New methods of estimating stochastic volatility and the stock return

Moawia Alghalith

28 January 2010

NEW METHODS OF ESTIMATING STOCHASTIC VOLATILITY AND STOCK RETURN

ABSTRACT. We present a new method of estimating the asset stochastic volatility and return. In doing so, we overcome some of the limitations of the existing random walk models, such as the GARCH/ARCH models.

1 Introduction

Theoretical literature on stochastic optimization and stochastic volatility is vast. Examples include Alghalith (2009), Castaneda-Leyva and Hernandez-Hernandez (2006), Focardi and Fabozzi (2004), and Cvitanic and Zapatero (2004). Empirical and statistical literature on stochastic volatility and asset pricing is also vast. Examples include Ferulano (2009), Carr and Hirsa (2007), and Geman (2007).

It is well-known that previous literature on estimating volatility and the asset rate of return adopted various versions of random walk models, such as the GARCH or ARCH models. Clearly, these models are only suitable for time series data. Moreover, they are suitable for discrete time models since the current variable is assumed to be determined by the previous value of the variable. In addition, these models omit the other determinants of volatility and rate of return, such as the stochastic economic factors.

Consequently, in this paper, we present a new method of estimating stochastic volatility and the rate of return of the asset (portfolio) that overcomes some of the limitations of the random walk models. In doing so, firstly, we present a model that is suitable for both cross-sectional and time series data.

Secondly, this model is appropriate for both continuous and discrete time models. Moreover, this model identifies the determinants of the volatility and asset return, and provides a link between the volatility and return.

2 The model

We use a two-dimensional standard Brownian motion $\{W_{1s}, W_{2s}, \mathcal{F}_s\}_{t\leq s\leq T}$ based on the probability space $(\Omega, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t\leq s\leq T}$ is the augmentation of filtration. Similar to previous models, we consider a risky asset, a risk-free asset and a random external economic factor. The risk-free asset price process is given by $S_0 = e^{\int_t^T r(Y_s)ds}$, where $r(Y_s) \in C_b^2(R)$ is the rate of return and Y_s is the economic factor.

The dynamics of the risky asset price are given by

$$dS_s = S_s \{\mu(Y_s)\, ds + \sigma(Y_s)\, dW_{1s}\}, \tag{1}$$

where $\mu(Y_s)$ and $\sigma(Y_s)$ are the rate of return and the volatility, respectively. The economic factor process is given by

$$dY_s = g(Y_s)\, ds + \rho dW_{1s} + \sqrt{1-\rho^2} dW_{2s}, Y_t = y, \tag{2}$$

where $|\rho| < 1$ is the correlation factor between the two Brownian motions and $g(Y_s) \in C^1(R)$.

The wealth process is given by

$$X_T^\pi = x + \int_t^T \{r(Y_s) X_s^\pi + (\mu(Y_s) - r(Y_s))\pi_s\} ds + \int_t^T \pi_s \sigma(Y_s) dW_s, \quad (3)$$

where x is the initial wealth, $\{\pi_s, \mathcal{F}_s\}_{t \leq s \leq T}$ is the portfolio process with $\int_t^T \pi_s^2 ds < \infty$. The trading strategy $\pi_s \in \mathcal{A}(x, y)$ is admissible.

The investor's objective is to maximize the expected utility of the terminal wealth and consumption

$$V(t, x, y, \sigma, \mu - r) = \sup_\pi E[U(X_T^\pi) \mid \mathcal{F}_t], \quad (4)$$

where $V(.)$ is the indirect utility function, $U(.)$ is continuous, bounded and strictly concave utility function.

We can rewrite (4) as (and suppressing the notations)

$$V\left(t,x,y,\sigma,\mu-r,a\right)=\sup_{\pi}E\left[U\left(ax+\int_{t}^{T}\left\{rX^{\pi}+\left(\mu-r\right)\pi\right\}ds+\int_{t}^{T}\pi\sigma dW_{s}\right)\mid\mathcal{F}_{t}\right], \tag{5}$$

where a is a shift parameter with initial value equals one (see Alghalith (2008)). Differentiating both sides of (5) with respect to a and x, respectively, we obtain

$$V_{a}\left(.\right)=xE\left[U'\left(.\right)\mid\mathcal{F}_{t}\right], \tag{6}$$

$$V_{x}\left(.\right)=aE\left[U'\left(.\right)\mid\mathcal{F}_{t}\right], \tag{7}$$

