MPRA
Munich Personal RePEc Archive

Market Based, Segregated Exchanges with Default Risk

Kilenthong, Weerachart and Townsend, Robert

University of the Thai Chamber of Commerce, MIT

1 November 2007

Online at https://mpra.ub.uni-muenchen.de/20724/
MPRA Paper No. 20724, posted 16 Feb 2010 15:40 UTC

Market Based, Segregated Exchanges in Securities with Default Risk☆

Weerachart T. Kielnthong[1], Robert Townsend[1]

Abstract

This paper studies a competitive general equilibrium model with default and endogenous collateral constraints. Even though all collateralized contracts are allowed, the possibility and desirability of trade in spot markets (or the equivalent trade in ex ante asset backed securities) creates externalities, as spot prices (or security prices) and the bindingness of collateral constraints interact. We show that if agents are allowed to contract ex ante on market fundamentals determining the state-contingent spot price, over and above contracting on true underlying states of the world, then competitive equilibria with bundled securities and commodities and with endogenous collateral constraints are equivalent with Pareto optima. Examples show that it is possible to have multiple market fundamentals in equilibrium. Equivalently, it is possible for there to be segregation into distinct competitive securities exchanges with endogenous (positive and negative) entry fees. Fees accrue to borrowers who are otherwise collateral constrained.

Keywords: default; endogenous collateral; externalities; segregated exchanges; Walrasian equilibrium; limited commitment

1. Introduction

This paper uses a competitive general equilibrium with directly-collateralized and asset-backed securities to analyze the interaction between the endogenous valuation of collateral and corresponding default decisions. The interaction creates an "externality", which causes a collateral-constrained equilibrium to be inefficient. The externality exists because the consumption feasibility set of an agent directly depends on the decisions of other agents in the contracting period through the spot prices and the prices of asset backed securities. The impact on the feasible set in turn affects the allocations of all agents whenever the collateral or borrowing constraints of some agents are binding.

The primary contribution of this paper is not the discovery of the externality, as this is clear from the literature, (e.g., Geanakoplos and Polemarchakis, 1986; Geanakoplos, 2003;

☆This draft is November 12, 2009. This paper previously circulated under the title "Endogenous Valuation of Collateral and Contingent Trade in Price Fundamentals." We would like to thank seminar participants at MIT, UCSB, 2008 SED conference, Bewley conference, and LAFE conference. Tee Kilenthong is grateful to financial support from David Marshall-Merrill Lynch Faculty Fellowship. Robert Townsend is grateful to financial support from NSF, NICHD, the John Templeton Foundation, and the Consortium on Financial Systems and Poverty at the University of Chicago through a grant from the Bill & Melinda Gates Foundation. The findings and conclusions contained in the report are those of the author(s) and do not necessarily represent the views of the funders.

Caballero and Krishnamurthy, 2001; Lorenzoni, 2008), but rather a solution to the problem using market-based, segregated exchanges in securities. This approach should be applicable more generally to problems in which the inefficiency of competitive equilibrium is a result of frictions depending on future spot prices or terms-of-trade, a pecuniary externality, e.g., moral hazard with retrading.

We take it as a primitive that default is possible or equivalently that collateral is required to make borrowers (or issuers of securities) repay their loans. A borrower may choose to default on a particular loan, or a particular state-contingent promise, and in doing so would lose the value of collateral backing that particular loan or security. A rational borrower will base her default decision security by security on the value of the collateral backing each liability, compared to the original promise to pay. Of course the value of the collateral good at the time of repayment decisions (called the execution period) and in the market for asset backed securities (in the contract period) is an equilibrium phenomenon. Yet this market-clearing price of collateral determines whether borrowers default or not and the overall amount of debt.

A contract or security consists of two items, a state-contingent promise and the collateral backing that promise. Contracts which do not default have to be backed by a sufficient, minimum level of collateral, again depending on the promise and the value of collateral. Likewise asset backed securities which are issued have to be backed in collateral by an equivalent value of asset backed securities acquired. For every set of securities which actually default, handing over collateral, there is another set which is equivalent, with the same overall payoff and no default. Adding up all such promises, over state-contingent security promises directed backed by collateral and over state-contingent securities backed by the promises of others, generates a state-contingent collateral constraint on trades in the ex ante contract market. But contracts which do default also require collateral that is to be handed over when the borrower does not repay. That is, partially collateralized securities are still intimately associated with the exactly amount of collateral which serves a backing. Again, by rescaling contracts, these constraints can be shown to be equivalent. We label such constraints collateral constraints, for brevity.

The externality problem is considered as a missing-market problem; that is, the markets for contracts over the "market fundamentals", those aspects of the environment which determine the spot-market-clearing price, are missing. The market fundamentals are, in general, defined by the distribution of the resources across types of agents. In this paper, with identical homothetic preferences, the market fundamental is simply the aggregate ratio of a pair of physical commodities. Note again that a market fundamental is endogenous; that is, it depends on aggregate saving, which is a result of the actions in the trading period of all agents as a group.

In this paper, we extend the commodity space so that contracts are contingent on these market fundamentals and contingent contracts can be bought or sold, over and above contracts contingent on true underlying states of the world, though securities contingent on underlying states and market fundamentals are bundled together. Allowing agents to contract ex-ante on market fundamentals determining the state-contingent spot-market-clearing price in effect allows them to contract on the price, and internalizes the externality. We thus prove that the competitive equilibria with endogenous collateral constraints in the extended commodity space are equivalent with Pareto optima. One could view these results as normative, indicative of the need for a systematic but market-determined way for traders to unwind commitments.

A price island is a way to conceptualize the consistent execution of the contingencies on fundamentals. That is, a price island specifies the spot price, and the set of agents that end up there have to support that price. This is like a club constraint in other literature, e.g., Prescott and Townsend (2006). Agents can carry in goods in such a way that their pre-trade ratio of endowments in a spot market deviates from the market fundamental, but the sum of the deviations must, by the definition of consistency, be zero. Lotteries are then a way to assign agents to price islands jointly with other decisions such as security holdings and end-of-trading-period collateral. In a decentralized equilibrium, in which prices are taken as given, all price islands including out-of-equilibrium price islands are available for agents to purchase. Specifically, we internalize the externality by making household types pay or be paid for their influence on the spot market prices, when the pre-trade endowment ratio is different from the market fundamental. On another interpretation, ex post spot trades are replaced by ex ante trade in asset backed securities. In this interpretation, a household has to pay or be paid for the rights to trade in a particular security exchange. This is related to the consumption right in Bisin and Gottardi (2006). Here the coupling of asset backed securities with directly backed securities is even more natural.

We do not require that the markets keep track of individual trades and contracts, only that the over all composition of traders be such as to deliver the contracted price. This takes a certain commitment to prevent retrading across the "price islands." In other words, trades are segregated across mutually-distinct security markets. With these institutions in place, prices will direct trade and traders efficiently.

More generally, endogenous collateral constraints generate a non-convexity problem, as prices reflecting assignments and collateral decisions interact multiplicatively. As Prescott and Townsend (1984b); Hansen (1985); Rogerson (1988) have shown, lotteries can (weakly) improve on deterministic allocations when feasible sets are not convex. For computational purposes, we allow each variable to take only finite values (finite grids).

Importantly, the amount a household pays (or receives) in unit of account in the contract market does depend on its type, its individual endowment position relative to the market fundamental. Again, the amount that an agent will pay (or receive) depends on the difference between the market fundamental and endowment ratio of good-1 to good-2 (including collateral holding). If her endowment ratio is exactly equal to the market fundamental, then she does not pay (nor is paid). On the other hand, if she comes with a low (high) endowment ratio relative to the market fundamental, i.e., holding little of good-1 and lots of good-2 relative to the market fundamental, so that with good 1 as the numeraire, the spot price of her abundantly held good 2 is high (as in the market there is much of good-1), then she will pay (will be paid) for the right to trade at the specified market fundamental.

The collateralization structure in this model incorporates both "tranching" and "pyramiding" (see also Geanakoplos, 1997). With "tranching", a specific piece of collateral can be used to back up several contracts as long as their promises to pay are in different states, i.e., no conflicting claims. With"pyramiding", agents are allowed to use financial assets, the contracts for promises to receive goods of others, as collateral for their own promises. One could interpret these contracts as the asset-backed securities which are much in the news these days, e.g., the set of securitized mortgage obligations, promises ultimately but indirectly backed by underlying collateral asset, e.g., houses. What actually gets traded is an equilibrium phenomenon. This is different from the contract-specific collateralization structure as in Geanakoplos (2003), among others, where the collateral of a contract cannot be used as collateral for any other contract. On the other hand, our structure is similar to that of Lustig (2007), where several state-contingent contracts can be backed by the same collateral.

Of course agents are allowed to retrade in spot markets, and that is what delivers the spot-market-clearing prices. However, with pyramiding, agents are indifferent between ex-ante contracting versus retrading in spot markets. This is because anything which can be done in the spot market, trading one good for another, can be done in the ex ante contract market, with promises to receive one good backing promises to surrender the other. Agents do not need to retrade in spot markets, but they may well do so. It is worth noting, however, that most of results in this paper would obtain even without the "pyramiding" assumption. We do so to allow for realism and generality. Even without pyramiding, state-contingent collateral constraints generate an externality. This externality is what we seek to remedy.

Related Literature

This paper is related to the literature on decentralization with autarky as the penalty for reneging, e.g., Kehoe and Levine (1993); Kocherlakota (1996); Alvarez and Jermann (2000). Similar to our model, they allow ex ante complete contracts, and focus on decentralization of constrained optimal allocations. On the other hand, the punishment mechanism is different from ours, as for us defaulting agents will only lose their collateral.

The second class of related literature is on limited borrowing with assets or capital as collateral. Our environment is similar to the environment of Geanakoplos (1997, 2003); Kiyotaki and Moore (1997); Geanakoplos and Zame (2002); Lustig (2007); Kilenthong (2008), among others, but our contractual structure is different. These papers assume a contract-specific collateralization structure while we allow for both "tranching" and "pyramiding". These papers, except Lustig (2007); Kilenthong (2008), follow the tradition of Bewley (1986) and assumes exogenously incomplete markets as part of the specification. Our paper allows for all state-contingent contracts, but with limited collateral, so in this sense our contracts are endogenously incomplete.

This paper also relates to financial accelerator and crises literature, e.g., Bernanke and Gertler (1989); Lamont (1995); Bernanke et al. (1996); Kiyotaki and Moore (1997); Aghion et al. (1999); Krishnamurthy (2003); Cooley et al. (2004); Rampini (2004); Ranciere et al. (2008). These papers use endogenous borrowing constraints as amplification mechanisms. Our paper is closely related to Kiyotaki and Moore (1997) and Krishnamurthy (2003) in that they also focus on collateral constraints. The difference is that our focuses are the efficiency property of a competitive equilibrium and the decentralization of constrained optimal allocations.

There is also a related literature on a pecuniary externality that results from retrading in spot markets or from anonymous trading, when there is some impediment to exchange, (e.g., Jacklin, 1987; Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986; Bisin et al., 2001; Caballero and Krishnamurthy, 2001, 2004; Allen and Gale, 2004; Golosov and Tsyvinski, 2007; Farhi et al., 2009; Lorenzoni, 2008). Similarly to Geanakoplos and Polemarchakis (1986); Caballero and Krishnamurthy (2001); Farhi et al. (2009); Lorenzoni (2008), we are explicit about the source of the externality in our context, but we go on to propose a market-based solution to the problem. This is also related to Makowski (1980); Pesendorfer (1995); Acemoglu and Zilibotti (1997) showing that interaction among firms/intermediaries may not create an efficient set of securities because they do not necessarily internalize complementarities in innovating securities. On the other hand, competitive broker dealers in our model solve an externality problem by creating new securities/commodities costlessly. Our results are robust to the presence of marginal or unit costs, however.

The remaining of the paper proceeds as follows. Section 2 describes the primitive ingredients of the model. We establish the existence of the externality in section 3. Section 4 defines the extended commodity space with lotteries. The constrained feasible allocation and the Pareto program are formulated in this section. Section 5 introduces a competitive equilibrium with lotteries and contracts over market fundamentals. In section 6, the first and second welfare theorems, and the existence theorem, are proved. The collateralization structure, namely tranching and pyramiding, is articulated in section 7. Some properties of equilibrium prices are discussed in section 8. Numerical examples are shown in section 9. Section 10 concludes the paper. Appendix A contain proofs. With limited space, some theoretical results and derivations that are omitted from the main text are presented in the supplementary materials.

2. The Model Economy

This is a two-period economy, $t = 0, 1$. All contracts are traded in period-0, henceforth called the "contracting period". In addition, in period-0, both of two consumption goods can be traded and consumed, and one of them can be saved. All contracts will be executed in period-1, henceforth called the "execution period". There are a finite number S of possible states of nature in period-1, i.e., $s = 1, 2, ..., S$. This allows $S = 1$ so there is only intertemporal trade. Let $0 < \pi_s < 1$ be the objective and commonly assessed probability of state s occurring, where $\sum_s \pi_s = 1$. The two goods can be traded and consumed in each state s. We refer to these as spot markets.

Again there are two goods, called good-1 and good-2. Good-1 cannot be stored (is completely perishable) from $t = 0$ to $t = 1$, while good-2 is storable. The good-2 that can be stored is collateralizable, i.e., can serve as collateral to back promises. Henceforth, good-2 and collateral good will be used interchangeably. Furthermore, good-1 will be the numeraire good in every date and state.

