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Abstract We show that the introduction of unfunded public pensions in a Cobb-Douglas economy with overlapping generations and endogenous fertility may cause complex economic cycles when individuals are short-sighted. In particular, the risk of cyclical instability increases with both the individual degree of thriftiness and the relative weight of individual fertility in the pension system. Our results provide a possible explanation of the occurrence of persistent cycles in an overlapping generations context and represent a policy warning about the dramatic destabilising effects of a fertility-related pension reform.

Keywords Endogenous fertility; Fertility-related pensions; Myopic foresight; OLG model

JEL Classification C62; H55; J14; J18; J26

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1. Introduction

Social security is a pillar of the welfare state in several developed countries, and is essentially based on pay-as-you-go (PAYG) public pensions, i.e. current workers finance benefits to current pensioners. The fertility crisis that has affected and indeed still affects a lot of countries around the world (e.g., Germany, Italy, Japan and Spain) is threatening the viability of public pension budgets, as the number of young contributors is steadily falling and the number of old beneficiaries is steadily rising (due to also the reduced adult mortality). Motivated by the thrift of both ageing and below-replacement fertility on the existence of the widespread PAYG systems, pension reforms are currently high on the political agendas of many governments, especially in Europe (see, e.g., Boeri et al., 2001, 2002; Blinder and Krueger, 2004).

As a remedy against the potential negative effects of the fertility crisis on PAYG pensions, it has been suggested, amongst other things, to incentive families to have more children in order to increase the ratio of economically active to total population, for instance through the public provision of child allowances (van Groezen et al, 2003; van Groezen and Meijdam, 2008). Moreover, linking the size of the pension arrangement received when by the old-aged to the number of children raised when young may be another interesting instrument that might be used to promote the fertility recovery as well as for optimality purposes (see, Kolmar, 1997; Abio et al, 2004; Fenge and Meier, 2005, 2009).

While a growing body of literature on the relationship between pensions, fertility, longevity and economic growth has been developed in the last decades (see, amongst many others, Zhang et al., 2001, 2003; Pecchenino and Pollard, 2005), less attention has been paid to the dynamical effects of public PAYG pensions in an economy with overlapping generations (OLG) and endogenous fertility.

As is known, cyclical behaviour can occur in many-good OLG models (Grandmont, 1985) as well as in the one-good Diamond-type OLG context (Farmer, 1986; Reichlin, 1986), but only when production factors are relatively complement. Moreover, with myopic foresight, the steady state may be oscillatory and exhibit deterministic complex cycles (Michel and de la Croix, 2000, de la Croix and Michel, 2002; Fanti and Spataro, 2008), but only when the inter-temporal elasticity of substitution in the utility function is higher than unity (i.e., higher than in the case of Cobb-Douglas preferences).

The aim of this paper is to provide a deeper understanding of the stability effects of public PAYG pensions in a textbook OLG economy (e.g. Diamond, 1965) when fertility is endogenous and utility and production functions are Cobb-Douglas. It is show that when individuals are short-sighted, the introduction of a fertility-related component in the pension formula may have dramatic destabilising effects and deterministic chaos appears even for very small-sized PAYG schemes. In such a case, in fact, the relative weight of the public pensions in capital accumulation is higher than in the case of a pure PAYG scheme. Fertility-related pensions, therefore, act as an economic de-stabiliser in overlapping generations economies.

The remainder of the paper is organised as follows. In section 2 we develop the model. In section 3 the dynamical features are analysed and discussed. Section 4 concludes.

2. The model

2.1. Government

The government redistributes across generations with PAYG transfers from the young to the old that are partially or totally linked to the number of children raised when young. At time t, therefore, current workers finance pensions to current pensioners, and the fertility-related pay-as-you-go (FR-PAYG henceforth) pension accounting rule in per worker terms reads as

$$p_t = \theta w_t \cdot \left[(1-\omega)\overline{n}_{t-1} + \omega n_{t-1} \right], \tag{1}$$

the left-hand side (p_t) being the pension expenditure and the right-hand side the tax receipts. In particular, w_t is the wage earned by the young workers at time t, $0 < \theta < 1$ is the fixed contribution rate and $0 \le \alpha \le 1$ is a weighting parameter of the different distribution rules for total contribution to PAYG pensions. In particular, it measures the importance of the individual number of children relative to the average number of children in the PAYG system (see, for instance, Kolmar, 1997; Abio et al., 2004; Fenge and Meier, 2005, 2009; Fenge and von Weizsäcker, 2010). The polar cases $\alpha = 0$ and $\alpha = 1$ imply a pure PAYG scheme and a PAYG scheme totally linked to individual fertility, respectively. Therefore, Eq. (1) shows that at time t PAYG pensions depend on (*i*) the individual rate of fertility at time $t-1$, n_{t-1}, with a share α of the contribution, and (*ii*) the average rate of fertility in the whole economy at time $t-1$, $\bar{n}_{t-1}$, with a share $1-\alpha$ of the contribution. Following Fenge and Meier (2005, p. 34), we define the policy variable α "the child factor".

2.2. Individuals

Consider an overlapping generations (OLG) economy populated by identical individuals. Life is divided into childhood and adulthood. In the former period each individual does not make economic decisions. In the latter period she works and bears children when young and she is retired when old.

Only young individuals (of measure N_t) join the workforce. They are endowed with one unit of time supplied inelastically on the labour market, while receiving a unitary wage income at the competitive rate w_t. This income is used to consume, to save, to bear children and to finance material consumption of the elderly through the public pension scheme Eq. (1). Raising children is costly, and the amount of resources that parents need to take care of them is given by a monetary cost $q\,w_t$ per child, with $0 < q < 1$ being the percentage of child-rearing cost on working income.[1] Therefore, the budget constraint faced by an individual of the young (child bearing) generation at t reads as:

$$c_{1,t} + s_t + q\,w_t n_t = w_t (1-\theta), \tag{2}$$

i.e. wage income – net of contributions paid to transfer resources from work time to retirement time – is divided into material consumption when young, $c_{1,t}$, savings, s_t, and the cost of bearing children, $q\,w_t n_t$.