where the subscripts denote partial derivatives; thus

$$\frac{V_{a}\left(.\right)}{V_{x}\left(.\right)}=x. \tag{8}$$

Consider the following Taylor's expansion of $V(.)$

$$
\begin{aligned}
V(t, x, \sigma, \mu - r, a) \quad = \quad & V + V_x x + V_y y + V_\sigma \sigma + V_{\mu - r}(\mu - r) + V_a a \\
& + \frac{1}{2}\left(V_{xx} x^2 + V_{yy} y^2 + V_{\sigma\sigma}\sigma^2 + V_{(\mu-r)(\mu-r)}(\mu - r)^2 + V_{aa} a^2\right) + \\
& + V_{xy} xy + V_{x\sigma} x\sigma + V_{x(\mu-r)} x(\mu - r) + V_{xa} xa + V_{y\sigma} y\sigma \\
& + V_{y(\mu-r)} y(\mu - r) + V_{ya} ya + V_{\sigma(\mu-r)}\sigma(\mu - r) + V_{\sigma a}\sigma a \\
& + V_{a(\mu-r)} a(\mu - r). \qquad (9)
\end{aligned}
$$

Differentiating (9) with respect to a and x, respectively, we obtain

$$V_a(.) = V_a + V_{aa} + V_{xa} x + V_{ay} y + V_{a\sigma}\sigma + V_{a(\mu-r)}(\mu - r), \qquad (10)$$

$$V_x(.) = V_x + V_{xa} + V_{xx} x + V_{xy} y + V_{x\sigma}\sigma + V_{x(\mu-r)}(\mu - r). \qquad (11)$$

Substituting $(10) - (11)$ into (8), we obtain

$$V_a + V_{aa} + V_{xa} x + V_{ay} y + V_{a\sigma}\sigma + V_{a(\mu-r)}(\mu - r) = x\left\{V_x + V_{xa} + V_{xx} x + V_{xy} y + V_{x\sigma}\sigma + V_{x(\mu-r)}(\mu - r)\right\} \qquad (12)$$

and thus

$$\sigma = \frac{-V_a - V_{aa} + (V_x - V_{ax})\,x - V_{ay}y - V_{a(\mu-r)}\,(\mu - r) + V_{xx}x^2 + V_{xy}xy + V_{x(\mu-r)}x\,(\mu - r)}{V_{a\sigma} - V_{x\sigma}x}. \tag{13}$$

The above equation can be rewritten as

$$\sigma = \frac{-\beta_0 + \beta_1 x - \beta_2 y - \beta_3\,(\mu - r) + \beta_4 x^2 + \beta_5 xy + \beta_6 x\,(\mu - r)}{\beta_7 - \beta_8 x}, \tag{14}$$

where β_i is a parameter that needs to be estimated; (8) can be easily estimated by a non-linear regression. Similarly, using the above procedure, we can estimate the rate of return of the asset (portfolio) using the following regression equation

$$\mu = \frac{-\beta_0 + \beta_1 x - \beta_2 y + \beta_4 x^2 + \beta_5 xy + \beta_8 x\sigma - \beta_7\sigma}{\beta_3 - \beta_6 x} + (\beta_3 - \beta_6 x)\,r.$$

Contrary to previous literature, this method can also be applied to cross-sectional data. Moreover, it links volatility to the current risk premium, economic factor and wealth.

References

[1] Alghalith, M. (2009). "A new stochastic factor model: general explicit solutions." *Applied Mathematics Letters*, **22**, pp 1852-1854.

[2] Alghalith, M. (2008). "Recent applications of theory of the firm under uncertainty." *European Journal of Operational Research*, **186**, pp 443-450.

[3] Carr, P. and A. Hirsa (2007). "Forward evolutions equations for knock-out options." in Advances in Mathematical Finance, Birkhauser, Boston, pp 195-218.

[4] Castaneda-Leyva, N. and D. Hernandez-Hernandez (2006). "Optimal consumption-investment problems in incomplete markets with random coefficients." *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC apos;05. 44th IEEE Conference*, pp 6650 - 6655.

[5] Cvitanic, J. and Zapatero, F. (2004). Introduction to the economics and mathematics of financial markets, MIT Press, Cambridge, MA.

[6] Ferulano, R. (2009). "A mixed historical formula to forecast volatility." Journal of Asset Management, **10**, pp 124-136.

[7] Focardi, F. and F. Fabozzi (2004). " The Mathematics of Financial Modeling and Investment Management. " Wiley E-Series.

[8] Geman, H. (2007). "Mean reversion versus random walk in oil and natural gas prices." in Advances in Mathematical Finance, Birkhauser, Boston, pp 219-230.