There is a continuum of agents of measure one. The agents are divided into H types, each of which is indexed by $h = \{1, 2, \cdots, H\}$. Each type h consists of $\alpha^h \in (0, 1)$ fraction of the population such that $\sum_h \alpha^h = 1$. Each agent type h is endowed with good-1 and good-2, $\mathbf{e}_0^h = \left(e_{10}^h, e_{20}^h\right)$ in period-0 and $\mathbf{e}_s^h = \left(e_{1s}^h, e_{2s}^h\right)$, in each state $s = 1, \cdots, S$. Let $\mathbf{e}^h = \left(\mathbf{e}_0^h, \cdots, \mathbf{e}_S^h\right)$ be the endowment profile of agent type h over period-0 and all states s in period-1. Heterogeneity of agents originates in part from the endowment profiles $\mathbf{e}^h$. As a notational convention, vectors or matrices will be represented by bold letters.

Let $k^h \in \mathbb{R}_+$ denote the collateral holding (equivalent to the holding of good-2) of an agent type h at the end of period-0. Note that this collateral allocation does not need to be equal to his initial endowment of good-2. In particular, since good-2 can be exchanged in the contracting period (at date $t = 0$), k^h will be equal to the net-position in the collateral good after trading in period-0. The collateral good as legal collateral backing claims is assumed to be kept in escrow, and cannot be taken away either by borrowers or lenders. However, the holding of good-2 can also include normal saving. The storage technology of good-2 whether in collateral or normal saving is linear but potentially with a random return. In some applications, it is natural to treat the returns as a constant, and focus on how collateral interacts with intertemporal trade. In other applications, the risk is in the collateral itself, i.e., what happens if housing values could fall (as if in a small open economy). Each unit of good-2 stored will become R_s units of good-2 in state $s = 1, \cdots, S$. Specifically, storing I units of good-2 at date-0 will deliver $R_s I$ units of good-2 in state s. It is noteworthy that the results in this paper are valid even if the technology R is not random. In most of the exposition, uncertainty originates in the endowment, primarily.

Preferences are identically homothetic. The preferences of agent type h are represented by the utility function $U\left(c_1^h, c_2^h\right) : \mathbb{R}_+^2 \to \mathbb{R}$, where $\left(c_1^h, c_2^h\right)$ are the consumption of good-1 and good-2 of agent h, respectively. Let $0 < \beta \leq 1$ be the common discount factor. The discounted von Neumann-Morgenstern expected utility of h is thus

$$\mathscr{U}^h\left(\mathbf{c}^h\right) \equiv U\left(c_{10}^h, c_{20}^h\right) + \beta \sum_{s=1}^{S} \pi_s U\left(c_{1s}^h, c_{2s}^h\right)$$

where, as with the notation for endowments, $\mathbf{c}^h = \left(\mathbf{c}_0^h, \cdots, \mathbf{c}_S^h\right)$ is the consumption allocation with $\mathbf{c}_0^h \equiv \left(c_{10}^h, c_{20}^h\right) \in \mathbb{R}_+^2$ and $\mathbf{c}_s^h \equiv \left(c_{1s}^h, c_{2s}^h\right) \in \mathbb{R}_+^2$ for $s = 1, \ldots, S$ as the consumption of good-1 and good-2 in period-0, in state s, respectively. The utility function satisfies

Assumption 1. *For each agent type h, common utility function $U\left(c_1^h, c_2^h\right)$ is homothetic, continuous, strictly concave, strictly increasing in both arguments, and satisfies the usual Inada conditions.*

Homotheticity will allow us closed form solutions in the determination of spot prices. Risk aversion with random endowments motivates trade in state-contingent securities. Heterogeneous intertemporal endowments motivates trade in bonds. We will on occasion put superscript h on the utility function for clarity, but preference heterogeneity is not an essential part of what we do here.

2.1. Market Fundamentals

Agents can trade in spot markets in each state s. In principle, the market-clearing prices in these spot markets depend on the distribution of pre-trade (before ex-post spot trade) endowments or the composition of agents. To be precise, let z_s be a *market fundamental* that determines the spot-market-clearing price in state s, and accordingly the *spot-price function* is defined by $p(z_s)$.

With identical homothetic preferences, the aggregate ratio of good-1 to good-2 in state s is the market fundamental in state s; that is, $z_s = \frac{\sum_h \alpha^h e_{1s}^h}{R_s K + \sum_h \alpha^h e_{2s}^h}$, where $K = \sum_h \alpha^h k^h$ is the aggregate (endogenous) saving including collateral. Here then the spot price function can be represented by a single-valued function $p(z_s)$ such that $p(z_s) = p(z_s')$ implies that $z_s = z_s'$. In other words, the market fundamental is necessary and sufficient to pin down the spot price. This ensures that working with spot prices is equivalent to working with market fundamentals. We summarize:

Lemma 1. *With identical homothetic preferences, the market fundamental in state s is given by*

$$z_s = \frac{\sum_h \alpha^h e_{1s}^h}{R_s K + \sum_h \alpha^h e_{2s}^h} \tag{1}$$

Market clearing price $p(z_s)$ is a one-to-one function, i.e. $p(z_s)$ is a single-valued, and $p(z_s) = p(z_s')$ implies that $z_s = z_s'$. In addition, with strictly concavity of $U(\cdot)$, $p(z_s)$ is strictly monotone increasing.

Condition (1) is called a *consistency constraint*. It ensures that the market fundamental is consistently well-defined. That is, $p(z_s)$ is exactly the spot price that constitutes a spot market equilibrium. This is an implication of the homotheticity assumption.

2.2. Collateralization Structure

A specific piece of collateral can be used to back up several contracts as long as their promises to pay are in different states. Thus, there is no conflict in a given state s. This is known as *tranching*. This is distinct from the contract-specific collateralization structure (in Geanakoplos, 2003, among others), in which the collateral of a given security cannot be used as collateral for any other security. For full generality here, we will consider state-contingent securities as the primitives and otherwise let the security structure be endogenous. Accordingly, we focus on securities paying in each state s with market fundamental z_s, one at a time.

A (contingent) security promising to pay one unit of good 1 in period-1 and state s with $\widehat{C}$ units of good 2 as collateral is a promise to pay a unit of good 1 if the state of nature is s and nothing otherwise. For notational convenience, we use $\widehat{}$ to distinguish securities paying in good 1, the numeraire, from securities paying in good 2. With limited commitment, that is, allowing default, the payoff of this security is given by

$$\widehat{D} = \begin{cases} \min\left(1, \widehat{C} R_s p(z_s)\right) & \text{if state is } s \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where this payoff is in units of good 1 in period $t = 1$, and $p(z_s)$ is the price of good 2 (in units of good 1) in state s. Note that this defaulting condition depends on the spot price $p(z_s)$.

Equation (2) captures the essence of limited commitment problem in this model. The issuer or "borrower" in period $t = 0$ may not wish to honor the state-contingent obligation. This creates the limited commitment problem; that is, she will keep the promise if that promise is no larger than the value of the collateral, i.e., $1 \leq \widehat{C} R_s p(z_s)$, and will "default" otherwise, $\widehat{C} R_s p(z_s) < 1$. In case of default, the payoff of the contract in state s is equal to the value of its collateral in that state, $\widehat{C} R_s p(z_s)$ units of good 1.

We will first show that there is no loss of generality in restricting attention to securities without default, and also in excluding over-collateralized securities, whose collateral value is strictly larger than the promise. We prove the result for a security paying in good 1 in state s with good 2 as collateral. Then, we will argue that the same logic applies for all other types of securities.

Now consider a contingent security *that will be in default* in state s, with collateral $\widehat{C} < \frac{1}{R_s p(z_s)}$. That is, an issuer of this security will "default" in state s. Hence, according to condition (2), the payoff of this security (in units of good 1) in state s is

$$\min\left(1, \widehat{C} R_s p(z_s)\right) = \widehat{C} R_s p(z_s) < 1 \tag{3}$$

We now argue that there is an alternative security that does not default but generates exactly the same total payoffs using the same amount of collateral overall. Consider a state-s contingent security with collateral amount $\widetilde{C} = \frac{1}{R_s p(z_s)}$. This security will not default. It is straightforward to show that the payoff of this security is one unit of good 1 in state s. Now consider $\widehat{C} R_s p(z_s)$ units of the *alternative* security. That collection of securities pays in state s one per unit or $\widehat{C} R_s p(z_s)$ in total. This is exactly the same as the payoff of the original security with default: see (3). In addition, the total collateral for $\widehat{C} R_s p(z_s)$ units of the alternative security with $\frac{1}{R_s p(z_s)}$ collateral per unit is $\widehat{C}$, which is exactly the same as the collateral level of the original security. Therefore, the alternative security can generate the same payoffs using the same total amount of collateral but without default.

A similar argument also applies to all other types of securities, including directly-collateralized securities paying in good-2 rather than good-1, and asset-backed securities, which will be precisely defined later. For now define an asset-backed security as a contract backed by *promises to receive* goods of others. This is termed *pyramiding*. The details of asset-backed securities will be discussed in Section 7.

The discussion is summarized in the following lemma.

Lemma 2. *For any state-contingent security, there exists a security with* no default *that can generate the* same total payoffs *using the* same amount of collateral.

Thus, there is no loss of generality in restricting attention to no-default securities only. Further, issuing securities that do default requires no less collateral than (an equivalent set of) securities that do not. In other words and this may seem counterintuitive, securities with default, i.e., with little collateral, do not economize on collateral. In addition, we also show in the supplementary materials that default cannot make collateral constraints, formally defined below, less binding.

In addition, with perfectly divisible collateral, there is no loss of generality in excluding over-collateralized securities, whose collateral value is strictly larger than the promise. More precisely, an over-collateralized security paying in good-1 in state s is a contract with a collateral $\widehat{C}$ such that $\widehat{C} R_s p(z_s) > 1$. The payoff of this security in state s is 1. This security is equivalent to a no-default security with $\frac{1}{R_s p(z_s)} < \widehat{C}$ units of good-2 as collateral, whose payoff in state s is also 1. A similar result applies to other types of securities as well.

It is worthy of emphasis, however, that own saving should not be interpreted as over-collateralization, as no securities are acquired from others; that is, each agent can save. This saving will result in the slackness of the collateral constraint (4) defined below. In particular, an agent may hold at the end of period-0 more collateral good than the (minimum) amount needed to collateralize all securities issued.

3. Collateral Constraints and Externality

As will be proved in Section 7 (see Lemma 4), there is no loss of generality in considering only two classes of securities; (i) $\hat{\theta}^h_s$ - securities paying in good 1 in state s, (ii) θ^h_s - securities paying in good 2 in state s. Here a positive number denotes the purchaser or holder , and negative the issuer. When negative, each of the state-contingent securities must be backed by the issuer either by good-2 or by purchased assets (other people's promises). In other words, $\hat{\theta}_s^h$ and θ_s^h include both directly collateralized and asset-backed securities. As well be established in Section 7, the collateral constraints for an agent type h take the intuitive form

$$p(z_s)R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h \quad \geq \quad 0, \ \forall s \tag{4}$$

The collateral constraint (4) states that, for each state s, the net-value of all assets, including collateral good and securities, must be non-negative. If $\hat{\theta}_s^h$ and θ_s^h were negative, we could write this as $p(z_s)R_s k^h \geq -\hat{\theta}_s^h - p(z_s)\theta_s^h$. That is, there is sufficient collateral in value in state s to honor the value of all promises. Since $\hat{\theta}_s^h$ and θ_s^h include asset-backed and directly backed securities, collateral types per se do not matter. In addition, we can show that the markets economize on collateral; that is, there is no gain from pooling collateral across agents type h (see Lemma 9 in Appendix A).

The collateral constraints (4) can be written in consumption space as follows. Suppose for the moment that securities are such that there is no spot trade in equilibrium (see Lemma 5). The consumption for an agent type h in state s is given by

$$c_{1s}^h \quad = \quad e_{1s}^h + \hat{\theta}_s^h \tag{5}$$
$$c_{2s}^h \quad = \quad e_{2s}^h + R_s k^h + \theta_s^h \tag{6}$$

Substituting these two equations into the collateral constraint (4) yields

$$c_{1s}^h + p(z_s)c_{2s}^h \quad \geq \quad e_{1s}^h + p(z_s)e_{2s}^h \tag{7}$$

This condition implies that, due to limited commitment and the possibility of default, the market value of consumption in a state s of an agent cannot be lower than the market value of her endowment (without collateral k^h) in the same state (related to Kehoe and Levine, 1993; Golosov and Tsyvinski, 2007, among others). Intuitively, if this constraint were to be violated, an agent type h would have promised to deliver some part of the value of her endowments, over and above her consumption, but such promises require collateral.

The interaction between the bindingness of collateral constraints and spot prices generates an externality. Technically, there is an externality because the consumption feasibility set of an agent type h depends on other agents' choices of saving $k^{\tilde{h}}$ through the spot price. This dependency results from the collateral constraints[1] (7), or borrowing constraints in general. If there were no collateral constraint, the consumption feasibility set would be independent of other agents' choices (and therefore there would be no externality). Intuitively, an infinitesimal agent has no influence on aggregate saving. On the other hand, a

[1]This is true even we allow for directly-collateralized securities paying in good 2, ψ_s^h only. As precisely described in Section 7, each agent will also face a spot trade constraint:

$$\tau_{1s}^h + p(z_s)\tau_{2s}^h = 0$$

where τ_{is}^h is the amount of good i an agent h received from trading in the spot market in state s. Combining this spot trade constraint with (4), (5), and (6) leads to the collateral constraint (7).

constrained planner knows she can influence the resource allocation in period 1 through period 0 assignments, namely saving. The asymmetry between the influence of the planner versus induced agents generates an inefficiency when a collateral constraint is binding. We now present the formal statement below. For simplicity, we focus on identical allocations for each type.

