a risk free asset as one in which $r(t,\omega) = 0$[15] and using the Karatzas-Shreve notation we consider the interval $\theta_0 < 0 < \theta_1$ and $r(t,\omega) \in [\theta_0, \theta_1]$. An investor wants to minimize the Arrow-Pratt risk associated with the asset--which is discounted exponentially at rate R. Therefore, she wants to solve the following control problem for the risk process starting at $r(0,\omega) = x$

(71) $$J(x; r^*) = \min_{r \in \mathcal{U}} \int_0^\infty e^{-Rs} r(s)\,ds$$

for all x. Where $\mathcal{U}$ is a set of admissible Arrow-Pratt risk measures. Karatzas and Shreve (1991, p.438) show that the optimal drift for risk aversion is given by

(72) $$r^*(x) = \begin{cases} \theta_1; & x \le \delta \\ \theta_0; & x > \delta \end{cases}$$

where

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[14] Those authors solved an array of control problems outside the scope of this paper.

[15] Recall that we write $r(t,\omega)$ for $r(X(t,\omega)$ for notational convience.

(73) $$\delta = \frac{1}{\sqrt{\theta_1^2 + 2R} + \theta_1} - \frac{1}{\sqrt{\theta_0^2 + 2R} - \theta_0}$$

Since negative Arrow-Pratt risk measure connotes risk seeking, a conservative investor would like to keep her risks below δ which is a function of interest rate and boundary values. However, being too conservative could diminish returns. So she would like to control risk so that it stays as close as possible to the critical value of $r^* = \delta$. Karatzas and Shreve (1991) also show how to compute transition probabilities for $r(t,\omega)$ in this context. The literature on controlled diffusion is both rick and large. In order not to overload this paper, we direct the reader to Krylov (1980). Perhaps most important, by introducing a separate stochastic process for Arrow-Pratt risk measure, we hope to provide a richer solution space for problems involving risk and uncertainty.

D. Stochastic Differential Utility Functional

We mention in passing that since utility functions can be recovered from AP risk measure, from definition we have

(74) $$r(x) = \frac{d}{dx}\log\left(u'(x)\right)$$

Whereupon we get

(75) $$u(x) = u(x_0)e^{-R(x_0)}\int_{x_0}^{x} e^{\int_0^w r(v)dv}\, dw$$

And $R(x_0) = \int_0^{x_0} r(x)dx$ for the "process" starting at x_0. Recall that $Z(t,\omega)$ is a Markov process and that $r(Z(t,\omega))$ is the associated AP risk. In that set-up, "stochastic differential utility" is obtained by

(76) $$u\big(Z(t,\omega)\big) = u(z_0)e^{-R(z_0)}\int_{z_0}^{Z(t,\omega)} e^{\int_0^w r(v)dv}\, dw$$

For the process starting at $Z(t_0, \omega) = z_0$ at time t_0. Equation (68) is a differentiable stochastic utility functional representation of the process $\{Z(t, \omega), \mathfrak{F}_t; 0 \leq t < \infty\}$.

Assuming $u(z_0) = 0$, the Ito expansion of (76) is given by

$$u\big(Z(t,\omega)\big) = u_z dZ(t,\omega) + \frac{1}{2} u_{zz} \big(dZ(t,\omega)\big)^2 \tag{77}$$

$$= (u_z r(Z(t,\omega) + u_{zz})dt + \sqrt{2} u_z dV(t,\omega)$$

where u_z and u_{zz} are the first and second derivatives of u in an Ito expansion. This result compares favourably to Duffie and Epstein (1992, p. 415) who characterized stochastic differential utility by the equation

$$dV_t = \left[-f(c_t, V_t) - \frac{1}{2} A(V_t) \|\sigma_V(t)\|^2\right] dt + \sigma_V(t) dB_t \tag{78}$$

Where V_t is a discounted recursive utility, c_t is consumption, $f(\cdot)$ is an aggregator function, $A(V_t)$ is Arrow-Pratt risk measure and $\sigma_V(t)$ is the volatility of V_t. In fact, a simple equation of coefficients between (77) and (78) establishes functional equivalence[16]

[16] It should be noted that V_t is recursive, i.e., discounted in a Bellman type equation while $Z(t, \omega)$ is discounted and non-recursive. It would be interesting to see if the utility of a discounted Markov process is functionally equivalent in a probabilistic sense to recursive utility of the same Markov process employed in Bellman's program.

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