Old individuals are retired and live with the amount of resources saved when young plus the expected interests accrued at the rate $r^e{}_{t+1}$ and the expected public pension benefit $p^e{}_{t+1}$. At time $t+1$, therefore, the budget constraint of an old retired person started working at t is:

$$c_{2,t+1} = \left(1 + r^e{}_{t+1}\right)s_t + \theta\, w^e{}_{t+1} \cdot \left[(1-\omega)\overline{n}_t + \omega n_t\right], \tag{3}$$

i.e. material consumption when old, $c_{2,t+1}$, is the sum of private savings plus the expected interest and the expected public pension benefit.

Each adult individual of generation t draws utility from young-aged consumption ($c_{1,t}$), old-aged consumption ($c_{2,t+1}$) and the number of children she wishes to raise (n_t).[2] Assuming logarithmic preferences, the representative individual entering the working period at t solves the following problem:

$$\max_{\{c_{1,t}, c_{2,t+1}, n_t\}} U_t\left(c_{1,t}, c_{2,t+1}, n_t\right) = \ln\left(c_{1,t}\right) + \beta \ln\left(c_{2,t+1}\right) + \phi \ln\left(n_t\right), \tag{4}$$

subject to Eqs. (2) and (3), where $0 < \beta < 1$ is the subjective discount factor or, alternatively, the individual relative degree of thriftiness, and $0 < \phi < 1$ captures the parents' taste for children.

The first order conditions for an interior solution are given by:

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[1] This child cost structure is similar to that adopted by, amongst many others, Wigger (1999) and Boldrin and Jones (2002).

[2] See Eckstein and Wolpin (1985) and Galor and Weil (1996).

$$\frac{c_{2,t+1}}{c_{1,t}} \cdot \frac{1}{\beta} = 1 + r^e_{t+1}, \tag{5}$$

$$\frac{c_{1,t}}{n_t} \cdot \phi = q\, w_t - \omega\, \theta \frac{w^e_{t+1}}{1 + r^e_{t+1}}. \tag{6}$$

Eq. (5) equates the marginal rate of substitution between working period consumption and retirement period consumption to their relative prices (i.e. the expected interest rate determined on the capital market). Eq. (6) equates the marginal rate of substitution between working period consumption and the number of children to the expected marginal cost of raising an extra child. This cost is given by the difference between the monetary cost of bearing an additional child and the present value of the expected pension benefit weighted by the child factor. The higher the child factor, the lower the expected net marginal cost of raising an extra child. If $\omega = 0$ (pure PAYG pensions), the cost of child rearing is only determined as a share of the working income. In contrast, if $0 < \omega \leq 1$ (FR-PAYG pensions), a positive inter-generational effect exists that causes a reduction in the gross monetary cost of children due to the higher benefit received by each pensioner, i.e. individuals want to substitute young-aged consumption with children.

Now, combining Eqs. (5) and (6) with the individual lifetime budget constraint gives the demand for children and the saving rate, respectively:

$$n_t = \frac{\phi\, w_t (1 - \theta)}{(1 + \beta + \phi) q\, w_t - [(1 + \beta)\omega + \phi]\theta \frac{w^e_{t+1}}{1 + r^e_{t+1}}}, \tag{7}$$

$$s_t = \frac{w_t (1 - \theta)}{(1 + \beta + \phi) q\, w_t - [(1 + \beta)\omega + \phi]\theta \frac{w^e_{t+1}}{1 + r^e_{t+1}}} \left[\beta\, q\, w_t - (\beta\, \omega + \phi)\theta \frac{w^e_{t+1}}{1 + r^e_{t+1}} \right]. \tag{8}$$

Eq. (7) determines the individual number of children in a partial equilibrium context. A rise in the child factor causes a positive inter-generational effect that reduces the marginal cost of child bearing and thus increases fertility ($\partial n_t / \partial \omega > 0$). Eq. (8), instead, determines the saving rate in a partial equilibrium context. It reveals that the child factor plays a twofold counterbalancing role: (*a*) it reduces the saving rate because individuals will expect a higher pension benefit as long as the number of their descendant raises (i.e. the expected *public pension component* – the second term in square brackets of Eq. 8 – increases, while keeping the *private saving component* unaffected – the first term in square brackets of Eq. 8), and (*b*) it increases the saving rate since a higher child factor makes more convenient to substitute young-aged consumption with children at time t (i.e. reduces the denominator of Eq. 8). However, the final (partial equilibrium) effect of a rise in the child factor on savings is negative ($\partial s_t / \partial \omega < 0$), that is the positive saving-effect (*b*) is always dominated by the negative saving-effect (*a*).

2.3. Firms

Firms are identical and act competitively on the market. Aggregate production at time t (Y_t) takes place by combining capital (K_t) and labour ($L_t = N_t$ in equilibrium) according to the constant returns to scale Cobb-Douglas technology $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where $A > 0$ is a scale parameter and $0 < \alpha < 1$ is the output elasticity of capital. Defining $k_t := K_t / N_t$ and $y_t := Y_t / N_t$ as capital and output per worker, respectively, the intensive form production function may be written as $y_t = Ak_t^{\alpha}$. Assuming total depreciation of capital at the end of each period and normalising the price of final output to unity, profit maximisation implies that factor inputs are paid their marginal products, that is:

$$r_t = \alpha A k_t^{\alpha-1} - 1, \quad (9)$$

$$w_t = (1-\alpha) A k_t^{\alpha}. \quad (10)$$

2.4. Equilibrium

Given the government budget Eq. (1) and knowing that population evolves according to $N_{t+1} = n_t N_t$, market-clearing in goods and capital markets is expressed (in per worker terms) as

$$n_t k_{t+1} = s_t \,. \tag{11}$$

Using Eqs. (7) and (8) to substitute out for n_t and s_t, respectively, equilibrium implies:

$$k_{t+1} = \frac{\beta}{\phi} q w_t - \frac{\beta\omega + \phi}{\phi} \theta \frac{w^e{}_{t+1}}{1 + r^e{}_{t+1}} \,. \tag{12}$$

Eq. (12) shows that the equilibrium stock of capital at $t+1$ is determined as the difference between the private saving component and the expected public pension component at t, both divided by the taste or children. The former (the first addendum on the right-hand side of Eq. 12) exclusively depends on the willingness to save out of wage income – given the assumption of Cobb-Douglas preferences. The latter (the second addendum on the right-hand side of Eq. 12) depends on the expected values of both the wage and interest rates.