3.1. Collateral Constrained Optimality

Attainable allocations are those that can be achieved by exchanges of securities and collateral in date 0 and exchanges of consumption goods in date 1 at state s, respecting spot prices $p(z_s)$. Accordingly, attainable allocations are defined using the spot-price function $p(z_s)$. As will be later proved in this section (see Lemma 5), the asset-backed securities in this model are simply substitutes for spot markets. Henceforth, we let asset-backed securities play this role and shut down active trade in spot markets.

Definition 1. An allocation $\left(\mathbf{c}_0^h, k^h, \hat{\theta}_s^h, \theta_s^h\right)_h$ is *attainable* if

(i) it satisfies resource constraints:

$$\sum_h \alpha^h c_{10}^h \leq \sum_h \alpha^h e_{10}^h \tag{8}$$

$$\sum_h \alpha^h \left[c_{20}^h + k^h\right] \leq \sum_h \alpha^h e_{20}^h \tag{9}$$

$$\sum_h \alpha^h \hat{\theta}_s^h = 0, \ \forall s \tag{10}$$

$$\sum_h \alpha^h \theta_s^h = 0, \ \forall s \tag{11}$$

(ii) for each h, it satisfies the collateral constraints (4),

(iii) the consistency constraints (1) hold for all s.

With non-constant, s-contingent spot-price function, the attainable set is non-convex. The non-constant price condition is typical. For instance, this is the case with identical homothetic and strictly concave preferences. The main source of the non-convexity is the product of spot-price function and the sum of collateral and contract allocations, $p(z_s)\left(R_s k^h + \theta_s^h\right)$, in the collateral constraints (4).

Lemma 3. *With identical homothetic and concave preferences, the attainable set is non-convex.*

A constrained optimal allocation is characterized using the following planner's problem. Let $\overline{\mathscr{U}}^h$ be the reservation utility level for an agent type h.

Definition 2. The Pareto Program without Lotteries:

$$\max_{\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)_h} U\left(c_{10}^1, c_{20}^1\right) + \beta \sum_s \pi_s U\left(e_{1s}^1 + \hat{\theta}_s^1, e_{2s}^1 + R_s k^1 + \theta_s^1\right) \tag{12}$$

subject to (1), (4), (8)-(11), the participation constraint for each $h = 2, \cdots, H$,

$$U\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s U\left(e_{1s}^h + \hat{\theta}_s^h, e_{2s}^h + R_s k^h + \theta_s^h\right) \quad \geq \quad \overline{\mathscr{U}}^h, \tag{13}$$

and non-negativity constraints for consumption and collateral allocations.

For expositional reasons, we focus only on interior solutions; that is, the non-negativity constraint for k^h is neglected without loss of generality here. Let μ_{cc-s}^h, and $\mu_{\bar{u}}^h$ denote the Lagrange multipliers for the collateral constraint (4) for agent h in state s, and for the participation constraint (13) for agent h, respectively. For notational convenience, let $\mu_{\bar{u}}^1 = 1$. A necessary condition[2] for constrained optimality related to collateral allocation k^h is given by, for any h,

$$\frac{U_{20}^h}{U_{10}^h} = \sum_s \pi_s \beta \frac{U_{2s}^h}{U_{10}^h} R_s \quad + \quad \sum_s \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h U_{10}^h} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu_{\bar{u}}^h U_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}} \tag{14}$$

where $U_{i0}^h = \frac{\partial U^h\left(c_{10}^h, c_{20}^h\right)}{\partial c_{i0}}$, $U_{is}^h = \frac{\partial U^h\left(c_{1s}^h, c_{2s}^h\right)}{\partial c_{is}}$ for $i = 1, 2$, $p'(z_s) = \frac{\partial p(z_s)}{\partial z_s}$, and $K = \sum_h \alpha^h k^h$. See the derivation in the proof of Lemma 10 in the supplementary materials.

Of special interest, the last term depends not only on the bindingness of collateral constraints for h but also the bindingness of other agents' collateral constraints. This implies that if an agent's collateral constraint is binding, it will impact everyone. This is the source of the externality.

3.2. Collateral Equilibrium

Let $\widehat{P}_{as}$ and P_{as} be the prices of securities paying in good 1 and in good 2 in state s, respectively. For notational convenience, we denote the vectors of security prices by $\widehat{P}_a = \left(\widehat{P}_{as}\right)_s$ and $P_a = (P_{as})_s$. A collateral equilibrium is defined:

Definition 3. A collateral equilibrium is a specification of prices of good 2 in period-0, P_{20}, the prices of securities paying in good 1, $\widehat{P}_a$, and the prices of securities paying in good 2, P_a, the spot price of good 2 in each state s, $p(z_s)$, and an allocation $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)_h$ such that

(i) taking prices as given, for any h, $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)$ solves

$$\max_{\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h\right)} U\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s U\left(e_{1s}^h + \hat{\theta}_s^h, e_{2s}^h + R_s k^h + \theta_s^h\right) \tag{15}$$

subject to the collateral constraints (4), and the budget constraint at $t = 0$:

$$c_{10}^h + P_{20}\left(c_{20}^h + k^h\right) + \widehat{P}_a \cdot \hat{\theta}^h + P_a \cdot \theta^h \quad \leq \quad e_{10}^h + P_{20} e_{20}^h \tag{16}$$

[2]Given that the constraint set is not convex (Lemma 3), this optimality condition is necessary but may not be sufficient. Nevertheless, this does not cause any problem to our externality argument, as we simply need to show that a collateral equilibrium cannot be constrained optimal, i.e. does not satisfy the necessary optimal condition (14).

(ii) all markets clear: (8)-(11) hold,

(iii) the consistency constraints (1) hold for all s.

The necessary optimal condition for a collateral equilibrium that is comparable to the optimal condition for a constrained optimality (14) is given by, for any h,

$$P_{20} = \frac{U_{20}^h}{U_{10}^h} = \sum_s \pi_s \beta \frac{U_{2s}^h}{U_{10}^h} R_s + \sum_s \frac{\gamma_{cc-s}^h}{U_{10}^h} p(z_s) R_s \tag{17}$$

where γ_{cc-s}^h is the Lagrange multiplier for the collateral constraint of contracts paying in state s for agent h. See the derivation in the proof of Lemma 10 in the supplementary materials.

3.3. The Externality

Note that an infinitesimal agent takes a spot price, $p(z_s)$, as invariant. To the contrary, the constrained planner can influence the spot prices $p(z_s)$ through collateral assignments, k^h, for the agents of type h in period-0, which affect the market fundamentals z_s. This key influence is the term in $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K}$. The difference between the impact of the planner and that of the agents creates the externality and causes an inefficiency.

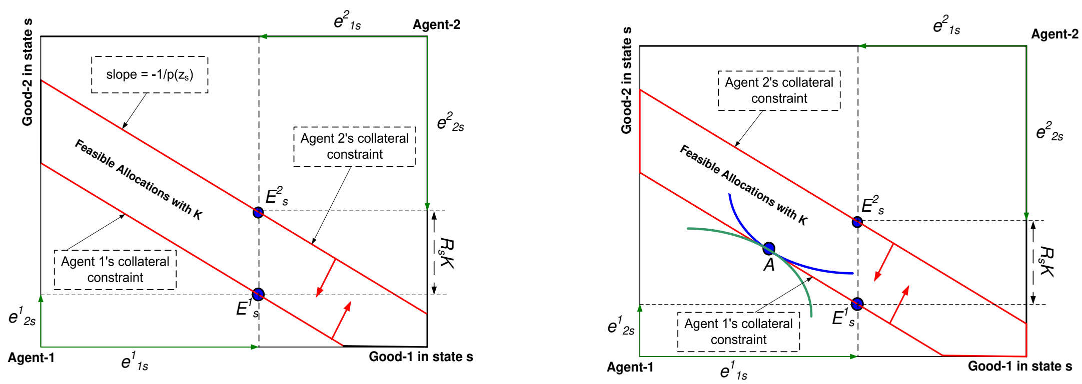
Note that if the last term in (14) is zero and we set $\gamma_{cc-s}^h = \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h}$, then condition (17) is exactly the same as (14). On the other hand, if the last term in (14), $\sum_s \frac{\alpha^h}{\mu_{\bar{u}}^h U_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \hat{\theta}_s^{\tilde{h}}$, is positive, as in all numerical examples in this paper, the equilibrium price of good-2 in period-0 will be too high in the collateral equilibrium relative to the (constrained) optimal one. Intuitively, the planner can do better by lowering the aggregate saving or collateral (see Example 1).

The last term in (14) could be zero if either $\mu_{cc-s}^{\tilde{h}} = 0$ for all $\tilde{h}$ or $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} = 0$. With a strictly concave utility function, the spot price varies with the market fundamental[3] (is not constant), i.e. $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \neq 0$. As a result, when the collateral constraints are binding, i.e., $\mu_{cc-s}^{\tilde{h}} > 0$ for some $\tilde{h}$, the last term in (14) will be non-zero in general. With this non-zero term, a collateral equilibrium will not be constrained efficient. It is the interaction between the bindingness of collateral constraints and spot prices that is the key. As an exceptional case, a collateral equilibrium could be a full first-best optimum. That is, the environment could be such that despite the focus of the paper we could ignore the collateral constraint. But, otherwise, the collateral equilibrium must be constrained *sub*optimal. The result is summarized in the following theorem. The proof is in the supplementary materials. In particular, we argue that a collateral equilibrium is first-best optimal when all collateral constraints are slack.

[3] If the utility function is linear in both goods, then the spot price is constant, i.e. $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} = 0$. This clearly results in constrained efficiency. Similarly, if the amount of aggregate saving is fixed exogenously, then the market fundamental (the ratio of good-1 to good-2) is fixed. This also implies that the last term is zero, and so constrained efficiency.

Theorem 3.1. *If a collateral equilibrium is not first-best optimal, then it is constrained* suboptimal.

Intuitively, consider an economy with two representative agents. Let $K = k^1 + k^2$ be the total saving. See figure 1a as an example of an attainable set in state s. $E_s^h = \left(e_{1s}^h, e_{2s}^h\right)$ is the endowment of agent h in state s. Note that when $K > 0$, the endowments points of the two agents will not coincide, and the vertical distance between them is $R_s K$. Indeed, it is as if agent 2 were holding all the collateral. The collateral-constrained attainable set is represented by the area between the parallel lines. The lower (upper) border of the attainable set corresponds to the bindingness of agent 1's (agent 2's) collateral constraint (7). As a result, when agent 1's collateral constraint is binding, the resulting consumption allocation will be on the lower border (e.g., point A in figure 1b), and vice versa.

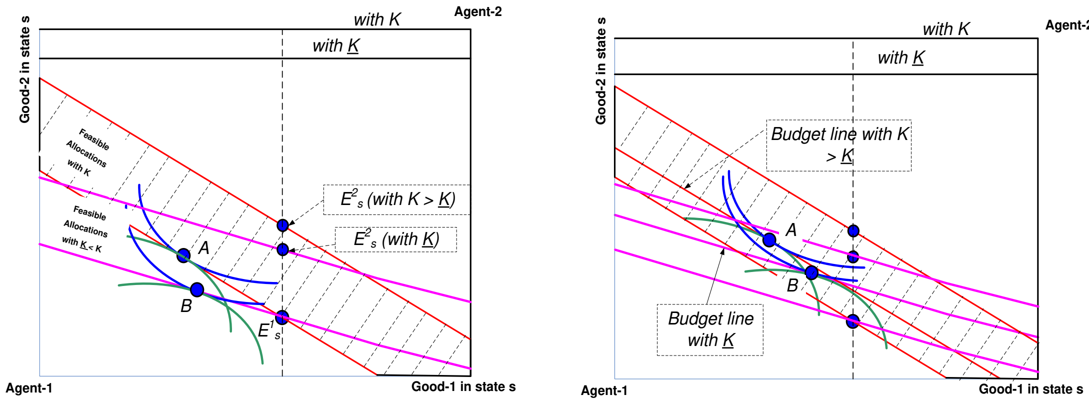


(a) (b)

Figure 1: Collateral-constrained attainable sets. (a) Attainable allocations (the shaded area) depends on the spot price $p(z_s)$ which depends on total saving K. (b) The resulting consumption allocation will be on the lower border of the attainable set when agent 1's collateral constraint is binding.

Figure 2a illustrates that different spot prices lead to different attainable sets. Note that lower total saving ($\underline{K} < K$) implies lower slope of the boundaries of the attainable set. This is because lower total saving implies higher spot price of good 2 relative to good 1, which in turn implies that the slope, $-\frac{1}{p(z_s)}$, is lower in absolute value. As shown in the figure, when two attainable sets with different spot prices are overlapped, they are not contained in one another. Hence, when agent 1's collateral constraint is binding, the marginal change in total saving (say, from K to $\underline{K}$) could lead to a consumption allocation that is not attainable under the original level of total saving (the graph features a discrete change moving from point A to new point B which is not attainable with the overall total saving K). This marginal effect, therefore, is not priced in the collateral equilibrium considered thus far. That is, there is an externality.

On the other hand, if none of the collateral constraints were binding, this dependency on the spot price will not generate the externality. See figure 2b. With no binding collateral constraints, the resulting consumption allocations will be strictly inside (interior of) the



(a) (b)

Figure 2: Collateral-constrained feasible sets. (a) Different levels of (endogenous) aggregate saving imply different (overlapped) feasible sets. When agent 1's collateral constraint is binding, changing total saving from K to $\underline{K}$ moves the resulting consumption from A to B which is not feasible with K. (b) When none of the collateral constraints is binding, changing total saving from K to $\underline{K}$ moves the resulting consumption from A to B, which is feasible with K.

attainable set. This implies that its neighborhood is also in the attainable set. As a result, the marginal change in total saving will lead to the resulting consumption allocation that is still feasible under the original level of total saving (A and B are both feasible with total saving K). This implies that the collateral equilibrium prices out this marginal effect. Hence, there is no externality in this case.

