Profit Maximization and the Threshold Price

Mynbaev, Kairat

International School of Economics, Kazakh-British Technical University

1 January 1998

Profit Maximization and the Threshold Price

Kairat T. Mynbaev

International School of Economics, Kazakh-British Technical University

Tolebi 59, 050000

Almaty, Kazakhstan

Kairat.Mynbaev@fulbrightmail.org

Abstract. If the output market is perfectly competitive and the firm's production function is not concave, an increase in the output price may lead to an explosive increase in firm's profits at some point. We explore the properties of this point, called a threshold price. We derive the formula for the threshold price under very general conditions and show how it helps to study correctness of the profit maximization problem, without explicit assumptions about returns to scale or convexity/concavity of the production function.

JEL Classification: D2, D4

Keywords: threshold price, profit maximization, production function, cost function, Cobb-Douglas function, returns to scale

Since the insightful work of Shephard (1970) many developments in the production theory have been related to the duality between the production and cost functions. See Färe and Grosskopf (1994), First et al. (1993) and Martinez-Legaz et al. (2005) for an account of more recent results. Mynbaev's (1996, 1998) study of the duality principle has resulted in a new notion, the threshold price, which is the boundary between output prices yielding finite profits and those yielding infinite profits. The threshold price shows at a relatively elementary level what kind of problems may arise in the general equilibrium theory when there are production technologies with increasing returns to scale.

Here we provide an improved exposition of Mynbaev (1998). We derive the formula for the threshold price under very general conditions. We show how it helps to study correctness of the profit maximization problem both in the competitive and monopolistic cases, without explicit assumptions about returns to scale or convexity/concavity of the production function.

An inequality $x' \geq x$ between two vectors means that $x'_i \geq x_i$ for all coordinates. By definition, $p > 0$ if and only if $p_1, ..., p_m > 0$. R_+^m is the set of all vectors $p > 0$. $\|x\| = \left(x_1^2 + ... + x_n^2\right)^{1/2}$ denotes the Euclidean norm of a vector x and (x, y) stands for the scalar product.

Consider a firm whose production technology is described by a correspondence f such that for each n-dimensional input vector $x > 0$ the set of m-dimensional output vectors is $f(x) \subset R_+^m$. Let f^{-1} denote the inverse correspondence defined on the image $\Re(f)$ such that $f^{-1}(y)$ means the set of inputs able to produce output y. The product market is assumed to be perfectly competitive and the price of the output bundle $y \in R_+^m$ is (p, y) where $p \in R_+^m$. The inputs market is not necessarily competitive and the inputs price function $W: R_+^n \to R_+$ may be nonlinear: for any $x > 0$, $W(x)$ means its price (not per unit!) In case of a perfectly competitive inputs market the inputs price is given by the scalar product $W(x) = (w, x)$ where w is a constant vector.

Denoting by $c(y,W) = \inf\{W(x) : x \in f^{-1}(y)\}$ the cost function, by $F(x, p)$ $= \sup\{(p, y) : y \in f(x)\}$ the revenue function and by $\pi(p,W)$ $= \sup\{(p, y) - W(x) : y \in f(x), x \in R_+^n\}$ the profit function we have equivalently $\pi(p,W) = \sup\{F(x, p) - W(x) : x \in R_+^n\}$ and $\pi(p,W)$ $= \sup\{(p, y) - c(y,W) : y \in \Re(f)\}$.

Definition. Let normalized price vectors p^0 satisfy $\sum_{i=1}^m p_i^0 = 1$ and consider proportional prices $p = tp^0$. Since the profit function is non-decreasing in output prices, the function $\pi(tp^0,W)$ is non-decreasing along the ray $\{tp^0 : t > 0\}$. If $\pi(tp^0,W)$ is finite at some t, it is finite for all $t' < t$, and if it happens to be infinite at some t, it will be infinite for all $t'' > t$. Therefore there exists $t_0 = t_0(p^0)$ which separates all prices tp^0 yielding finite profits from those yielding infinite profits. We call the number $T(p^0) = t^0$ a *threshold factor* and the price $T(p^0)p^0$ a *threshold price in the direction of* p^0.

It follows that $T(p^0) = \sup\{t > 0 : \pi(tp^0,W) < \infty\} = \inf\{t > 0 : \pi(tp^0,W) = \infty\}$ for any p^0 belonging to the simplex $S = \{p > 0 : \sum p_i = 1\}$. Below we explore various properties of the threshold price. Simpler theorems are called statements.