The existence of a fertility-related component in the PAYG system ($0 < \omega \le 1$) has two important effects on capital accumulation: first, it makes the crowding out effect of public pensions on private savings much stronger than the case of pure PAYG pensions ($\omega = 0$); second, it makes the individual degree of thriftiness (β) as a potential destabilising parameter. A rise in degree of parsimony, in fact, increases both the private saving component and the public pension component and, hence, its final effect on capital accumulation may be ambiguous.

Below we study how the dynamic path of capital accumulation evolves depending on whether individuals have either perfect or myopic expectations.

2.4.1. Perfect foresight

With perfect foresight, the expected interest and wage rates depend on the future value of the per worker stock of capital, that is

$$\begin{cases} 1 + r^e{}_{t+1} = \alpha A k_{t+1}{}^{\alpha-1} \\ w^e{}_{t+1} = (1-\alpha) A k_{t+1}{}^{\alpha} \end{cases}. \tag{13}$$

Combining Eqs. (9), (10), (12) and (13), the dynamic equilibrium sequence of capital can be written as

$$k_{t+1} = \frac{q\beta\alpha(1-\alpha)A}{\alpha\phi + \theta(1-\alpha)(\beta\omega + \phi)} \cdot k_t^{\alpha}. \tag{14}$$

Steady-state implies $k_{t+1} = k_t = k^*$, so that:

$$k^* = \left[\frac{q\beta\alpha(1-\alpha)A}{\alpha\phi + \theta(1-\alpha)(\beta\omega + \phi)}\right]^{\frac{1}{1-\alpha}}. \tag{15}$$

2.4.2. Myopic foresight

With myopic foresight, the expected interest and wage rates depend on the current value of the per worker stock of capital, that is

$$\begin{cases} 1 + r^e{}_{t+1} = \alpha A k_t{}^{\alpha-1} \\ w^e{}_{t+1} = (1-\alpha) A k_t{}^{\alpha} \end{cases}. \tag{16}$$

Combining Eqs. (9), (10), (12) and (16), the dynamic path of capital accumulation is now given by:

$$k_{t+1} = \frac{\beta}{\phi} q(1-\alpha) A k_t^{\alpha} - \theta \cdot \frac{\beta\omega + \phi}{\phi} \cdot \frac{1-\alpha}{\alpha} k_t. \tag{17}$$

The steady-state is still determined by Eq. (15), see Michel and De La Croix (2000).

Despite Eq. (17) is a simple first order non-linear difference equation, the dynamics of capital may be highly non-linear and endogenous fluctuations may emerge. The local stability properties of a double Cobb-Douglas economy with endogenous fertility, FR-PAYG pensions and myopic expectations are analysed in the next section.[3]

3. Local stability with myopic expectations

From Eqs. (15) and (17), the following proposition holds:

Proposition 1. *In a double Cobb-Douglas OLG economy with endogenous fertility, FR-PAYG pensions and short-sighted individuals the dynamics of capital is the following.*

(1) Let $0 < \alpha < \alpha_3$ *hold. Then* $\underline{\theta} < \overline{\theta} < 1$*, and:*

(1.1) if $0 < \theta < \underline{\theta}$*, the dynamics of capital is monotonic and convergent to* k^**;*

(1.2) if $\underline{\theta} < \theta < \overline{\theta}$*, the dynamics of capital is oscillatory and convergent to* k^**;*

(1.3) if $\theta = \overline{\theta}$*, a flip bifurcation emerges;*

(1.4) if $\overline{\theta} < \theta < 1$*, the dynamics of capital is oscillatory and divergent to* k^**.*

(2) Let $\alpha_3 < \alpha < \alpha_1$ *hold. Then* $\underline{\theta} < 1$*,* $\overline{\theta} > 1$*, and:*

(2.1) if $0 < \theta < \underline{\theta}$*, the dynamics of capital is monotonic and convergent to* k^**;*

(2.2) if $\underline{\theta} < \theta < 1$*, the dynamics of capital is oscillatory and convergent to* k^**.*

(3) Let $\alpha_1 < \alpha < 1$ *hold. Then* $\overline{\theta} > \underline{\theta} > 1$*, and the dynamics of capital is monotonic and convergent to* k^* *for any* $0 < \theta < 1$*,*

[3] The (local) stability properties of an economy with perfect foresight is briefly presented in Appendix A. Different from the case with myopic expectation, with rational expectations the economy does not exhibit any interesting dynamical feature.

where

$$\underline{\theta} = \underline{\theta}(\alpha, \beta, \phi, \omega) := \frac{\alpha^2}{(1-\alpha)^2} \cdot \frac{\phi}{\beta\omega + \phi}, \qquad (18)$$

$$\overline{\theta} = \overline{\theta}(\alpha, \beta, \phi, \omega) := \frac{\alpha(1+\alpha)}{(1-\alpha)^2} \cdot \frac{\phi}{\beta\omega + \phi} = \underline{\theta} \cdot \frac{1+\alpha}{\alpha}, \qquad (19)$$

$$\alpha_1 = \alpha_1(\beta, \phi, \omega) := \frac{1}{\beta\omega}\left[\beta\omega + \phi - \sqrt{\phi(\beta\omega + \phi)}\right], \quad 1/2 < \alpha_1 < 1, \qquad (20)$$

$$\alpha_3 = \alpha_3(\beta, \phi, \omega) := \frac{1}{2\beta\omega}\left[2\beta\omega + 3\phi - \sqrt{\phi(8\beta\omega + 9\phi)}\right], \quad 1/3 < \alpha_3 < \alpha_1. \qquad (21)$$

Proof. Differentiating Eq. (17) with respect to k_t and using Eq. (15) gives:

$$\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t = k^*} = \alpha - \theta \frac{(1-\alpha)^2}{\alpha} \cdot \frac{\beta\omega + \phi}{\phi}. \qquad (22)$$

Monotonic and non-monotonic dynamics

From Eq. (22), the condition $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t = k^*} \gtreqless 0$ implies