4. Internalizing The Externality: The Economy with "Price-Islands"

Let $\mathbf{z} = (z_s)_{s=1}^S$ denotes a vector of the market fundamentals in all states, each of which is called a *price-island* in state s. The composition of agents determines the market fundamental. We can go further and interpret a price-island z_s as a *segregated exchange* institution in which the composition of agents forms in such a way as to deliver the market fundamental z_s.

Being in island-z_s in state s means that an agent h can trade in spot markets at spot price $p(z_s)$, as determined by the market fundamental z_s. Equivalently, even if the spot markets were shut down[4], an agent on an island z_s can accomplish the same thing by trading in ex-ante securities $(\hat{\theta}_s, \theta_s)$, that is, trading in segregated security exchanges, in which the prices depend on z_s.

[4]As proved in Lemma 5 in Section 7, the spot markets are redundant in that agents are indifferent between trading in ex-ante contracts or in spot markets. Importantly, the spot markets are opened.

Let $\Delta_s \in \mathbb{R}$ define "type h's *deviation from the market fundamental*[5]" in state s.

$$\Delta_s = z_s \left(e^h_{2s} + R_s k^h\right) - e^h_{1s} = \left(e^h_{2s} + R_s k^h\right)\left(z_s - \frac{e^h_{1s}}{e^h_{2s} + R_s k^h}\right), \ \forall s \tag{18}$$

Note that if $\Delta_s = 0$, then $\frac{e^h_{1s}}{e^h_{2s}+R_s k^h} = z_s$ and type h's pre-trade endowment is exactly equal to the market fundamental. If $\Delta_s > 0$, then type h holds a relative low amount of good-1 and abundant amount of good-2, that is, relative to z_s. Adding one unit of good-2 (via collateral k) adds to the deviation Δ_s by exactly $z_s R_s$ (see Eq.(18)). This is the same for all agent types. But note also that there is a part of Δ_s over which h has no control, namely her endowments. The "type h deviation from the fundamental" will be priced in a competitive equilibrium. In addition, the price of the right to trade in each island will be proportional to the "type h deviation from the fundamental".

Now suppose it were possible to assign agents to different islands even in state s as if by a lottery. Security trades are also bundled into this potentially random assignment. Island assignments, by lottery or not, are still state-contingent. Importantly, a resident of an island-z_s can trade securities with other residents in the same island only. Securities are executed at spot prices within each island only.

Thus, for each agent type h, let $x^h(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta) \geq 0$ denote a probability measure on $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, where Δ_s satisfies (18) for all s. In other words, $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ is the probability of receiving period-0 consumption, $\mathbf{c}_0 \equiv (c_{10}, c_{20})$, collateral, k, securities paying in good-1, $\hat{\theta}_s$, securities paying in good-2, θ_s, and being in island-z_s in state s where all securities are executed and all spot-trade takes place also. Recall that a positive (negative) amount of trade means receiving (transferring out) the specified good.

As a probability measure, a lottery of an agent type h satisfies

$$\sum_{(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta)} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = 1 \tag{19}$$

With a continuum of agents, $x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ can be interpreted as the fraction of agents type h assigned to a bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. More formally, with all choice object gridded up as an approximation, the commodity space L is assumed to be a finite n-dimensional linear space[6].

For notational purposes, let $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ be a typical commodity, called a bundle. We will use b and $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ interchangeably. Accordingly, we can write $\mathbf{x}^h \equiv \left[x^h(b)\right]_b \in \mathbb{R}^n_+$ as a typical lottery for an agent type h.

[5]If we were in the underlying spaces of k and z_s, they would enter multiplicatively, hence and so we would have a non convexity problem. This is not a problem with lotteries, however.

[6]The limiting arguments under weak-topology used in Prescott and Townsend (1984a) can be applied to establish the results if L is not finite.

4.1. Consumption Possibility Set

A holder of a bundle $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ will receive k units of collateral and hold portfolio of securities $\left(\hat{\theta}, \theta\right)$. With limited commitment, each bundle b will be feasible only if the collateral and security assignments satisfy the collateral constraint (4) which we repeat here:

$$p\left(z_s\right) R_s k + \hat{\theta}_s + p\left(z_s\right)\theta_s \geq 0,\ \forall s \tag{20}$$

Accordingly, we impose the following condition on a probability measure $x^h(b)$.

$$\begin{aligned} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) &\geq 0 \text{ if } \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \text{ satisfies (18) and (20)} \\ &= 0 \text{ if otherwise} \end{aligned} \tag{21}$$

In words, a positive measure can be defined only on feasible bundles, which have to satisfy conditions (18) and (20). More formally, the consumption possibility set of an agent type h is defined by

$$X^h = \left\{\mathbf{x}^h \in \mathbb{R}^n_+ : \sum_b x^h(b) = 1, \text{ and for any } b, x^h(b) \text{ satisfies (21)}\right\} \tag{22}$$

Let $\mathbf{x}^h$ be a typical element of X^h. Note that $X^h \subset L$ is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one on each agent's endowment.

4.2. Attainable Allocations

An attainable allocation with lotteries is defined in an analogous manner to the ones without lotteries. In particular, an allocation $\mathbf{x} \equiv \left(\mathbf{x}^h\right)_h$ is attainable if $\mathbf{x}^h \in X^h$ for all h, and it satisfies the following feasibility constraints.

Recall that good-1 cannot be stored; only good-2 is storable. The aggregate endowment of good-1 in period-0 is $\sum_h \alpha^h e^h_{10}$. Therefore, the resource constraint for good-1 in period-0 is given by

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} \alpha^h x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{10} \leq \sum_h \alpha^h e^h_{10} \tag{23}$$

Similarly, the feasibility constraint for good-2 in period-0 is given by

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} \alpha^h x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \left[c_{20} + k\right] \leq \sum_h \alpha^h e^h_{20} \tag{24}$$

Note that the nonnegativity constraint on k guarantees that the aggregate saving is non-negative.

Recall that all securities are executed within each assigned island only. In particular, for an island-z_s in state s, the net supply of a security paying in good-1 in state s, $\hat{\theta}_s$ must be zero

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \hat{\theta}_s = 0\ , \forall s, z_s \tag{25}$$

where $\mathbf{z}_{-s} = (z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_S)$ is a vector of market fundamentals in all states but state s. This feasibility condition holds for every state s and every island z_s. Similarly, the feasibility or market-clearing constraint for a security paying in good-2 is given by

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \theta_s = 0 \ , \forall s, z_s \tag{26}$$

Similar to the economy without lotteries, the market fundamental in each island must be consistent. In other words, the planner will choose the composition of agents to set the market fundamental for each island to its specified level. With identical homothetic preferences, the consistency constraint for an island-z_s is that the aggregate ratio of good-1 to good-2 within the island-z_s be exactly z_s:

$$z_s = \frac{\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \left(e_{1s}^h + \hat{\theta}_s\right)}{\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \left(e_{2s}^h + R_s k + \theta_s\right)} \tag{27}$$

Using the feasibility conditions for securities within each island, (25)-(26) and the definition of "type h deviation from the fundamental" (18), these consistency constraints can be rewritten as

$$\sum_h \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}_{-s}, z_s, \Delta\right) \Delta_s \quad = \quad 0, \ \forall s, z_s \tag{28}$$

where the last equation follows the definition of Δ_s.

Definition 4. An allocation $\mathbf{x} \equiv \left(\mathbf{x}^h\right)_{h=1}^H \in X^1 \times \ldots \times X^H$ is said to be *attainable* if $\mathbf{x}^h \in X^h$ for every h, and it satisfies (23)-(26) and (28).

Let X denote the set of all attainable allocations. With finite linear weak-inequality constraints, the attainable set X is compact and convex. In addition, the assumption that the endowment is on the grids also ensures that X is nonempty.

4.3. Constrained Optimal Allocations

A constrained optimal allocation is an attainable allocation such that there is no other attainable allocation that can make at least one agent type strictly better off without making any other agent type worse off. To be precise, the expected utility of an agent type h, holding a lottery $\mathbf{x}^h$, is given by

$$\mathscr{U}^h\left(\mathbf{x}^h\right) = \sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \left\{U(c_{10}, c_{20}) + \beta V^h \left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right\}$$

where $V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right) = \sum_s \pi_s U\left(e_{1s}^h + \hat{\theta}_s, e_{2s}^h + R_s k + \theta_s\right)$ is the "indirect" utility of agent h that is derived from a bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. This indirect utility is the result of the assignment of the bundle that is executed in period-1 over state s. In other words, it summarizes all actions happening to the holder of the bundle in period-1. In each state s, a holder type h receives $\hat{\theta}_s$ units of good-1 as the net-payment from portfolio $\hat{\theta}$, θ_s units of good-2 as the net-payment from portfolio θ, $R_s k$ units of good-2 from the collateral good, and also e^h_{1s} units of good-1 and e^h_{2s} units of good-2 as endowments.

Definition 5. An attainable allocation $\mathbf{x}^* \in X$ is said to be a *constrained optimal* allocation if there is no another attainable allocation $\mathbf{x} \in X$ such that

$$\mathscr{U}^h\left(\mathbf{x}^h\right) \geq \mathscr{U}^h\left(\mathbf{x}^{*h}\right) \text{ for every } h \text{, and } \mathscr{U}^{\bar{h}}\left(\mathbf{x}^{\bar{h}}\right) > \mathscr{U}^{\bar{h}}\left(\mathbf{x}^{*\bar{h}}\right) \text{ for some } \bar{h}$$

We characterize constrained optimality using the following Pareto program. Let $\lambda^h \geq 0$ be the Pareto weight of agent type h. There is no loss of generality to normalize the weights such that $\sum_h \lambda^h = 1$. A constrained Pareto optimal allocation $\mathbf{x}^*$ solves the following Pareto program.

Definition 6. The Pareto Program with Lotteries:

$$\max_{\left(\mathbf{x}^h \in X^h\right)_h} \sum_h \lambda^h \alpha^h \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} x^h(b)\left\{U^h(c_{10}, c_{20}) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right\} \tag{29}$$

subject to (23)-(26), (28).

Note again that we already embedded the collateral constraints (20) and the "individual deviations from the fundamental" (18) into the consumption possibility sets X^h.

It is clear that the objective function now is linear in x^h. Thereby it is continuous and weakly concave. As discussed earlier, the feasible set X is non-empty, compact, and convex. Therefore, a solution to the Pareto program for given positive Pareto weights exists and is a global maximum. The proof of the equivalence between Pareto optimal allocations and the solutions to the program is omitted for brevity (see Prescott and Townsend, 1984b, for a similar proof).

5. Decentralized Equilibrium

Let P_{20} be the price of good-2 in period-0, and $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ be the price of a bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. Note that the price of good-1 in period-0 is $P_{10} = 1$ as good-1 is the numeraire good. Each agent is infinitesimally small relative to the entire economy and will take all prices as given. The broker-dealers introduced below will also act competitively. Note as well that Δ is also priced.

Consumers: Each agent h, taking prices, $P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, as given, chooses $\mathbf{x}^h$ in period $t = 0$ to maximize its expected utility:

$$\max_{\mathbf{x}^h} \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} x^h\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)\left\{U(c_{10}, c_{20}) + \beta V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)\right\} \tag{30}$$

subject to $\mathbf{x}^h \in X^h$, and period-0 budget constraint

$$e_{10}^h + P_{20} e_{20}^h \geq \sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \tag{31}$$

The period-0 budget constraint (31) states that the agent sells all her endowments[7] including good-2 at price P_{20} and uses this income to buy lotteries $\mathbf{x}^h$, which includes consumption in period-0, $\left(c_{10}^h, c_{20}^h\right)$.

In state-s, a type-h holder of bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ receives in addition to her endowments of good-1 and good-2, $\left(e_{1s}^h, e_{2s}^h\right)$, $\hat{\theta}_s$ units of good-1 as the net-payment of portfolio $\hat{\theta}$, $R_s k$ units of good-2 from the collateral good, θ_s units of good-2 as the net-payment of portfolio θ. Of course, if $\hat{\theta}_s$ and θ_s are negative, these are promise to pay. Again, the result of the actions in period-1 is summarized in $V^h\left(k, \hat{\theta}, \theta, \mathbf{z}\right)$. It is worthy of emphasis that the agent will reside in island z_s, where she can in principle trade good-1 and good-2 at price $p(z_s)$ in spot markets. (Again in the equilibrium under consideration it will not be necessary to trade even though they believe they could.)

Broker-Dealers: The primary role of a broker-dealer is to put together deals, i.e., buying consumption goods and collateral and selling the bundle, including securities backed by collateral. In order to do so, the broker-dealer issues (sells) $y(b) \in \mathbb{R}_+$ units of each bundle $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, at the unit price $P(b)$. Note that the broker-dealer can issue any non-negative number of a bundle b; that is, the number of bundles issued does not have to be between zero and one and is not a lottery. It is simply the number of bundles, a real number. Let $\mathbf{y} \in L$ be the vector of the number of bundles issued as one move across b. With constant returns to scale, the profit of a broker-dealer must be zero and the number of broker-dealers becomes irrelevant. Therefore, without loss of generality, we assume there is one representative broker-dealer, which takes prices as given.

By issuing or selling bundle $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$, the broker-dealer promises to deliver k units of collateral good to a holder, i.e., provide collateral to back promises. In order to do so, the broker-dealer needs to acquire a sufficient amount of collateral. In particular, it buys I units of good-2 at price P_{20} (in terms of good-1) in period-0, and distributes it according to $\mathbf{y}$.

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) k = I \tag{32}$$

This constraint states that the broker-dealer uses I as collateral for the promises to deliver the collateral good.