Denote $g(x) = \sup\{\| y \| : y \in f(x)\}$. Obviously, $F(x, p) \le \| p \| g(x)$. g is called locally bounded if it is bounded on each sector $\{x \in R_+^n : \| x \| \le r\}, r > 0$. Everywhere we assume that g is locally bounded, W is non-negative and locally bounded. The next theorem contains two versions of the main formula: one is expressed in terms of the revenue function and the other in terms of the cost function. Equality of the two expressions is a manifestation of duality.

Theorem 1. (a) If $\lim_{\|x\|\to\infty} W(x) = \infty$, then for all $p \in S$ the threshold factor equals $T(p) = \liminf_{\|x\|\to\infty} W(x) / F(x, p)$.

(b) If $\lim_{\|x\|\to\infty} g(x) = \infty$, then the threshold factor equals $T(p) = \liminf_{\|y\|\to\infty} c(y,W)/(p, y)$ for any $p \in S$.

Proof. (a) The proof uses only homogeneity of $F(x, p)$ of degree 1 in p and the equation $\pi(tp^0,W) = \sup\{tF(x, p^0) - W(x) : x \in R_+^n\}, p^0 \in S.$ Denote $M_+ = \{x : tF(x, p^0) - W(x) \ge 0\}$, $\alpha = \liminf_{\|x\|\to\infty} W(x) / F(x, p^0) \in [0,\infty]$. To make sure that $\pi(tp^0,W) < \infty$ it is sufficient to verify that $\sup\{tF(x, p^0) - W(x) : x \in M_+\} < \infty$. The formula for $T(p)$ will be obtained as a consequence of two bounds: upper and lower. The lower bound is of the form $T(p^0) \geq \alpha$. Since the case $\alpha = 0$ is trivial, we assume $\alpha > 0$ and consider two subcases: $\alpha < \infty$ and $\alpha = \infty$.

If $\alpha < \infty$, then by definition for any $0 < \varepsilon < \alpha$ there is $r > 0$ such that $W(x)/F(x, p^0) \geq \alpha - \varepsilon \ \forall \|x\| > r$. Suppose that $t < \alpha$ and let ε be so small that $t < \alpha - 2\varepsilon$. Then for $\|x\| > r$ we have

$$tF(x, p^0) - W(x) = F(x, p^0)\left(t - \frac{W(x)}{F(x, p^0)}\right) < F(x, p^0)(\alpha - 2\varepsilon - \alpha + \varepsilon) < 0$$

so that $M_+ \subset \{x : \|x\| \leq r\}$. $\pi(tp^0, W) < \infty$ because F is locally bounded and

$$\sup\{tF(x, p^0) - W(x) : x \in M_+\} \leq t \sup\{F(x, p^0) : \|x\| \leq r\} < \infty.$$

If $\alpha = \infty$, then for any $N > 0$ there exists $r > 0$ such that $\|x\| > r$ implies $W(x)/F(x, p^0) \geq N$. Fixing any $t > 0$, we can choose $N = t + 1$ and obtain

$$tF(x, p^0) - W(x) = F(x, p^0)\left(t - \frac{W(x)}{F(x, p^0)}\right) \leq F(x, p^0)(N - 1 - N) < 0 \ \forall \|x\| > r.$$

Again, $M_+ \subset \{x : \|x\| \leq r\}$ and $\pi(tp^0, W) < \infty$. We have shown that $\pi(tp^0, W) < \infty$ for any $t < \alpha \, (\leq \infty)$. This proves the lower bound.

The upper bound looks like this: $T(p^0) \leq \alpha$. Assume $\alpha < \infty$ to avoid triviality. By definition, for any $\varepsilon > 0$ there is a sequence $\{x^N\}$ such that $W(x^N)/F(x^N, p^0) \leq \alpha + \varepsilon$, $\|x^N\| \to \infty$. Take any $t > \alpha$ and select ε which satisfies $t \geq \alpha + 2\varepsilon$. Then $t - W(x^N)/F(x^N, p^0) \geq \alpha + 2\varepsilon - \alpha - \varepsilon = \varepsilon$. Hence,

$$\pi(tp^0, W) \geq \sup_N F(x^N, p^0)\left(t - \frac{W(x^N)}{F(x^N, p^0)}\right) \geq \varepsilon \sup_N F(x^N, p^0) \geq \frac{\varepsilon}{\alpha + \varepsilon} \sup_N W(x^N) = \infty$$

because $W(x^N) \to \infty$. Thus, $\pi(tp^0, W) = \infty$ for all $t > \alpha$ and the upper bound obtains.