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} \cdot \frac{\beta\omega + \phi}{\phi} \gtreqless 0 \Leftrightarrow \theta \lesseqgtr \underline{\theta}, \qquad (23)$$

where $\theta = \underline{\theta}$ (defined by Eq. 18) represents the value of the contribution rate below (beyond) which the dynamics of capital is monotonic (non-monotonic). In particular, $\underline{\theta} < 1$ ($\underline{\theta} > 1$) for any $0 < \alpha < \alpha_1$ ($\alpha_1 < \alpha < 1$). Moreover, $\underline{\theta} < 1$ if and only if $\alpha < \alpha_1$ and $\alpha > \alpha_2$, where α_1 is defined by Eq. (20) and $\alpha_2 = \alpha_2(\beta, \phi, \omega) := \frac{1}{\beta\omega}\left[\beta\omega + \phi + \sqrt{\phi(\beta\omega + \phi)}\right]$. Since $1/2 < \alpha_1 < 1$ and $\alpha_2 > 1$ for any β, ϕ and $0 < \alpha \leq 1$, then the case $\alpha > \alpha_2$ can be ruled out.

Now, $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ gives

$$\alpha-\theta\frac{(1-\alpha)^2}{\alpha}\cdot\frac{\beta\omega+\phi}{\phi}<1\Rightarrow\theta>-\frac{\alpha}{1-\alpha}\cdot\frac{\phi}{\beta\omega+\phi}. \tag{24}$$

Therefore, in the case of monotonic dynamics the economy always converges to the stationary state irrespective of the size of the pension system, i.e. $0<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ for any $0<\theta<1$.

Non-monotonic dynamics: stability analysis

The condition $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}\underset{<}{\overset{>}{-}}-1$ implies:

$$\alpha-\theta\frac{(1-\alpha)^2}{\alpha}\cdot\frac{\beta\omega+\phi}{\phi}\underset{<}{\overset{>}{-}}-1\Rightarrow\theta\underset{>}{\overset{<}{-}}\overline{\theta}, \tag{25}$$

where $\theta=\overline{\theta}>\underline{\theta}$ (defined by Eq. 19) is the flip bifurcation value of the contribution rate, i.e. the threshold value of θ below (beyond) which the steady state is stable (unstable). In particular, $\overline{\theta}<1$ ($\overline{\theta}>1$) for any $0<\alpha<\alpha_3$ ($\alpha_3<\alpha<1$). Moreover, $\overline{\theta}<1$ if and only if $\alpha<\alpha_3$ and $\alpha>\alpha_4$, where α_3 is defined by Eq. (21), $\alpha_4=\alpha_4(\beta,\phi,\omega):=\frac{1}{2\beta\omega}\left[2\beta\omega+3\phi+\sqrt{\phi(8\beta\omega+9\phi)}\right]$ and $\alpha_3<\alpha_1$. Since $1/3<\alpha_3<\alpha_1$ and $\alpha_4>1$ for any β, ϕ and $0<\omega\leq1$, then the case $\alpha>\alpha_4$ can be ruled out.

Therefore,

(*i*) if $0<\alpha<\alpha_3$ then $\underline{\theta}<\overline{\theta}<1$ and (*1.1*) $0<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<1$ for any $0<\theta<\underline{\theta}$, (*1.2*) $-1<\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<0$ for any $\underline{\theta}<\theta<\overline{\theta}$, (*1.3*) $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}=-1$ if and only if $\theta=\overline{\theta}$, and (*1.4*) $\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*}<-1$ for any $\overline{\theta}<\theta<1$. This proves point (1);

(*ii*) if $\alpha_3 < \alpha < \alpha_1$ then $\underline{\theta} < 1$, $\overline{\theta} > 1$ and (*2.1*) $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t = k^*} < 1$ for any $0 < \theta < \underline{\theta}$, and (*2.2*) $-1 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t = k^*} < 0$ for any $\underline{\theta} < \theta < 1$. This proves point (2);

(*iii*) if $\alpha_1 < \alpha < 1$ then $\overline{\theta} > \underline{\theta} > 1$ and $0 < \frac{\partial k_{t+1}}{\partial k_t}\Big|_{k_t = k^*} < 1$ for any $0 < \theta < 1$. This proves point (3). **Q.E.D.**

Proposition 1 can easily be interpreted as follows: the stock of capital installed at time $t+1$ is determined as the saving rate divided by the number of children at time t (see Eqs. 7, 8 and 11). Therefore, the accumulation of capital depends on difference between the private saving component and the public pension component, both divided by the taste for children (see Eq. 12). With Cobb-Douglas utility, the private saving component exclusively depends on the marginal willingness to save out of wage income, and reflects the positive effect on capital accumulation of a higher working income following a rise in k_t. In contrast, the public pension component depends on both the expected pension benefit and the expected interest rate, and reflects the negative (crowding out) effect on capital accumulation following a rise in k_t. If the private saving component dominates (is dominated by) the public pension component, the dynamics of capital is monotonic (non-monotonic). When production is relatively labour-oriented and the contribution rate is low enough, the private saving component dominates and thus the dynamics of the economy is monotonic and the steady state is always stable, i.e., the so-called saddle node bifurcation can never occur. A rise in the contribution rate increases the relative weight of the public pension component and a non-monotonic unstable dynamics emerges in that case. In contrast, when production is relatively capital-oriented the dynamics is always monotonic irrespective of the size of the PAYG system.

We now perform a sensitivity analysis of the critical values of the contribution rate which discriminates between monotonic and non-monotonic dynamics (see Eq. 18), as well as between non-monotonic stable and unstable dynamics (see Eq. 19) in the cases of both FR-PAYG pensions ($0 < \omega \leq 1$) and pure PAYG ($\omega = 0$) pensions.