Similarly, the broker-dealer will also deliver (c_{10}, c_{20}) units of good-1 and good-2 to a holder of a bundle $b = \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. In order to do so, the broker-dealer acquires C_1

[7] It is worthy of emphasis that we can write an equivalent problem specifying consumption transfers in period-0, instead of consumption allocation. By doing so, agents do not need to sell their entire endowments but simply buy and sell consumption transfers. In other words, it is not restrictive that we make agents sell their entire endowments and buy consumption allocation through lotteries.

units of good-1 and C_2 units of good-2, and distributes them according to $\mathbf{y}$:

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)c_{10} \;=\; C_1 \tag{33}$$

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)} y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)c_{20} \;=\; C_2 \tag{34}$$

In conclusion, the total (market) cost to the broker-dealer from buying good-1 and good-2 for collateral backing and consumption is equal to $C_1 + P_{20}C_2 + P_{20}I$.

Furthermore, it also delivers or assigns a portfolio of claims or securities $\left(\hat{\theta},\theta\right)$ to the holder. We do not require the broker-dealer to offer collateral backing for securities delivered to consumers, $\hat{\theta},\theta < 0$, and done with collateral k as that is up to the consumers themselves. Neither does the broker-dealer itself hold any securities or net position, but simply distributes securities according to $\mathbf{y}$. Hence, the net-supply of each contract from the broker-dealer must be zero:

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\hat{\theta}_s \;=\; 0,\ \forall s,z_s \tag{35}$$

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\theta_s \;=\; 0,\ \forall s,z_s \tag{36}$$

As we shall see below, these constraints will be equivalent to the market-clearing constraints for contracts in the competitive equilibrium with $y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right) = \sum_h \alpha^h x^h\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$, essential the market-clearing condition for lotteries.

The broker-dealer's technology also requires that the sum of all "type h deviations from the fundamental" must be zero in each island-z_s for every state s:

$$\sum_{\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},\Delta\right)} y\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z}_{-s},z_s,\Delta\right)\Delta_s \;=\; 0\ ,\forall s,z_s \tag{37}$$

This constraint is the counter part of the consistency constraint (28) in the Pareto program. In particular, using the market-clearing condition for lotteries, we can show that this consistency constraint is identical to (28). Hence, it is also called the consistency constraint for an island-z_s for every state s.

The objective of the broker-dealer is to maximize its profit by choosing $(\mathbf{y},C_1,C_2,I)$, taking prices, $P_{20}, P\left(\mathbf{c}_0,k,\hat{\theta},\theta,\mathbf{z},\Delta\right)$, as given:

$$\begin{array}{ll}\max\limits_{(\mathbf{y},C_1,C_2,I)} & \sum\limits_b y\left(b\right)P\left(b\right) - \left[C_1 + P_{20}C_2 + P_{20}I\right] \\ \text{s.t.} & (32)-(37)\end{array} \tag{38}$$

where the first term is the total revenue of the broker-dealer and the bracketed term denotes its total (market) cost.

The existence of an optimum to the broker-dealer's problem requires, that for any bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$,

$$P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \le c_{10} + P_{20}c_{20} + P_{20}k + \widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta \tag{39}$$

where $\widehat{P}_a(\mathbf{z}) \equiv \left(\widehat{P}_a(z_s, s)\right)_{s=1}^{S}$, $P_a(\mathbf{z}) \equiv (P_a(z_s, s))_{s=1}^{S}$, and $P_\Delta(\mathbf{z}) \equiv (P_\Delta(z_s, s))_{s=1}^{S}$ are the vectors of Lagrange multipliers for the market-clearing constraints for contracts paying in good-1 (35), for the market-clearing constraints for contracts paying in good-2 (36), and for consistency constraints (37), respectively. In particular, for an island z_s in state s, $\widehat{P}_a(z_s, s)$, $P_a(z_s, s)$, and $P_\Delta(z_s, s)$ are the shadow prices of a contract paying in good-1 and good-2, respectively, and the shadow price of "type h deviations from the fundamental" in the island-z_s. Condition (39) holds with equality if $y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) > 0$. Here $P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ is the revenue from the sale of one unit of bundle $\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$. This condition is in fact the necessary and sufficient condition for the saddle-point profit maximization problem.

Define $c_{10} + P_{20}c_{20} + P_{20}k$ as the market cost of the bundle, and $\widehat{P}_a(\mathbf{z}) \cdot \hat{\theta} + P_a(\mathbf{z}) \cdot \theta + P_\Delta(\mathbf{z}) \cdot \Delta$ as its shadow cost. The broker-dealer considers the sum of both market cost and shadow cost as its total cost for issuing a bundle. The optimal condition (39) states that the broker-dealer will issue a bundle only if it does not cause a total loss (the revenue is strictly less than the sum of market cost and shadow cost). On the other hand, the revenue of a bundle cannot be strictly larger than the total cost. Otherwise, the broker-dealer will issue an unbounded number of such a bundle, which cannot be an equilibrium. In addition, the total market profit of the broker-dealer in equilibrium is zero.

Market Clearing: In period-0, the market-clearing condition for good-1 is

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) c_{10} = \sum_h \alpha^h e_{10}^h \tag{40}$$

Similarly, the market-clearing condition for good-2 in period-0 is

$$\sum_{\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)} y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) [c_{20} + k] = \sum_h \alpha^h e_{20}^h \tag{41}$$

The market-clearing conditions for lotteries in period-0 are

$$\sum_h \alpha^h x^h \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) = y\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right), \ \forall \left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right) \tag{42}$$

Definition 7. A competitive equilibrium is a specification of allocation $(\mathbf{x}, \mathbf{y})$, and the prices $P_{20}, P\left(\mathbf{c}_0, k, \hat{\theta}, \theta, \mathbf{z}, \Delta\right)$ such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves (30) subject to (31), taking prices as given,

(ii) for the broker-dealer, $\left\{\mathbf{y}, \widehat{P}_a(\mathbf{z}), P_a(\mathbf{z}), P_\Delta(\mathbf{z})\right\}$ solves (38), taking prices as given,

(iii) in period-0, markets for good-1, good-2 and lotteries clear, i.e., (40)-(42) hold,

6. Existence and Welfare Theorems

As in the classical general equilibrium model, the economy is a well-defined convex economy, i.e., the commodity space is Euclidean, the consumption set is compact and convex, the utility function is linear. As a result, the first and second welfare theorems hold, and a competitive equilibrium exists. In particular, this section proves that the competitive equilibrium is constrained optimal and any constrained optimal allocation can be supported by a competitive equilibrium with transfers. Then, we will use Negishi's method to prove the existence of a competitive equilibrium. Since all the proofs in this section are quite standard, we put them in the supplementary materials.

The standard contradiction argument will be used to prove the following first welfare theorem. We also assume that there is no local satiation point in the consumption set.

Assumption 2. *For any $\mathbf{x}^h \in X^h$, there exists $\tilde{\mathbf{x}}^h \in X^h$ such that*

$$\mathscr{U}^h\left(\tilde{\mathbf{x}}^h\right) > \mathscr{U}^h\left(\mathbf{x}^h\right) \tag{43}$$

where $\mathscr{U}^h\left(\mathbf{x}^h\right)$ is the expected utility of agent h derived from allocation $\mathbf{x}^h$.

This assumption is easily satisfied using reasonable specifications of the grid of consumption allocation in period-0. For example, with a strictly increasing utility function, if we include a very large consumption allocation in period-0 into the grid (larger than what can be attained with endowments and storage), then the local nonsatiation assumption will be satisfied.

Theorem 6.1. *With local nonsatiation of preferences (Assumption 2), a competitive equilibrium allocation is (constrained) Pareto optimal.*

The Second Welfare theorem states that any Pareto optimal allocation, corresponding to strictly positive Pareto weights, can be supported as a competitive equilibrium with transfers, precisely defined in the supplementary materials. The standard approach applies here. In particular, we will first prove that any constrained optimal allocation can be decentralized as a compensated equilibrium (precisely defined in the supplementary materials). Then, we will use a standard cheaper-point argument (see Debreu, 1954) to show that any compensated equilibrium is a competitive equilibrium with transfers.

Theorem 6.2. *With Assumption 1, any Pareto optimal allocation corresponding with strictly positive Pareto weights $\lambda^h > 0, \forall h$ can be supported as a competitive equilibrium with transfers.*

We will use Negishi's mapping method (Negishi, 1960) to prove the existence of competitive equilibrium. The proof benefits from the second welfare theorem. Specifically, a part of the mapping applies the theorem in that the solution to the Pareto program is a competitive equilibrium with transfers. We will show that a fixed-point of the mapping exists and it represents a competitive equilibrium without transfers.

Theorem 6.3. *For any positive endowments, with Assumption 1, a competitive equilibrium exists.*

7. Details of the Building Blocks: Asset-Backed Securities, Collateral Constraints, and Spot Markets

This section precisely defines asset-back securities (pyramiding), and derives the unified collateral constraints (4) by considering the collateral constraints of each type of securities one at a time and adding them up (and disaggregating back down). In addition, we also prove that the spot markets are equivalent to ex-ante asset-backed securities.

7.1. Collateral Constraints on Directly Collateralized Securities

To generalize a bit, let $\hat{\psi}^h \equiv \left(\hat{\psi}_s^h\right)_{s=1}^S \in \mathbb{R}^S$ and $\psi^h \equiv \left(\psi_s^h\right)_{s=1}^S \in \mathbb{R}^S$ denote agent h's portfolios of securities demanded, held at the end of period-0 paying in good-1 and in good-2, both with good-2 as collateral directly, respectively. Again, we adopt the convention that positive means demand and negative means sale. So, holding a positive amount of a security paying good 2 in state s, $max\left(0, \psi_s^h\right) = \psi_s^h$, a positive number, is equivalent to buying that security (or lending) while holding a negative amount of a security, $min\left(0, \psi_s^h\right) = \psi_s^h$, a negative number, is equivalent to selling that security (or borrowing). In short, the max and min operators pick off demand and supply, respectively. A wedge is created by the need to back the supply by collateral but not the demand.

More generally, a security paying a unit of good-1 in state s backed by good-2 pays the minimum of 1 unit of good-1 or the value of its collateral in state s. By an argument similar to the one given earlier, the minimum no-default collateral is $\frac{1}{p(z_s)R_s}$ per unit. Similarly, with no-default and no-over-collateralization, a security paying in good-2 in state s requires $\frac{1}{R_s}$ units of good-2 as collateral. The results so far are summarized in the first two rows of the Table 1 with collateral requirement in the last column.

Table 1: Collateral requirements for each type of securities.

	payment unit	collateral unit	issued liabilities	purchased assets available as collateral	total collateral requirement for no default securities
$\hat{\psi}_s^h$	good-1	good-2	$-\min\left(0, \hat{\psi}_s^h\right)$	$\max\left(0, \hat{\psi}_s^h\right)$	$-\left(\frac{1}{R_s p(z_s)}\right)\min\left(0, \hat{\psi}_s^h\right)$
ψ_s^h	good-2	good-2	$-\min\left(0, \psi_s^h\right)$	$\max\left(0, \psi_s^h\right)$	$-\left(\frac{1}{R_s}\right)\min\left(0, \psi_s^h\right)$
$\hat{\sigma}_s^h$	good-1	securities paying in good-2	$-\min\left(0, \hat{\sigma}_s^h\right)$	$\max\left(0, \hat{\sigma}_s^h\right)$	$-\left(\frac{1}{p(z_s)}\right)\min\left(0, \hat{\sigma}_s^h\right)$
σ_s^h	good-2	securities paying in good-1	$-\min\left(0, \sigma_s^h\right)$	$\max\left(0, \sigma_s^h\right)$	$-p(z_s)\min\left(0, \sigma_s^h\right)$
$\hat{\nu}_s^h$	good-1	securities paying in good-1	$-\min\left(0, \hat{\nu}_s^h\right)$	$\max\left(0, \hat{\nu}_s^h\right)$	$-\min\left(0, \hat{\nu}_s^h\right)$
ν_s^h	good-2	securities paying in good-2	$-\min\left(0, \nu_s^h\right)$	$\max\left(0, \nu_s^h\right)$	$-\min\left(0, \nu_s^h\right)$

For securities $\left(\hat{\psi}_s^h, \psi_s^h\right)$ with good-2 as collateral, paying in good-1 and good-2, respectively, agent h must hold good-2 at the end of period-0 no less than the collateral requirement in any state (shown in Table 1):

$$k^h \quad \geq \quad -\min\left(0, \hat{\psi}_s^h\right)\left(\frac{1}{R_s p(z_s)}\right) - \min\left(0, \psi_s^h\right)\left(\frac{1}{R_s}\right), \ \forall s \tag{44}$$

which can be rewritten as

$$p(z_s)R_s k^h + \min\left(0, \hat{\psi}_s^h\right) + p(z_s)\min\left(0, \psi_s^h\right) \quad \geq \quad 0, \ \forall s \tag{45}$$

These are *state-contingent collateral requirement constraints* with directly collateralized contracts. We incorporate asset-backed securities in the next section.

Note that when an agent h's collateral requirement constraints (44) are not binding for every state s (i.e., the LHS of (44) exceeds its RHS or (44) holds with strict inequality for every state s), then the agent h holds collateral k^h more than needed to back issued securities. The extra part of collateral is normal saving.

7.2. Pyramiding: Asset-Backed Securities

In real world economies, agents are allowed to use the *promises to receive* goods of others as collateral to back their own promises. This is termed *pyramiding*. In other words, there are two types of collateral, good-2 itself (described in the preceding section) and "assets" backed by such collateral. The prototypical example of an asset-backed promise in this paper is an ex-ante agreement for an agent to give up good-1 in the spot market in state s backed by someone else's promise, a receipt of good-2, or vice versa. The promise of receipt is the asset, and this backs the promise to pay. Indeed, if the planned spot-market trade is at equilibrium price of $p(z_s)$, then one is moving along a budget line and so the value of collateral, the good to be recovered, exactly equals the promise and there is no need for additional underlying collateral.