(b) Denote $M_+ = \{y \in \Re(f) : t(p^0, y) - c(y, W) \geq 0\}$, $\alpha = \liminf_{y \in \Re(f), \|y\| \to \infty} c(y, W)/(p^0, y)$. Obviously, $\pi(tp^0, W) = \sup\{t(p^0, y) - c(y, W) : y \in M_+\}$.

First we prove that $T(p^0) \geq \alpha$. To avoid triviality, assume $\alpha > 0$.

(i) Let $\alpha < \infty$. Take any $t \in [0, \alpha)$ and choose $\varepsilon > 0$ so that $t < \alpha - 2\varepsilon$. For this ε there exists $r > 0$ such that $c(y, W)/(p^0, y) \geq \alpha - \varepsilon$ for all $\|y\| > r$. Then for $\|y\| > r$ we have

$$t(p^0, y) - c(y, W) = (p^0, y)(t - c(y, W)/(p^0, y)) \leq (p^0, y)(\alpha - 2\varepsilon - \alpha + \varepsilon) < 0.$$

Hence, $M_+ \subset \{y : \|y\| \leq r\}$ and $\pi(tp^0, W) = t \sup\{(p^0, y) : y \leq r\} < \infty$.

(ii) Let $\alpha = \infty$. Fixing any $t > 0$ we can choose $N = t + 1$. Then for this N there exists $r > 0$ such that $\| y \| > r$ implies $c(y, W) / (p^0, y) \geq N$ and

$$t(p^0, y) - c(y, W) = (p^0, y)(t - c(y, W) / (p^0, y)) \leq (p^0, y)(N - 1 - N) < 0 .$$

Hence, $M_+ \subset \{y : \| y \| \leq r\}$ and $\pi(tp^0, W) < \infty$. We have shown that $\pi(tp^0, W) < \infty$ for any $t < \alpha \, (\leq \infty)$ which means $T(p^0) \geq \alpha$.

Next we prove that $T(p^0) \leq \alpha$. Without loss of generality we can consider $\alpha < \infty$. Let $t > \alpha$ and select $\varepsilon > 0$ such that $t \geq \alpha + 2\varepsilon$. For this ε there is a sequence $\{y^N\} \subset \Re(f)$ such that $c(y^N, W) / (p^0, y^N) \leq \alpha + \varepsilon$, $\| y^N \| \to \infty$. Then $t - c(y^N, W) / (p^0, y^N) \geq \varepsilon$ and

$$\pi(tp^0, W) \geq \sup_N (p^0, y^N)(t - c(y^N, W) / (p^0, y^N)) \geq \varepsilon \sup_N (p^0, y^N) = \infty .$$

Thus, $\pi(tp^0, W) = \infty$ for all $t > \alpha$ which proves the upper bound. □

Example 1. Using the Cobb-Douglas function $y = Ax_1^{\alpha} x_2^{\beta}$ with $\alpha, \beta > 0$ and a CES function $y = (a_1 x_1^{\rho} + a_2 x_2^{\rho})^{1/\rho}$ we can define a correspondence $f(x) = \{y_1, y_2 \geq 0 : (a_1 x_1^{\rho} + a_2 x_2^{\rho})^{1/\rho} \leq Ax_1^{\alpha} x_2^{\beta}\}$. In order for the set $f(x)$ to be strictly convex suppose $\rho > 1$. Then $F(x, p) = Ax_1^{\alpha} x_2^{\beta}(p_1 c_1 + p_2 c_2)$ where $c_1 = (a_1 c_0^{\rho} + a_2)^{-1/\rho}$, $c_2 = (a_1 + a_2 c_0^{-\rho})^{-1/\rho}$ and $c_0 = \left(p_1 a_2 / (p_2 a_1)\right)^{1/(\rho - 1)}$. Suppose W is homogeneous of degree $\gamma > 0$. Application of Theorem 1 gives: a) If $\alpha + \beta < \gamma$, then $T(p) = \infty$ for any $p \in S$; b) If $\alpha + \beta > \gamma$, then $T(p) = 0$ for any $p \in S$; c) In case $\alpha + \beta = \gamma$ denote $c_3(p) = A(p_1 c_1 + p_2 c_2) \sup\{x_1^{\alpha} x_2^{\beta} : W(x) = 1\}$. Then $T(p) = 1 / c_3(p)$, $p \in S$. □