Analysis of Eqs. (18) and (19) gives the following proposition:

Proposition 2. *The risk of cyclical instability with FR-PAYG pensions is higher than with pure PAYG pensions. A rise in the distributive capital share (α) monotonically reduces the risk of cyclical instability irrespective of the pension scheme. Moreover, while with pure PAYG pensions a change in the individual degree of thriftiness (β), and/or in the taste for children (ϕ) is neutral for stability, with FR-PAYG pensions a rise in the child factor (ω), and/or in the individual degree of thriftiness, and a reduction in the taste for children increases the risk of cyclical instability.*

Proof. First, in the case of pure PAYG pensions ($\omega = 0$) Eq. (18) becomes $\underline{\theta} = \underline{\theta}(\alpha) := \frac{\alpha^2}{(1-\alpha)^2}$ (i.e., the value of the contribution rate which discriminates between monotonic and non-monotonic dynamics is independent of both the subjective discount factor and taste for children), so that $\underline{\theta}(\alpha) < 1$ ($\underline{\theta}(\alpha) > 1$) for any $0 < \alpha < 1/2$ ($1/2 < \alpha < 1$). Therefore, with FR-PAYG pensions ($0 < \omega \leq 1$) the width of the parametric region in the space (α, θ) where non-monotonic dynamics are possible is larger than the corresponding region with pure PAYG pensions ($\omega = 0$). This means that when $0 < \omega \leq 1$, the threshold $\underline{\theta}(\alpha, \beta, \phi, \omega)$ can be smaller than unity even when $1/2 < \alpha < 1$. Second, in the case of pure PAYG pensions ($\omega = 0$) Eq. (19) becomes $\overline{\theta} = \overline{\theta}(\alpha) := \underline{\theta}(\alpha) \cdot \frac{1+\alpha}{\alpha}$ (i.e., the flip bifurcation value of the contribution rate is independent of both the subjective discount factor and taste for children), so that $\overline{\theta}(\alpha) < 1$ ($\overline{\theta}(\alpha) > 1$) for any $0 < \alpha < 1/3$ ($1/3 < \alpha < 1$). Therefore, with FR-PAYG pensions ($0 < \omega \leq 1$) the width of the parametric region in the space

(α,θ) where non-monotonic unstable dynamics are possible is larger than the corresponding region with pure PAYG pensions ($\alpha = 0$). This means that when $0 < \alpha \leq 1$, the flip bifurcation value $\bar{\theta}(\alpha,\beta,\phi,\omega)$ can be smaller than unity even when $1/3 < \alpha < 1$.

Moreover, from Eq. (19) we get:

$$\frac{\partial\bar{\theta}}{\partial\alpha} = \frac{\phi(1+3\alpha)}{(1-\alpha)^3(\beta\omega+\phi)} > 0, \tag{26}$$

for any $0 \leq \alpha \leq 1$, and

$$\frac{\partial\bar{\theta}}{\partial\omega} = -\frac{\alpha(1+\alpha)\phi\beta}{(1-\alpha)^2(\beta\omega+\phi)^2} < 0, \tag{27}$$

$$\frac{\partial\bar{\theta}}{\partial\beta} = -\frac{\alpha(1+\alpha)\phi\omega}{(1-\alpha)^2(\beta\omega+\phi)^2} < 0, \tag{28}$$

$$\frac{\partial\bar{\theta}}{\partial\phi} = \frac{\alpha(1+\alpha)\beta\omega}{(1-\alpha)^2(\beta\omega+\phi)^2} > 0, \tag{29}$$

for any $0 < \alpha \leq 1$. **Q.E.D.**

Figures 1 and 2 illustrate Proposition 2 and compare the parametric regions in the space (α,θ) that describe the (stable) monotonic and (stable and unstable) non-monotonic dynamics in the cases of pure PAYG pensions (Figure 1) and FR-PAYG pensions (Figure 2). It is clearly shown that while in a pure PAYG context cyclical instability arises only when $\alpha < 1/3$, in a FR-PAYG context the cyclical unstable region in the space (α,θ) is larger because of the destabilising effects played by the child factor, the individual degree of thriftiness and the taste for children.

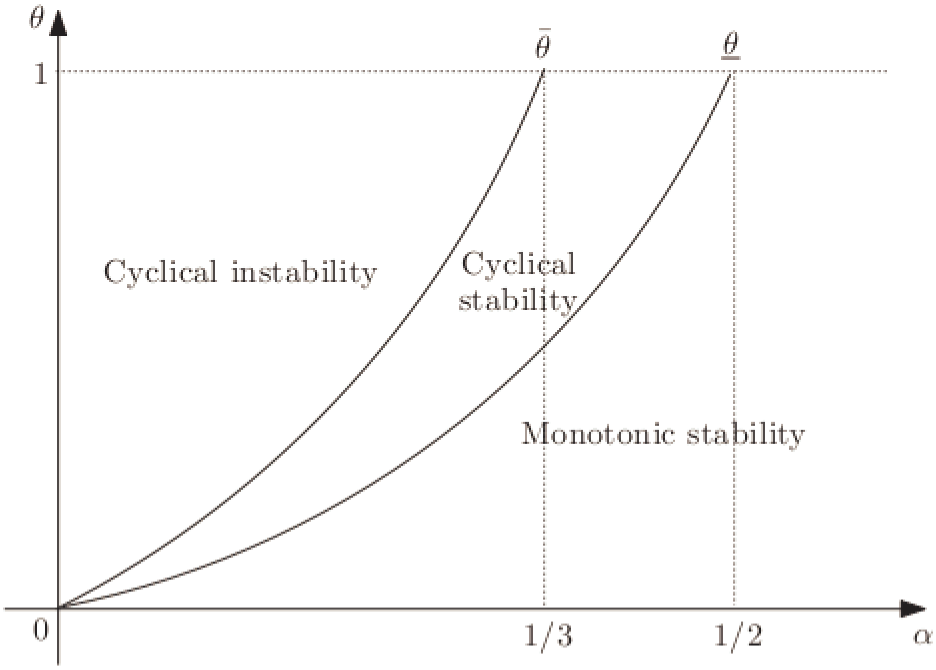

Figure 1. Case $\omega = 0$ (pure PAYG pensions). Stability and instability regions in the space (α, θ).