With two physical commodities, there are four possible types of asset-backed securities, summarized in the last four rows of Table 1. For example, a unit of an asset-backed security $\hat{\sigma}_s$ paying in good-1 in state s needs $\frac{1}{p(z_s)}$ units of assets paying in good-2 as collateral. The value of the payoff of $\frac{1}{p(z_s)}$ units of securities paying in good-2 in state s equals $p(z_s) \times \frac{1}{p(z_s)} = 1$ unit of good-1, which is exactly the face-value promise to pay. These collateral requirements are minimum no-default levels.

As shown in the third row of Table 1 (see the column titled total collateral requirement), an asset-backed security paying a unit of good-1 in state s, $\hat{\sigma}_s^h$, requires that the total amount of purchased assets paying in good-2 in state s is no less than $-\left(\frac{1}{p(z_s)}\right)\min\left(0, \hat{\sigma}_s^h\right)$. Similarly, an asset-backed security ν_s^h requires that the total amount of purchased assets paying in good-2 in state s is no less than $-\min\left(0, \nu_s^h\right)$ (see the last row of Table 1). On the other hand, the total amount of purchased assets paying in good-2 is $\max\left(0, \psi_s^h\right) + \max\left(0, \sigma_s^h\right) + \max\left(0, \nu_s^h\right)$, as shown in the second, fourth and last rows of Table 1 (see the next-to-last column titled purchased assets). Hence, the collateral requirement condition regarding issued securities $\hat{\sigma}_s^h$ and ν_s^h that require financial assets paying in good 2 as collateral can be written as, for any state s,

$$\max\left(0, \psi_s^h\right) + \max\left(0, \sigma_s^h\right) + \max\left(0, \nu_s^h\right) \quad \geq \quad -\left(\frac{1}{p(z_s)}\right)\min\left(0, \hat{\sigma}_s^h\right) - \min\left(0, \nu_s^h\right)$$

This states that the agent purchases enough assets or promises paying in good-2, $\theta_s^h, \sigma_s^h, \nu_s^h$, to back up her own asset-backed securities or issued promises $\hat{\sigma}_s^h, \nu_s^h$. The above condition can be rearranged as

$$p(z_s)\max\left(0,\psi_s^h\right)+p(z_s)\max\left(0,\sigma_s^h\right)+p(z_s)\nu_s^h \quad \geq \quad -\min\left(0,\hat{\sigma}_s^h\right) \tag{46}$$

where we applies the fact that $\max\left(0,\nu_s^h\right)+\min\left(0,\nu_s^h\right)=\nu_s^h$.

Similarly, the collateral requirement condition for issued securities that require financial assets paying in good 1 as collateral is given by

$$\max\left(0,\hat{\psi}_s^h\right)+\max\left(0,\hat{\sigma}_s^h\right)+\hat{\nu}_s^h \quad \geq \quad -p(z_s)\min\left(0,\sigma_s^h\right),\ \forall s \tag{47}$$

where the right-hand-side comes from the fourth and fifth rows of Table 1.

7.3. The Derivation of The Collateral Constraints

We now show that the collateral constraints (4) are equivalent to collateral requirement conditions (with three types of collateral), (45), (46), and (47). In other words, there is no loss of generality to use the collateral constraints (4); an allocation is attainable under (4) if and only if it is so under (45), (46), and (47).

To be more precise, let $\hat{\theta}_s^h=\hat{\psi}_s^h+\hat{\sigma}_s^h+\hat{\nu}_s^h$ and $\theta_s^h=\psi_s^h+\sigma_s^h+\nu_s^h$ be state-s contingent securities paying in good-1 and in good-2, respectively, which can be backed either by good-2 or purchased assets (other people's promises). Note that $\hat{\theta}_s^h$ and θ_s^h include both directly collateralized and asset-backed securities. An attainable allocation under (45), (46), and (47) can be defined similarly to the one under (4) by replacing (10)-(11) the following resource constraints:

$$\sum_h \alpha^h\hat{\psi}_s^h=\sum_h \alpha^h\psi_s^h=\sum_h \alpha^h\hat{\sigma}_s^h=\sum_h \alpha^h\sigma_s^h=\sum_h \alpha^h\hat{\nu}_s^h=\sum_h \alpha^h\nu_s^h=0,\ \forall s \tag{48}$$

The collateral constraint (4) results from summing (45), (46), and (47) altogether, and then applying $\max(0,x)+\min(0,x)=x$ to get rid of max and min operators. In addition, the proof of this lemma in Appendix A also shows how to recover contract allocation $\left(\hat{\psi}_s^h,\psi_s^h,\hat{\sigma}_s^h,\sigma_s^h,\hat{\nu}_s^h,\nu_s^h\right)_h$ from $\left(\hat{\theta}_s^h,\theta_s^h\right)$.

Lemma 4. *The following statements are true:*

(i) if $\left(\mathbf{c}_0^h,k^h,\hat{\psi}_s^h,\psi_s^h,\hat{\sigma}_s^h,\sigma_s^h,\hat{\nu}_s^h,\nu_s^h\right)_h$ is attainable, then the collateral constraint (4) and the market-clearing conditions (10)-(11) hold, and

(ii) if $\left(k^h,\hat{\theta}_s^h,\theta_s^h\right)_h$ is attainable, then there exists a collateral and security allocation $\left(k^h,\hat{\psi}_s^h,\psi_s^h,\hat{\sigma}_s^h,\sigma_s^h,\hat{\nu}_s^h,\nu_s^h\right)_h$ that satisfies collateral requirement conditions (45), (46), (47) and the market-clearing conditions (48).

7.4. Ex-ante Contracting versus Ex-post Spot Trading

Thus far we implicitly shut down trade in the spot markets in each state. This section shows that the spot markets are redundant when all types of contracts are available (see Lemma 5 below). In other words, agents do not need to trade in spot markets, though they may well do so. Importantly, the spot markets are open and deliver the spot price $p(z_s)$. In addition, we also show that the asset-backed securities are not necessary when the spot markets are open and active (see Lemma 6 below). Put differently, agents simply are indifferent between trading in spot markets or ex-ante asset-backed securities. The proofs are similar to the proof of Lemma 4, and therefore they are put in the supplementary.

When the spot markets are open, each agent h can trade $\hat{\tau}_s^h$ units of good 1 for τ_s^h units of good 2 at a spot price $p(z_s)$ according to the spot-trade constraint:

$$\hat{\tau}_s^h + p(z_s)\tau_s^h = 0 \tag{49}$$

Recall that the spot price function, $p(z_s)$, is the price such that the spot markets for both goods clear:

$$\sum_h \alpha^h \hat{\tau}_s^h = 0 \tag{50}$$

$$\sum_h \alpha^h \tau_s^h = 0 \tag{51}$$

Hence, an attainable allocation with the spot markets is defined by adding the spot-trade constraint (49) and market-clearing constraints (50)-(51) to Definition 1.

To be more precise, an allocation is said to be *equivalent* to an attainable allocation if it is attainable and generates the same consumption allocation and market fundamental in each state s as the original attainable allocation.

Lemma 5. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\theta}^h, \theta^h, \hat{\tau}^h, \tau^h\right)_h$, *there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\theta}'^h, \theta'^h, \hat{\tau}'^h, \tau'^h\right)_h$ *such that*

$$\hat{\tau}_s'^h = \tau_s'^h = 0, \forall s, h \tag{52}$$

Condition (52) in Lemma 5 implies that the spot markets in period-1 are redundant when all securities are allowed; that is, anything that can be done through the spot markets and one set of securities is feasible under another set of securities without spot markets. Henceforth (and previously), the ex-post spot trade transfers will be (were) set to zero, ($\hat{\tau}^h = 0, \tau^h = 0$ as in (52)) and the spot-trade constraints (49) will be (were) neglected, unless stated otherwise.

Lemma 6. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h, \hat{\tau}^h, \tau^h\right)_h$, *there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0^h, k^h, \hat{\psi}_s'^h, \psi_s'^h, \hat{\sigma}_s'^h, \sigma_s'^h, \hat{\nu}_s'^h, \nu_s'^h, \hat{\tau}'^h, \tau'^h\right)_h$ *such that*

$$\hat{\sigma}_s'^h = \sigma_s'^h = \hat{\nu}_s'^h = \nu_s'^h = 0, \forall s, h \tag{53}$$

It is worthy of emphasis that Lemma 5 and Lemma 6 imply that the asset-backed securities that we need in this model are the ones that replicate spot markets. In other words, the asset-backed securities in this model (with tranching) are simply substitutes for spot markets. Henceforth, we let asset-backed securities play this role and shut down active trade in spot markets. The result is summarized in the following corollary.

Corollary 1. *Asset-backed securities and the spot markets are perfect substitute in this model.*

8. Analysis of Prices

This section characterizes systematic relationships among equilibrium prices.

8.1. Spot Markets and Contract Prices: No-Arbitrage Condition

The pyramiding mechanism puts a restriction on the prices of contracts traded within each price-island. The ratio of the equilibrium prices of the securities in island-z_s in state s, $\frac{P_a(z_s,s)}{\widehat{P}_a(z_s,s)}$, must be equal to the marginal rate of substitution or the spot price in the island, $p(z_s)$. Otherwise, there will be an arbitrage possibility (by keeping the collateral constraints satisfied with pyramiding). The result is summarized in the following lemma.

Lemma 7. *In a competitive equilibrium, for each s and z_s,*

$$P_a(z_s, s) = p(z_s)\widehat{P}_a(z_s, s) \tag{54}$$

Using the no-arbitrage condition (54), the collateral constraints (4) can be rewritten as

$$P\left(z_s, s\right) R_s k \geq -\widehat{P}_a(z_s, s)\hat{\theta}_s - P\left(z_s, s\right)\theta_s,\ \forall s \tag{55}$$

These constraints state that the value in units of good 1 at $t = 0$ of all ex ante securities held (RHS) cannot exceed the value of collateral held (LHS). These constraints are applicable when the spot markets are not available but the ex-ante asset-backed securities can be traded.

8.2. Prices of the Right to Trade

Trading in price islands also imposes a restriction on collateral, contract and price-island prices, $P_{20}, P_a(z_s, s), P_\Delta(z_s, s)$. Even though collateral and securities are indeterminate (see Lemma 11 in the supplementary), holding collateral additionally impacts the spot price $p(z_s)$. Therefore, the equilibrium price of collateral must reflect the role of collateral on the spot price in each price island.

Again a no-arbitrage condition requires that the prices of two different bundles that result in the same consumption allocation for an agent h must have the same prices. Using the profit maximization condition of a broker-dealer (39) and some algebra, we can prove the following equation must hold.

$$P_{20} + \sum_{s=1}^{S} P_\Delta\left(z_s, s\right) z_s R_s \quad = \quad \sum_{s=1}^{S} P_a\left(z_s, s\right) R_s \tag{56}$$

The RHS is the price of contracts paying R_s units of good-2 in every state s. On the other hand, the LHS is the total cost of the same return, received by buying and holding a unit of collateral. The first term on the LHS is the price of the collateral good. The second term on the LHS comes from the fact that holding more a unit of good-2 increases Δ in every state s by the amount $z_s R_s$. In particular, an agent holding an additional unit of collateral must pay for the marginal impact $z_s R_s$ at price $P_\Delta(z_s, s)$. This term prices the impact of collateral on the market fundamental. In equilibrium, these two values must be the same.

Lemma 8. *In a competitive equilibrium, for each set of islands $\mathbf{z} = (z_s)_s$, (56) holds.*

8.3. Trading in Price-Islands Generates Intertemporal Transfers

Trading in price-islands can generate additional intertemporal transfers. For example, a constrained agent would like to smooth consumption by issuing securities or borrowing to transfer future resources back to period-0 but cannot do so much because of the limited commitment. Trading in price-islands facilitates more consumption smoothing by generating period-0 transfer for a constrained agent.

For the sake of discussion, we will consider a case with two types of agents one of which is constrained, and without uncertainty (i.e., $S = 1$). In addition, as shown in the previous section, we assume that a constrained agent holds no collateral, $k = 0$. Using (18), (39), and (54), an agent type h's budget constraint (31) can be rewritten as

$$\sum_b x^h(b)\left[c_{10} + P_{20}c_{20}\right] \le e^h_{10} + P_{20}e^h_{20} + \sum_b x^h(b)\,\widehat{P}_a(z_1, 1)\left[-\hat{\theta}_1 - p(z_1)\theta_1\right]$$
$$+ \sum_b x^h(b)\,P_\Delta(z_1, 1)\left[\frac{e^h_{11}}{e^h_{21}} - z_1\right]e^h_{21}$$

The third term on the RHS is the revenue from borrowing via $\left(\hat{\theta}_1, \theta_1\right)$. Using the collateral constraint (20), $p(z_1)R_1k + \hat{\theta}_1 + p(z_1)\theta_1 \ge 0$. Since the constrained agent holds no collateral, $k = 0$, her collateral constraint becomes $\hat{\theta}_1 + p(z_1)\theta_1 = 0$. Of course, this constrained agent would like to go short on the contracts (i.e., having $\hat{\theta}_1 + p(z_1)\theta_1 < 0$) but cannot do so because she holds no collateral. In other words, with zero collateral, the agent cannot borrow from trading in contracts.

Of special interest, the last term on the RHS shows that the constrained agent could potentially receive positive period-0 wealth by trading in price-islands. In particular, a constrained agent could smooth consumption intertemporally by trading in price-islands in such a way that this term is positive, giving her more resources to purchase date zero consumption. For example, if $P_\Delta(z_s, s) > 0$, then the constrained agent will buy a price island-z_s whose market fundamental is lower than her own endowment, i.e., $z_s < \frac{e^h_{1s}}{e^h_{2s}}$, and vice versa (see also Examples in the next section).