Consider the profit function $\pi(tp, W)$ along the ray $\{tp : t > 0\}$ with $p \in S$ fixed. By the definition of the threshold factor profit is finite for all $t < T(p)$ and infinite for all $t > T(p)$. What happens when $t = T(p)$? Simple examples show that profit can be finite or infinite. The next statement describes precisely all the situations in which $\pi(T(p) p, W) < \infty$. We exclude the trivial cases $T(p) = 0$ and $T(p) = \infty$.

Statement 1. Suppose $\lim_{\|x\| \to \infty} W(x) = \infty$, $\lim_{\|x\| \to \infty} g(x) = \infty$ and let f and W be such that $0 < T(p) < \infty$. Consider any $p \in S$. Then the condition $\pi(T(p) p, W) < \infty$ is equivalent to each of the next conditions:

(i) $\limsup_{y \in \Re(f), \|y\| \to \infty} [(p, y) T(p) - c(y, W)] < \infty$ and (ii) $\limsup_{\|x\| \to \infty} [F(x, p) T(p) - W(x)] < \infty$.

Proof. We show that the condition $\pi(T(p) p, W) < \infty$ is equivalent to (i). Suppose that (i) is not true. Then there exists a sequence $\{y^N\} \subset \Re(f)$ such that $(p, y^N) T(p) - c(y^N, W) \to \infty$. It follows that

$$\pi(T(p) p, W) \geq \sup_N \{(p, y^N) T(p) - c(y^N, W)\} = \infty$$

which contradicts the assumption. Conversely, let (i) be true but $\pi(T(p)p,W)=\infty$. Then there exists a sequence $\{y^N\}\subset\Re(f)$ such that $(p,y^N)T(p)-c(y^N,W)\to\infty$. Since the function $(T(p)p,y)-c(y,W)$ is locally bounded, $\{y^N\}$ must be unbounded. But then (i) is violated. The proof of the equivalence of $\pi(T(p)p,W)<\infty$ to (ii) is equally simple. □

The threshold price has been defined as the product $T(p)p,\ p\in S$. Suppose that $W(x)=(w,x)$. Then the threshold price will depend on w: $T(p,w)p$. The next result shows that $T(p,w)p$ as a function of w possesses properties of a homogeneous production function of degree 1.

Statement 2. Let $W(x)=(w,x)$. Then $T(p,w)$ as a function of w is monotone, homogeneous of degree 1, concave and continuous.

Proof. We consider only case (a) of Theorem 1, case (b) being similar.
Monotonicity: if $w'\geq w$, then by Theorem 1

$$T(p,w')=\liminf_{\|x\|\to\infty}(w',x)/F(x,p)\geq\liminf_{\|x\|\to\infty}(w,x)/F(x,p)=T(p,w).$$

Homogeneity:

$$T(p,tw)=\liminf_{\|x\|\to\infty}(tw,x)/F(x,p)=tT(p,w).$$

Concavity: for any $t\in[0,1], w, w'$

$$T(p,tw+(1-t)w')=\liminf_{\|x\|\to\infty}(tw+(1-t)w',x)/F(x,p)$$
$$=\liminf_{\|x\|\to\infty}\left[t\frac{(w,x)}{F(x,p)}+(1-t)\frac{(w',x)}{F(x,p)}\right]\geq tT(p,w)+(1-t)T(p,w').$$

Continuity. Denote $\lambda(w,w')=\max_i w_i/w_i'$. Obviously,

$$T(p,w)=\liminf_{\|x\|\to\infty}\frac{\sum w_i x_i}{F(x,p)}=\liminf_{\|x\|\to\infty}\frac{\sum w_i' x_i(w_i/w_i')}{F(x,p)}\leq\lambda(w,w')T(p,w')$$

Since $\lambda(w,w')\to 1$ when $w\to w'$, we have $\limsup_{w\to w'}T(p,w)\leq T(p,w')$. By symmetry also $\limsup_{w'\to w}T(p,w')\leq T(p,w)$ which proves continuity. □