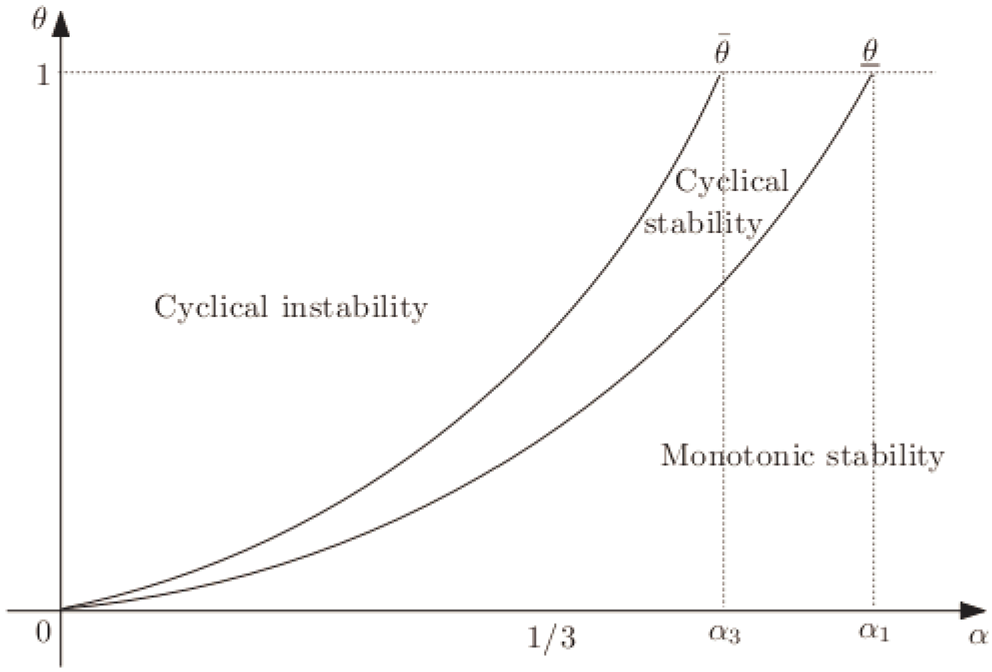

Figure 2. Case $0 < \omega \leq 1$ (FR-PAYG pensions). Stability and instability regions in the space (α, θ).

Table 1. Parametric instability regions ($0 < \overline{\theta} < 1$) under different PAYG systems.

Pure PAYG ($\omega = 0$)	Mixed FR-PAYG ($0 < \omega < 1$)	Pure FR-PAYG ($\omega = 1$)

$0 < \alpha < 1/3$	$0 < \alpha < \alpha_3(\beta, \phi, \omega)$	$0 < \alpha < \alpha_3(\beta, \phi, 1)$

Table 1 summarises for three different PAYG schemes the threshold values of the output elasticity of capital below which cyclically instability may emerge. Since $\alpha_3(\beta, \phi, 1) > \alpha_3(\beta, \phi, \omega) > 1/3$, it is evident that persistent cycles more likely occurs when the weight of individual fertility in the PAYG system is high.

Moreover, from Proposition 2 we may derive the following results as regards the effects of the preference parameters on the stability of the economy:

Result 1. *To the extent that fertility is low because the preference for children is low (e.g. developed countries), the introduction of FR-PAYG pensions (* $0 < \omega \leq 1$ *) generates a higher risk of cyclical instability than when fertility is high because the preference for children is high (e.g. under-developed or developing countries).*

Result 2. *To the extent that the degree of thriftiness is high because the financial education of individuals is high (e.g. developed countries), the introduction of FR-PAYG pensions (* $0 < \omega \leq 1$ *) generates a higher risk of cyclical instability than when the degree of thriftiness is low because the financial education of individuals is low (e.g. under-developed or developing countries).*

Results 1 and 2 lead to a rather paradoxical policy effect: since the introduction of FR-PAYG pensions is essentially advocated in economies with low fertility in order to overcome the sustainability issue of the widespread PAYG systems, our results imply that in economies where the taste for children is relatively low the instability risk induced by a pension reform that links the pension arrangement received when old to the number of children chosen when young is high. This result holds because a reduction in the taste for children increases the relative weight of the public pension component in equilibrium and thus contributes to destabilise the economy, while keeping the private saving component unaffected (see Eq. B1 in Appendix B).

Another paradoxical result can be derived about the effect of the parameter that describes the financial education of individuals when FR-PAYG pensions exist. A rise in subjective discount factor, in fact, means that individuals wish to smooth consumption over the retirement period and, hence, save more when young. This apparently causes a positive stabilising effect. However, the analysis of the local stability properties of the steady state reveals that β is neutral on the private saving component while increasing the relative weight of the public pension component, and thus acts as a destabilising device. Therefore, in a country where the individual degree of thriftiness is high because the financial education is high (e.g. developed countries which, unfortunately, are those most plagued by under-population and then prone to consider FR pension reforms), the introduction of a FR-PAYG scheme may cause unstable cycles and, as shown in the next section, even chaotic motions.

3.1. Chaotic dynamics: a numerical experiment

We are now interested in showing the emergence of deterministic chaos in a double Cobb-Douglas economy with FR-PAYG pensions.

In Figures 3-5 we depict the bifurcation diagrams for the parameter θ (on the horizontal axis), with respect to three different values of the child factor (ω). We take the following parameter set (only for illustrative purposes): $A = 10$, $\alpha = 0.25$, $\beta = 0.60$, $\phi = 0.05$, $q = 0.15$ and $k_0 = 0.10$ (the initial value of the stock of capital). The vertical axis shows the limit points of the equilibrium sequence of capital, k^*. When the contribution rate is relatively low a unique limit point exists. When the contribution rate raises a period doubling bifurcation emerges. Larger PAYG pensions imply that period doubling bifurcations appear more and more rapidly, thus bringing the economy into the chaotic region. As it is evident, the chaotic behaviour generated by FR-PAYG pensions more likely appears when the weight of children in determining the size of the pension arrangement is high. In fact, the flip bifurcation value of the contribution rate dramatically shrinks from $\overline{\theta} = 0.5555$ to $\overline{\theta} = 0.0427$ when the social security system shifts from a pure PAYG scheme to a pure FR-PAYG scheme. This of course increases the risk of cyclical instability. In fact, when PAYG pensions are fully linked to individual fertility, a small-sized pension system ($\overline{\theta} = 0.054$) directly brings the economy into the chaotic region.

Therefore, although fertility-related pensions are often advocated as a possible remedy against the peril of the future sustainability of unfunded public pensions as well as for optimality purposes (see Abio et al., 2004), the transition from a pure PAYG system (Figure 3) to a PAYG system partially (Figure 4) or totally (Figure 5) linked to individual fertility may easily open the route to deterministic chaos even in presence of small-sized public pensions.