On the other hand, an unconstrained agent will potentially hold strictly positive amount of collateral, $k > 0$. She will in fact transfer out period-0 wealth from trading in price-islands. For example, if a constrained agent has $z_s < \frac{e^h_{1s}}{e^h_{2s}}$, then the consistency constraint (28) implies that an unconstrained agent in island-z_s must have $z_s > \frac{e^h_{1s}}{R_s k + e^h_{2s}}$. Hence, if $P_\Delta(z_s, s) > 0$, the last term on the RHS will be negative for an unconstrained agent.

9. Numerical Examples

Two economies without uncertainty and one economy with uncertainty are discussed in this section. As formally shown in Lemma 11 in the supplementary, all equilibria presented here have constrained agents holding zero collateral, $k = 0$. For brevity, several details are omitted but available in the supplementary.

The first economy consists of two types of agents. A collateral equilibrium with an externality and a competitive equilibrium with price-islands (without externality) are both presented. We find naturally that the externality leads to a larger amount of aggregate saving (over all collateral) relative to the one in the competitive equilibrium with price-islands or segregated exchanges. Interestingly, the competitive equilibrium with price-islands has a *unique active price-island* even though all price-islands are available. This does not have to be true in general, however. In particular, the second environment with three types of agents illustrates a price-islands equilibrium with *multiple active price-islands*.

The effects of internalizing the externality on prices and allocations are discussed. We find that internalizing the externality could make (i) the price of good-2 fluctuates less over time relative to the collateral equilibrium, and (ii) someone is strictly better off and someone is strictly worse off than being in the collateral equilibrium. The first outcome suggests, again naturally enough, that the externality causes the collateral price to fluctuate too much. It is the good in short supply when there are borrowing constrained agents. The second outcome is a bit more surprising; trading in the price islands may generate a redistribution of wealth across agents. This is a general equilibrium effect of the model. See figure 3. Note that, as shown in example 3, the second statement may not be true in general. The redistribution of wealth could be very small or vanishing when the agents are quite similar. If so, internalizing the externality would lead to a Pareto improvement.

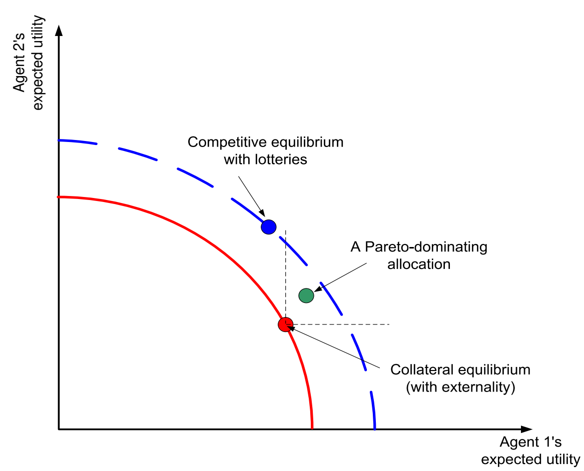


Figure 3: Bold curve: the Pareto frontier and collateral equilibrium without lotteries; Dash curve: the Pareto frontier and competitive equilibrium with lotteries.

Environment 1. There is a single state, $S = 1$, and two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function

$$U(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \frac{c_2^{1-\gamma}}{1-\gamma}, \quad \forall h \tag{57}$$

where $\gamma = 2$. Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor $\beta = 1$. The storage technology is given by $R = 1$. The endowment profiles of the agents are shown in Table 2 below. Recall that e^h_{it} is an agent h's endowment of good-i in period t. Note that endowments for both agents are symmetric. In particular, an agent type 1 is well endowed with both goods in period-0 and vice versa for type 2.

Table 2: Endowment profiles of the agents.

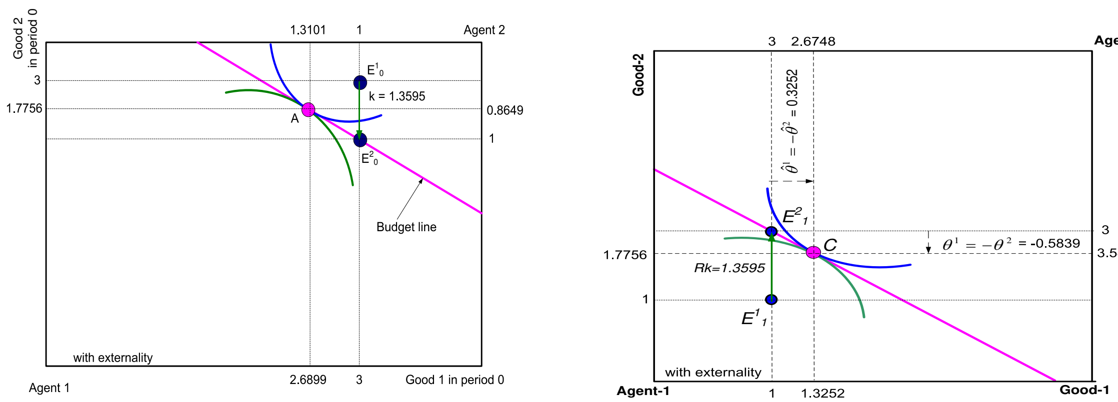
	endowments				first-best allocations				
	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}	k^h	c^h_{10}	c^h_{20}	c^h_{11}	c^h_{21}
$h = 1$	3	3	1	1	0	2	2	2	2
$h = 2$	1	1	3	3	0	2	2	2	2

The symmetry of endowments with $\beta = 1$, and of the utility function across goods implies that the first-best aggregate saving will be zero, and each agent gets the average 2 units of each good in each period (see Table 2). Accordingly, the first-best price of good-2 in period-0 is $P^{fb}_{20} = 1$, and the market fundamental is $z = 1$. Unfortunately, the first-best allocation is not attainable; that is, it violates the collateral constraints.

We now consider the economy with default and collateral (with an externality). The endowment profile and the first-best allocation suggest that agent 2 would like to move resources forward from $t = 1$ to $t = 0$, and therefore will be constrained. Hence, we will assume that agents type 2 hold no collateral, i.e. $k^1 = k$ and $k^2 = 0$. We will then solve for an equilibrium k.

As shown in the supplementary materials, the equilibrium collateral is $k = 1.3595$. As a result, the price of good-2 in period 0 is $P_{20} = \left(\frac{4}{4-k}\right)^2 = 2.2948$, and the market fundamental in period-1 is $z = \frac{4}{4+k} = 0.7463$, which implies that the spot price is $p(z) = z^\gamma = 0.5570$. Note that the collateral price at $t = 0$ is higher in the equilibrium with an externality, i.e., $P^{fb}_{20} = 1 < P_{20} = 2.2948$. On the other hand, the spot price of good-2 in period-1 is lower in the equilibrium with an externality, i.e., $p^{fb}(z) = 1 > p(z) = 0.5570$. In words, the collateral distortion makes the price of good-2 higher in the first period and lower in the second period relative to the first-best. Nevertheless, an agent type 1 is saving. This is because saving is the only way she can transfer resources to $t = 1$, given that a constrained agent holds zero collateral and cannot trade intertemporally.

Figure 4a illustrates period-0 equilibrium allocation with the externality. It shows that agent 1, the unconstrained agent, sells good 1 and buys good 2, and vice versa for agent 2. In addition, the allocation is on the budget line of constrained agent 2, which is the line passing through E^2_0. This implies that the unconstrained agent will effectively do all the saving, $k^1 = 1.3595$. Figure 4b illustrates period-1 equilibrium allocation with the externality, including security trades. Agent 1 buys $\hat{\theta}^1 = 0.3252$ and sells $\theta^1 = -0.5839$, and vice versa for agent 2. Equivalently, we can actually dispense with securities: agent 1 buys 0.3252 units of good 1 and sells 0.5839 units of good 2 at price $p(z) = 0.5570$ in spot markets, and vice versa. In other words, *all security trades are equivalent to spot trades.* In addition, the expected utility of an agent type 1 and type 2 are $\mathscr{U}^1 = -2.2527$ and $\mathscr{U}^2 = -2.5724$, respectively. The unconstrained agent is better off.



(a) (b)

Figure 4: The equilibrium (with externality) allocation. (a) Point A is period-0 equilibrium allocation with the externality where Agent 1 buys good-2, sells good-1, and saves good-2, and vice versa. (b) Point C is the period-1 equilibrium allocation with the externality where agent 1 buys good-1 and sells good-2, and vice versa.

We will now turn to a corresponding competitive equilibrium with price-islands. The equilibrium allocation reported in this paper is a numerical solution to the Pareto program (29) that corresponds to a competitive equilibrium (without transfers[8]).

There is *only one active island*, $z = 0.7729$, even though all price-islands are available for trade. We will now compare this equilibrium allocation without any externality to the one with an externality. The equilibrium average or per capita saving (without externality) is $\sum_h \sum_b \alpha^h x^h(b)k = \frac{1.1753}{2} = 0.5877$, which clearly smaller than the aggregate saving in the equilibrium with externality, though more than the first-best.

With lower aggregate saving, the price of good-2 in period 0 is lower ($P_{20} = 2.0073 < 2.2948$) but the spot price of good 2 is higher ($p(z) = 0.5974 > 0.5570$), relative to the one in the equilibrium with the externality. Thus, the price of good-2 varies less over time when the externality is internalized. Equilibrium fees of price-islands, including the fees of inactive (out-of equilibrium) islands are summarized in Table 3 below.

Notice that the fees of price islands are increasing with the market fundamentals; that is, the larger the specified market fundamental of a price island, the higher the fee of the

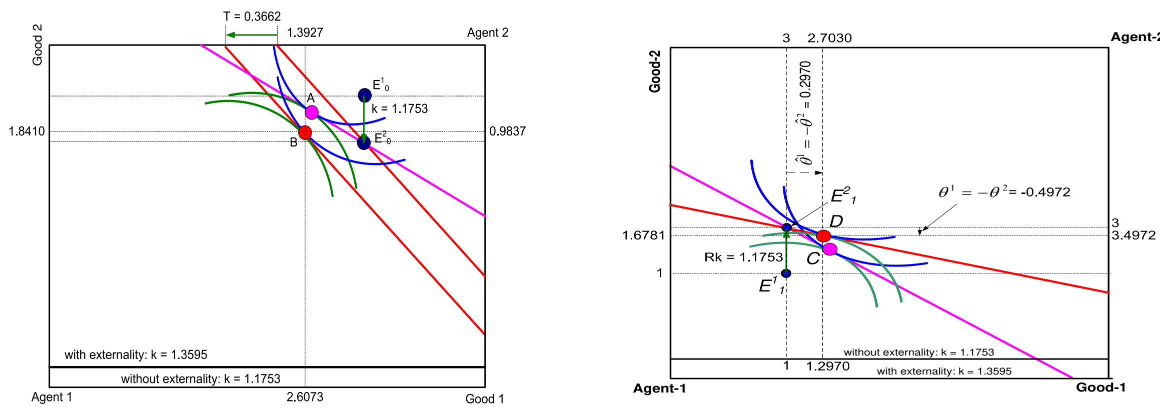
[8]The linearity of the problem again can potentially generate multiple equilibria; that is, there can be some configurations of Pareto weights that map a given endowment to different competitive equilibria with different prices and allocations (see the Working Paper version of Prescott and Townsend, 2006, for a similar discussion). In all of the examples, we search for a competitive equilibrium using Negishi's mapping method. The corresponding Pareto weight is $\lambda^1 = 0.7780, \lambda^2 = 0.2220$. It is computed using Matlab program on a personal computer with AMD Athlon 64X2 Dual Core Processor 3800+ 2.01 GHz, 3.87 GB RAM.

Table 3: Equilibrium fees of price-islands. The bold numbers are (actively traded) equilibrium prices.

	$z = 0.7479$	$\mathbf{z = 0.7729}$	$z = 0.7979$
$P_\Delta(z)$	0.4639	**0.5375**	0.6118

price island will be. Intuitively, the larger the market fundamental, the larger the price of good 2 relative to good 1. Hence, an agent with a larger amount of good 2 relative to good 1 will benefit more from being in a higher price island. Hence, it is optimal to require the beneficiary agents to pay more for trading in a higher price island, as in the row $P_\Delta(z)$. Note also that the prices of out-of-equilibrium (non-active) islands are set to make broker-dealers break even, but at such prices consumers do not want to buy them.

Since the equilibrium fee $P_\Delta(z) = 0.5375$ is positive in this example, agents with negative deviation from the fundamental (agent 2 in this case) must get paid for the access to the price island. In particular, a constrained agent ($h = 2$) with $\Delta = -0.6813$, is receiving transfer $-P_\Delta(z)\Delta = 0.3662$ in period $t = 0$ for being in the equilibrium price-island. Graphically, this shifts her budget line outward by $T = 0.3662$. See figure 1a. This implies how trading in price islands generates the redistribution of wealth in general equilibrium.



(a) (b)

Figure 5: The equilibrium (without the externality) allocation. (a) Point B is the period-0 equilibrium allocation without the externality where Agent 1 receives wealth transfer, $T = 0.366$, from trading in price-islands. (b) Point D is the period-1 equilibrium allocation without the externality.

In addition, the expected utility of an agent type 1 and type 2 are $\mathscr{U}^1 = -2.2936, \mathscr{U}^2 = -2.3905$, respectively. Recall that the expected utility in the externality equilibrium of an agent type 1 and type 2 are $\mathscr{U}^1 = -2.2527$ and $\mathscr{U}^2 = -2.5724$, respectively. This shows that internalizing the externality is beneficial to an agent type 2 (constrained agent) but harmful for an agent type 1. This is a (distributional) general equilibrium effect. Internalizing the externality not only improves efficiency of the economy, but also redistributes wealth. To induce welfare gain for all of agents, there must be lump sum transfers. All agents are benefit from the efficiency effect, which shifts the Pareto frontier outward as shown in figure 3. Some agents may be harmed by the distributional effect, however. Note also that similar to the collateral equilibrium, all security trades are equivalent to spot trades.