Example 2. Continuing Example 1, we get in case $W(x)=(w,x)$: a) If $\alpha+\beta<1$, then $T(p,w)p=\infty$ for all $p\in S$; b) if $\alpha+\beta>1$, then $T(p,w)p=0$ for all $p\in S$; and c) if $\alpha+\beta=1$, then $T(p,w)p=(A\alpha^\alpha\beta^\beta)^{-1}w_1^\alpha w_2^\beta p/(p_1c_1+p_2c_2)$ for all $p>0$ such that $p_1+p_2=1$. □

Next we study the geometry of product prices for which profit is finite. Define the sets FIN and INF by $FIN=\{p>0:\pi(p,W)<\infty\}$, $INF=\{p>0:\pi(p,W)=\infty\}$. It is clear that these sets are complementary. We say that the profit maximization (PM) problem is: 1) correct, if INF is empty, 2) semi-correct, if both FIN and INF are not empty, and 3) incorrect, if FIN is empty.

Statement 3. Suppose $\lim_{\|x\|\to\infty} W(x)=\infty$, $\lim_{\|x\|\to\infty} g(x)=\infty$. We can assert that

(a) The set FIN is convex. With every $p\in FIN$ this set contains the rectangle $\{p':0\le p'\le p\}$. FIN is limited by coordinate planes $\{p: p_i=0\}$, $i=1,\ldots,m$, and the surface $\left\{p \liminf_{\|x\|\to\infty} W(x)/F(x,p): \sum_{i=1}^m p_i=1\right\}$.

(b) There are three mutually exclusive possibilities (what we call a crowd principle):

(i) $T(p)=\infty$ for any $p\in S$, (ii) $0<T(p)<\infty$ for any $p\in S$, and (iii) $T(p)=0$ for all $p\in S$.

(c) The next equivalencies are true:

(i) The PM problem is correct $\Leftrightarrow$ $\liminf_{\|x\|\to\infty} W(x)/F(x,p)=\infty \quad \forall p\in S$

(ii) The PM problem is semi-correct $\Leftrightarrow$ $0<\liminf_{\|x\|\to\infty} W(x)/F(x,p)<\infty \quad \forall p\in S$

(iii) The PM problem is incorrect $\Leftrightarrow$ $\liminf_{\|x\|\to\infty} W(x)/F(x,p)=0 \quad \forall p\in S$

Proof. (a) Convexity of FIN follows from convexity of π in product prices. Since π is monotone in product prices, with every $p\in FIN$ this set contains the rectangle $\{p':0\le p'\le p\}$.

Along the ray $\{tp^0: t>0\}$, where $\sum p_i^0=1$, π is finite for $t<T(p)$ and infinite for $t>T(p)$. This means the set FIN is bounded by the surface $\{pT(p): \sum p_i=1\}$ in the northeast direction.

(b) Note that $\min_i p_i g(x)\le F(x,p)\le \|p\| g(x)$ and therefore

$$\frac{1}{\|p\|}\liminf_{\|x\|\to\infty}\frac{W(x)}{g(x)}\le T(p)=\liminf_{\|x\|\to\infty}\frac{W(x)}{F(x,p)}\le\frac{1}{\min p_i}\liminf_{\|x\|\to\infty}\frac{W(x)}{g(x)}$$

This implies the crowd principle. □

Now we wish to characterize all triples (p,f,W) for which $\pi(p,f,W)<\infty$.

Statement 4. Factor p as $p=tp^0$, $p^0\in S$.

(a) If $\lim_{\|x\|\to\infty} W(x)=\infty$, then $\pi(p,f,W)<\infty$ if and only if one of the conditions

(i) $t<\liminf_{\|x\|\to\infty}\dfrac{W(x)}{F(x,p)}$ or (ii) $t=\liminf_{\|x\|\to\infty}\dfrac{W(x)}{F(x,p)}$ and $\limsup_{\|x\|\to\infty}[tF(x,p)-W(x)]<\infty$

holds.

(b) Alternatively, if $\lim_{\|x\|\to\infty} g(x)=\infty$, then $\pi(p,f,W)$ is finite if and only if one of the conditions

(i) $t<\liminf_{y\in\Re(f),\|y\|\to\infty}\dfrac{c(y,W)}{(p,y)}$ or (ii) $t=\liminf_{y\in\Re(f),\|y\|\to\infty}\dfrac{c(y,W)}{(p,y)}$ and $\limsup_{y\in\Re(f),\|y\|\to\infty}[t(p,y)-c(y,W)]<\infty$

holds.