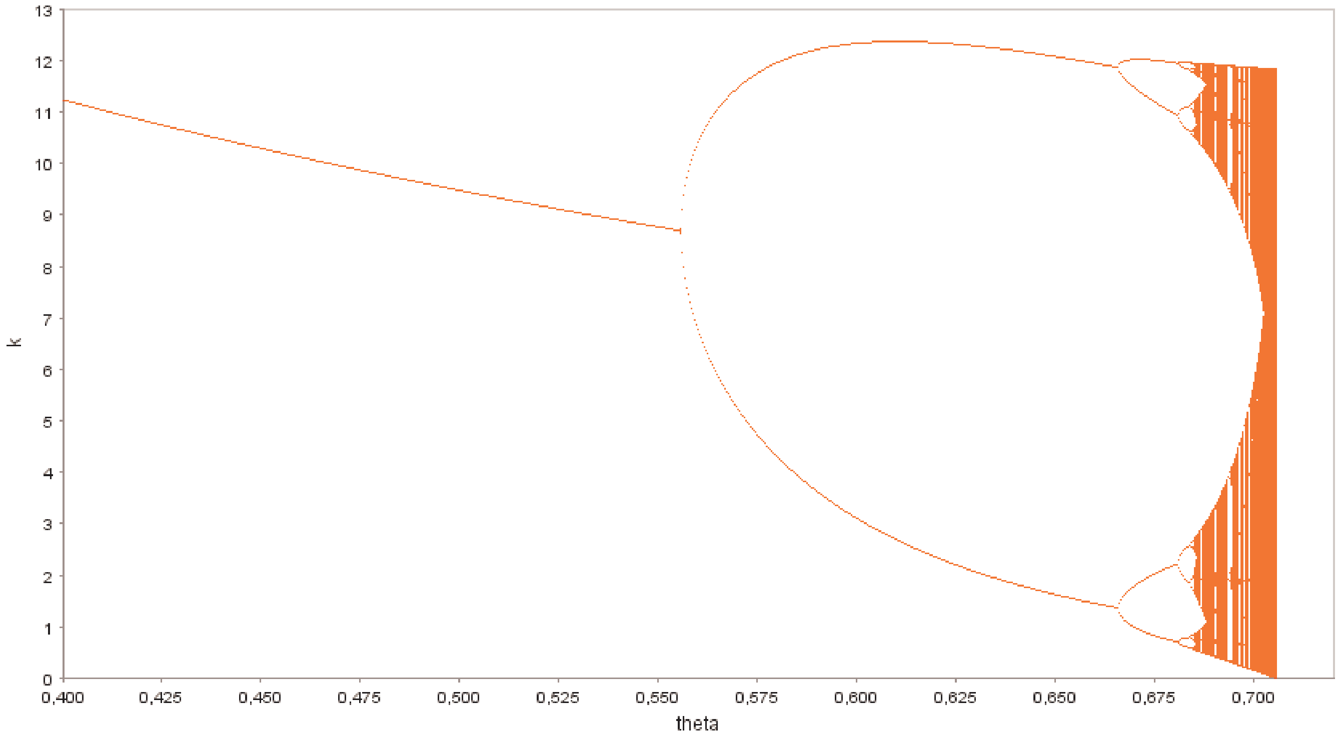

Figure 3. Case $\alpha = 0$ (pure PAYG scheme). Bifurcation diagram for θ ($\overline{\theta} = 0.5555$).

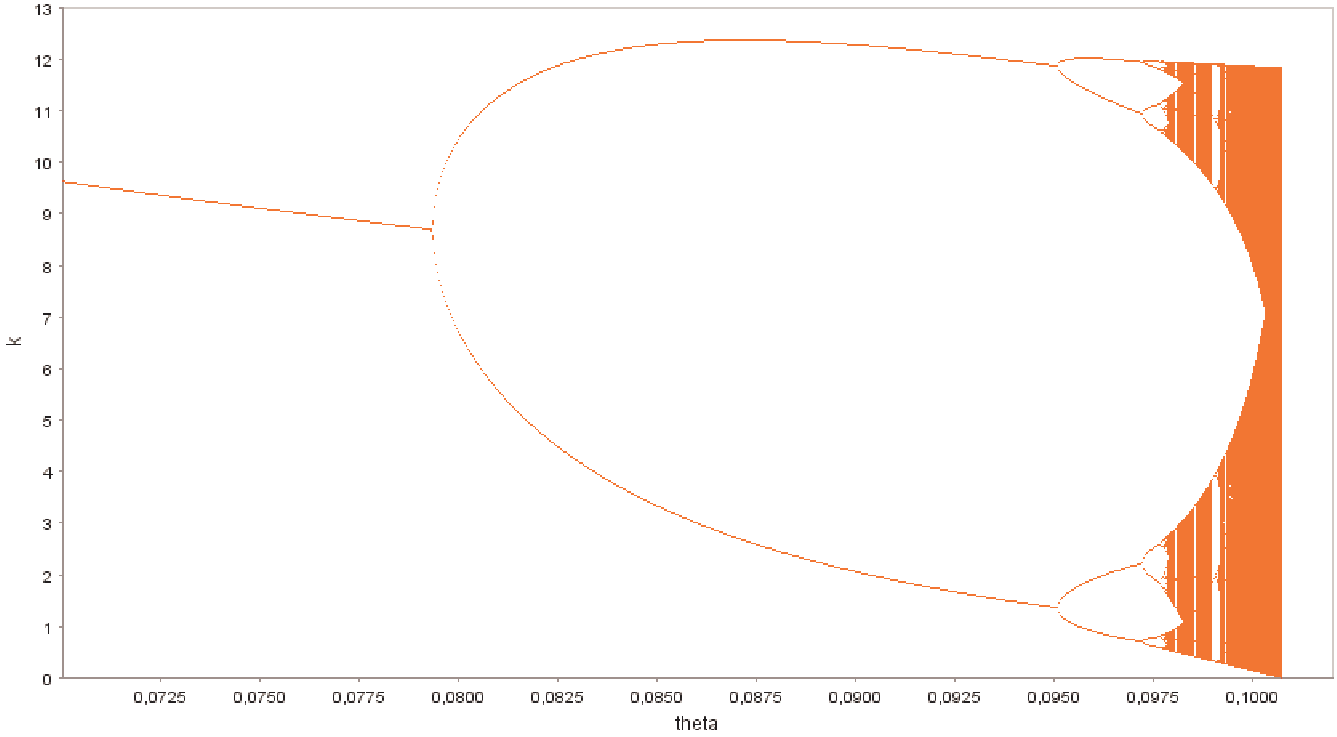

Figure 4. Case $\alpha = 0.50$ (mixed FR-PAYG scheme). Bifurcation diagram for θ ($\bar{\theta} = 0.0793$).

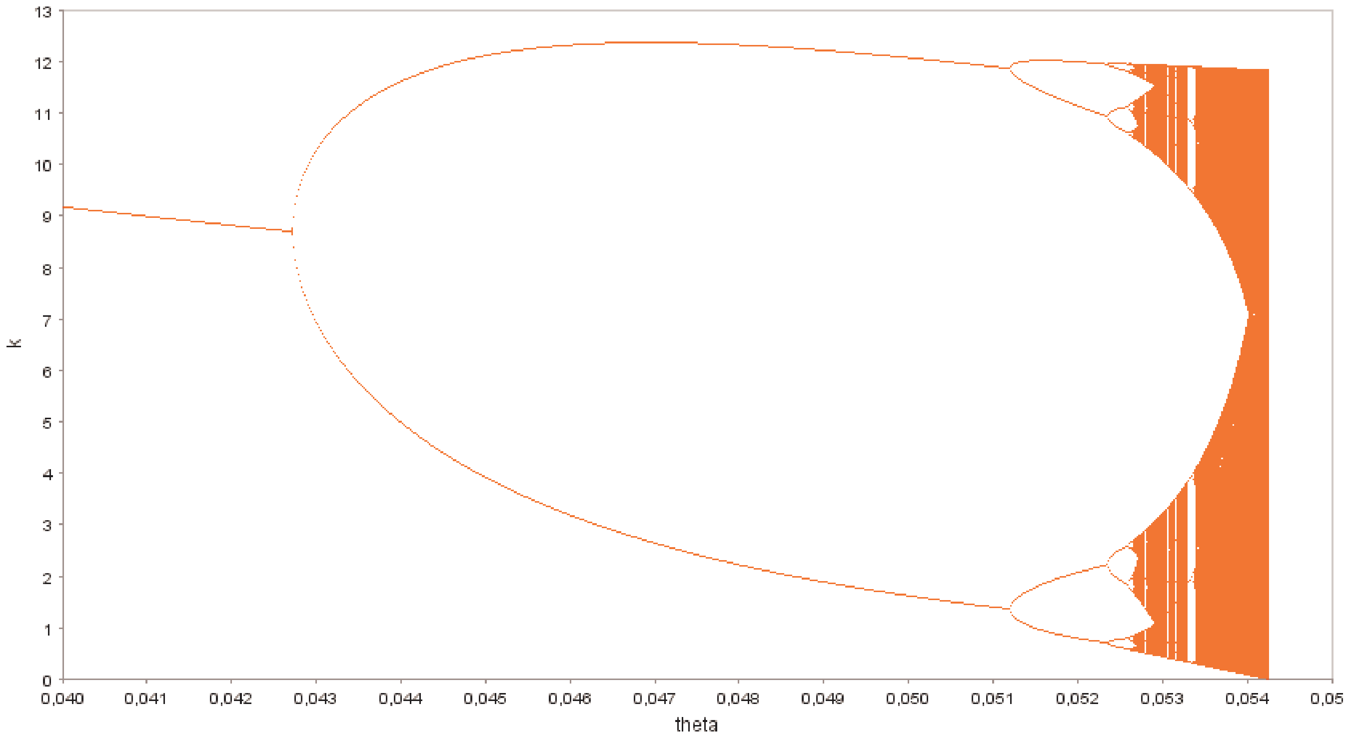

Figure 5. Case $\alpha = 1$ (pure FR-PAYG scheme). Bifurcation diagram for θ ($\bar{\theta} = 0.0427$).

4. Conclusions

We analysed the dynamics of an overlapping generations economy with endogenous fertility and fertility-related pay-as-you-go public pensions when individuals are myopic foresighted.

We showed that a fertility-related pension reform dramatically increases the risk of cyclical instability generated by the PAYG system. Moreover, the existence of a fertility-related component in the pension formula generates two counterintuitive policy effects: a rise in the individual degree of thriftiness and a reduction is taste for children both increase the area of cyclical instability. In fact, the capital accumulation function is divided into two components: the private saving component and the public pension component. What is important for stability is the relative size of the latter component. With FR-PAYG pensions, the more individuals wish to smooth consumption over their retirement period and the less is the taste for children, the higher is the relative weight of the public pension component, i.e. both parameters act as economic de-stabilisers.

Therefore, a double Cobb-Douglas economy with FR-PAYG pensions and myopic foresighted individuals contains in itself the possibility of deterministic complex cycles.

Our results have a twofold interpretation: (*i*) constitute a policy warning about the risks of (cyclical) instability caused by PAYG pension schemes in presence of realistic myopia of individuals, and (*ii*) they represent an explanation of the occurrence of persistent cycles in economies with endogenous fertility.

Appendix A

In this appendix we briefly show that the dynamics of a Cobb-Douglas OLG economy with FR-PAYG pensions and perfect foresight cannot be cyclical.

Proposition A.1. *The dynamics of capital in a double Cobb-Douglas OLG economy with FR-PAYG pensions and perfect foresighted individuals is always monotonic and convergent to* k^*.

Proof. Differentiating Eq. (14) with respect to k_t and using Eq. (15) we find:

$$\left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*} = \alpha \frac{q\beta\alpha(1-\alpha)A}{\alpha\phi+\theta(1-\alpha)(\beta\omega+\phi)}\left(k^*\right)^{\alpha-1} = \alpha . \tag{A1}$$

Therefore, $0 < \left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*} < 1$ for any $0<\theta<1$. **Q.E.D.**

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