The proof is immediate from Theorem 1 and Statement 1.

The next application of the threshold price is to characterization of decreasing returns to scale. Given f,W we say that the production unit exhibits decreasing returns to scale, if $\lim_{y\in\Re(f),\|y\|\to\infty} c(y,W)/(p,y)=\infty$ for any $p>0$.

Statement 5. Let $\lim_{\|x\|\to\infty} W(x)=\infty$, $\lim_{\|x\|\to\infty} g(x)=\infty$. The production unit exhibits decreasing returns to scale if and only if $\liminf_{\|x\|\to\infty} W(x)/F(x,p)=\infty$ for any $p>0$. In this condition, the words "for any" can be replaced by "for some". The condition is equivalent to the following: for any positive ε, p there exists a constant $C=C(\varepsilon,p)>0$ such that $F(x,p)\le\varepsilon W(x)+C$ for all $x\ge 0$.

When $W(x)=(w,x)$, the last condition means that $F(x,p)$ grows slower than any positively sloped linear function.

To prove Statement 5 it suffices to compare the two parts of Theorem 1.

The final statement relates to the monopolist case when the threshold price does not make sense but some ideas used above are still applicable. Suppose that the product market is described by inverse demand functions $P_1(y),...,P_m(y)$ such that $(P(y),y)$ gives the product price. We assume that $P(y)>0$ for all $y\ne 0$ and that $\sup\{P_i(y):\|y\|\ge r\}<\infty$ for all $r>0, i=1,...,m,$ that is, the prices $P_1(y),...,P_m(y)$ can increase to infinity when $y\to 0$. f and W will satisfy the same conditions as before (g is locally bounded, W is non-negative and locally bounded) and $\lim_{x\to\infty} W(x)=\infty$. Define $F(x,P)=\sup\{(P(y),y): y\in f(x)\}$. Note that $\limsup_{\|x\|\to 0} F(x,P)=\infty$ is possible.

Statement 6. Denote $T=\liminf_{\|x\|\to\infty} W(x)/F(x,P)$. Under the conditions just described the monopolist's profit $\pi(P,f,W)$ is finite if and only if $\limsup_{\|x\|\to 0} F(x,P)<\infty$ and one of the conditions

(i) $T>1$ or (ii) $T=1$ and $\limsup_{\|x\|\to\infty}[F(x,P)-W(x)]<\infty$

holds.

Proof. Note that the normalization $\sum p_i^0 = 1$ in Theorem 1 was arbitrary. Instead, the normalization $\sum p_i^0 = m$ can be used. Besides, the vector p^0 was fixed. In the present context we can put $p^0 = (1,...,1)$ and consider pseudo-prices $p = tp^0, t > 0$. By putting

$$F_1(x, tp^0) = \sup\left\{\sum_{i=1}^{m} tp_i^0 P_i(y) y_i : y \in f(x)\right\}$$

we obtain a function which is homogeneous of degree 1 in pseudo-prices. With $t = 1$ one has $F_1(x, p^0) = F(x, P)$. Theorem 1 and Statement 4 can be formally used (as if the firm in question was competitive) to obtain Statement 6.

Similar statements can be obtained for the situation when dependence on input prices is linear and on output prices – nonlinear. The threshold factor and price are defined by varying input prices. □

Acknowledgment. This research has been made possible by the Fulbright scholarship and hospitality of the Economics Department of the University of California at Riverside.

References

Färe, R. and S. Grosskopf, 1994, Cost and Revenue Constrained Production (Springer-Verlag, New York).

First, Z., S. T. Hackman and U. Passy, 1993, Efficiency estimation and duality theory for nonconvex technologies, Journal of Mathematical Economics 22, 295-307.

Martınez-Legaz, J. E., A. M. Rubinov and S. Schaible, 2005, Increasing quasiconcave co-radiant functions with applications in mathematical economics, Mathematical Methods of Operations Research 61, 261–280.

Mynbaev, K., 1996, Production and cost functions in connection with internal economies of scale, Revista Econômica do Nordeste, 27, 805-831.

Mynbaev, K., 1998, The profit function and threshold price (in Portuguese), in: 3º Encontro Regional de Economia, Revista Econômica do Nordeste, v. 29, Número especial, (Fortaleza, Ceara) 967-975.

Shephard, R.W., 1970, Theory of cost and production functions (Princeton University Press, Princeton, New Jersey).