The next environment demonstrates that it is possible to have multiple active islands or segregated exchanges. In particular, with three types of agents, two of which are constrained, there are two distinct active price islands in equilibrium, each of which consists of different composition of agents. Each island, in fact, consists of one constrained type and one unconstrained type. The equilibrium allows discrimination among constrained types.

Environment 2. There are three types of agents, and a single state. Each agent is given the same utility function as in (57) with $\gamma = 2$. Each type consists of $\frac{1}{3}$ fraction of the population, i.e. $\alpha^h = \frac{1}{3}$. Similar to the previous example, $\beta = 1$, and $R = 1$. The endowment profile is given in Table 4 below. Note that to conserve on space we do not present an equilibrium with externality of this economy.

Table 4: Endowment profiles of the agents.

Type of Agents	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}
$h = 1$	$\frac{12.5}{3}$	11.5	0.5	0.5
$h = 2$	$\frac{12.5}{3}$	0.5	7	5
$h = 3$	$\frac{12.5}{3}$	0.5	5	7

With large endowments in $t = 1$, agent 2 and agent 3 want to move resources forward to $t = 0$. The scarcity of collateral then implies that both of them will be collateral constrained. In addition, type 2 has the larger ratio of period-1 endowment of good 1 to good 2 $\frac{e^h_{11}}{e^h_{21}}$ than type 3. As shown below, this difference suggests that agent 2 will be in a higher price-island than agent 3 in equilibrium. In addition, in a competitive equilibrium with price-islands, an agent type 2 is better off than an agent type 3. Intuitively, agent 2 holds lots of good 1 in period 1 when it is valuable while agent 3 holds lots of good 2 in period 1 when it is not so valuable.

A competitive equilibrium allocation with price-islands is presented in Table 5 below. Note that the equilibrium reported here is a competitive equilibrium without lump sum transfers[9]. Interestingly, there are *two active price islands*, $z = 0.6028$ and $z = 0.8167$. The price island $z = 0.6028$ consists of some fraction of agents type-1, and all of agents type-3 (a constrained type). On the other hand, the price island $z = 0.8167$ consists of some fraction of agents type-1, and all of agents type-2 (a constrained type). Technically, lotteries are optimal because the collateral constraints create a non-convexity problem (see Lemma 3). In particular, a convex combination over both islands of the equilibrium allocations of Table 5 is not attainable.

Similar to the previous example, all security trades are equivalent to spot trades; that is, all of them can be replicated by trading in spot markets. In particular, agent 2 and agent 3 buy good 2 and sell good 1 in the spot markets, and vice versa for agent 1. In addition, agent 1's security trading varies across price-islands. This is because different islands have different spot prices. Notice also that there are more of agents type 1 in island $z = 0.8167$

[9]The Pareto weight for this particular equilibrium is given by $\lambda^1 = 0.5571, \lambda^2 = 0.3511, \lambda^3 = 0.0918$

Table 5: Equilibrium allocation of (non-zero-mass) lotteries. There are multiple active price islands; $z = 0.6028$ and $z = 0.8167$.

	$h=1$		$h=2$	$h=3$
k^h	6.5144	4.6287	0.0000	0.0000
$\hat{\theta}^h$	1.4022	1.6582	-1.3111	-0.2935
θ^h	-3.8588	-2.4861	1.9657	0.8077
z	0.6028	0.8167	0.8167	0.6028
mass: $x^h(b)$	0.2093	0.7907	1.0000	1.0000

than in island $z = 0.6028$. This follows from the consistency constraints; that is, the market fundamental in each island has to be equal to the specified level[10].

Interestingly, agent 2 and agent 3 have different consumption allocations in period 0 even though they have the same endowment in period 0. This is because they trade in different price-islands, and thereby receive different wealth transfers. In particular, using the allocation in Table 5 and the prices in Table 6, agent 2 receives $-P_\Delta(z = 0.8167)\Delta = -2.2098 \times (-2.9165) = 6.4449$ units of good 1 at $t = 0$ while agent 3 receives $-P_\Delta(z = 0.6028)\Delta = -0.7651 \times (-0.7804) = 0.5971$ units of good 1 at $t = 0$.

Equilibrium fees of price-islands, including the fees of inactive (out-of equilibrium) islands are summarized in Table 3 below.

Table 6: Equilibrium fees of price-islands. The bold numbers are (actively traded) equilibrium prices. Note: they are not adjacent.

	$z = 0.4500$	$\mathbf{z = 0.6028}$	$z = 0.7028$	$z = 0.7667$	$\mathbf{z = 0.8167}$	$z = 0.8667$
$P_\Delta(z)$	0.2717	**0.7651**	1.7619	1.8117	**2.2098**	2.7253

In addition, the expected utility of an agent type 1, type 2, and type 3 are $\mathscr{U}^1 = -1.3123, \mathscr{U}^2 = -0.9117, \mathscr{U}^3 = -1.4982$, respectively. Agent 2 is significantly better off than agent 3. Note that they have the same period 0 endowment. The difference comes from two sources. The first one is the difference in their period 1 endowments, as discussed earlier. This part is true even in the collateral equilibrium with the externality[11]. The second one is the large difference in receiving transfers in period 0 from trading in price island ($T_2 = 6.4449 > T_3 = 0.5971$). This part is only true in the competitive equilibrium with price-islands[12]. Notice also that a constrained agent type 2 is better off than being in the first-best world, where she will receive expected utility $\mathscr{U}^2 = -0.9600$. This implies that constrained does not necessary mean worse-off. This is the general equilibrium effect.

[10] For instance, the average of good 1 in island $z = 0.8167$ is $0.7907 \times 0.5 + 1.000 \times 7 = 7.3953$, and average of good 2 in island $z = 0.8167$ is $0.7907 \times (4.6287 + 0.5) + 1.000 \times 5 = 9.0553$. Hence, the market fundamental in island $z = 0.8167$ is $\frac{7.3953}{9.0553} = 0.8167$.

[11] The difference between agent 2 and agent 3 in the collateral equilibrium is not as large as in the competitive equilibrium with price-islands, i.e., $\mathscr{U}^2 = -1.6452, \mathscr{U}^3 = -1.9786$. Though, both of them are strictly better off in the competitive equilibrium with price-islands. See the supplementary materials for the detailed derivation.

[12] Note also that agent 2 benefits from the availability of the lotteries in that it allows her to be in a higher market fundamental island ($z = 0.8167$), which means a larger transfer, than otherwise ($z = 0.7133$).

The last economy illustrates an economy with uncertainty where directly collateralized securities, $\hat{\psi}$, are actively traded. All agents are constrained, but at different states. In particular, an agent will be binding in a state where her endowment is large. This is because she would like to transfer a part of such a large amount of wealth forward to $t = 0$ but cannot do so because of the collateral constraints.

Environment 3. The economy in this example is similar to the one in example 1, but there are two states, $S = 2$. There are two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function (57) with $\gamma = 2$. Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor $\beta = 1$. The storage technology is constant and given by $R_s = 1$ for $s = 1, 2$. The endowment profile is presented in Table 7. Note that the agents are ex-ante identical. But agent type 1 has relative more of both goods in state $s = 1$ than in state $s = 2$ and vice versa for agent type 2.

Table 7: Endowment profiles of the agents.

	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}	e^h_{12}	e^h_{22}
$h = 1$	2	2	3	3	1	1
$h = 2$	2	2	1	1	3	3

First, the symmetry of the endowments and preferences implies that period-0 allocation should be the same for all agents; that is, $c^h_{10} = c_{10}$ and $c^h_{20} = c_{20}$, for all h. Further, the indeterminacy between k^h and $\theta^h(s)$ implies that there is no loss of generality in considering the case with symmetric collateral allocation, i.e. $k^h = k$, for all h.

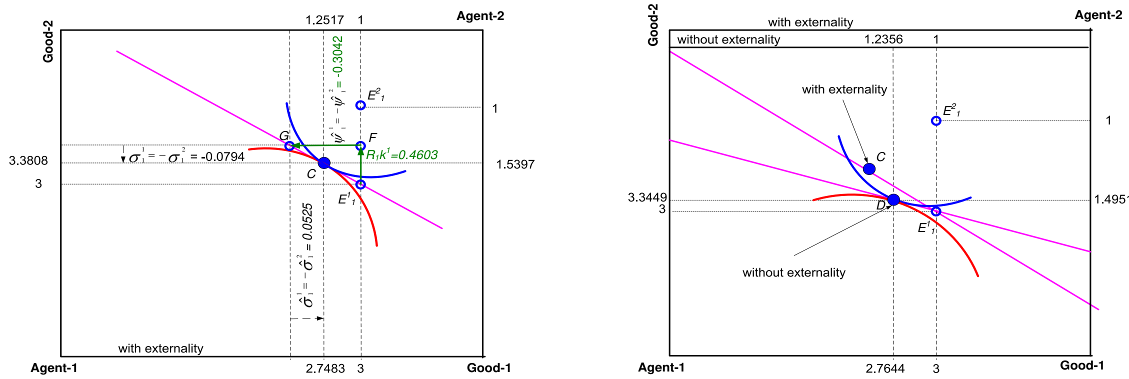
We will now solve for a competitive equilibrium with the externality. The detailed derivation is omitted and presented in the supplementary materials. The unique competitive equilibrium with the externality of this economy has $k^h = k \approx 0.4603$, for all h. Accordingly, the market fundamental and spot price are $z_s = 0.8129$ and $p(z_s) = 0.6608$, respectively, for all s. The price of good-2 in period 0, prices of contracts are $P^{ext}_{20} = 1.6872, \widehat{P}^{ext}_s = 1.2766, P^{ext}_s = 0.8436$, respectively ("ext" stands for externality).

Recall that $\hat{\theta}^h_s$ and θ^h_s include directly-collateralized and asset-backed securities. Nevertheless, we now can recover the positions of each securities[13]. Of special interest, directly-collateralized securities (or borrowing contracts) are $\hat{\psi}^1_1 = -\hat{\psi}^2_1 = -0.3042 = -\hat{\psi}^1_2 = \hat{\psi}^2_2$. In words, an agent $h = 1$ issues (borrows) $\hat{\psi}^1_1 = -0.3042$ units of directly-collateralized security paying good 1 at $s = 1$, and vice versa for an agent $h = 2$. Figure 3a illustrates securities traded in equilibrium. Here, as discussed earlier, all asset-back securities are equivalent to spot trades[14].

We now turn to the competitive equilibrium with price-islands. As mentioned earlier, the equilibrium allocation reported here is a numerical solution to the Pareto program (29) corresponding to a competitive equilibrium without transfers. Each type of agent holds the same amount of collateral good $k^h = 0.4200 < 0.4603$ which is less than the one in

[13]This can be done using equations (77)-(80) in the proof of Lemma 4

[14]Here agents trade only securities paying good 1. Though, as in the proof of Lemma 6, there is an equivalent allocation at which only securities paying good 2 are traded.



(a) (b)

Figure 6: (a) Point C is the period-1 equilibrium allocation with the externality. (b) Point D is the period-1 equilibrium allocation without the externality.

competitive equilibrium with the externality. As a result, the market fundamental and the spot price of good 2 in each state s is higher here, i.e., $z_s = 0.8285 > 0.8129$ and $p(z_s) = 0.6864 > 0.6608$. In addition, the price of good-2 in period 0 is $P_{20} = 1.5903 < P_{20}^{ext} = 1.6872$, which is again lower than the one in the competitive equilibrium with the externality.

We can recover the positions of each security using the same approach as in the competitive equilibrium with the externality. Directly-collateralized securities (or borrowing contracts) are $\hat{\psi}_1^1 = -\hat{\psi}_1^2 = -0.2872 = -\hat{\psi}_2^1 = \hat{\psi}_2^2$. In words, an agent $h = 1$ issues (borrows) $\hat{\psi}_1^1 = -0.2872$ units of directly-collateralized security paying good 1 at $s = 1$ in an island $z_1 = 0.8285$ at a unit price $\widehat{P}_a(z_1, 1) = 1.2936$, and vice versa for an agent $h = 2$. Again, all asset-back securities are equivalent to spot trades. The equilibrium without the externality is also illustrated in figure 3b. Note that agents trade less securities relative to the equilibrium with the externality. This is because the agents save less, and are issuing fewer securities. That is, the externality generates too much borrowing.

10. Concluding Remarks

As we write this, the world financial markets are in much turmoil. Needless to say we do not attempt here to model all possible problems and corresponding solutions. Rather we use theory to try to pinpoint one important aspect of what is going on: default, the consequent use of collateral which moves intertemporal endowments, and endogenous spot prices at the time of repayment decisions all interact to create an externality. It is in this sense that in our model markets do not function efficiently. Essentially all traders take spot prices as given when deciding what claims to buy and issue, and those that issue in the initial securities market need to back their promises with collateral, which determines subsequent spot market prices. This simultaneity happens in complete market set ups without frictions as well, but in our model the set of feasible trades for each agents depends on equilibrium spot market prices, so agents are imposing an externality on one another when they each make their own decisions.

Our solution to this problem is equally intuitive: create a market in the spot market price itself, that is allow agents to contract on what price they will unwind their contract commitments, over and above contracting on intertemporal or state-contingent exchange. Of course that price is still endogenous and the contract price must equal the spot market price at which supply equals demand, taking into account exogenous endowments, saving, and contract positions and who is in the market. So when agents contract on the spot price they essentially are counting on having the requisite number and types of traders around to support that contract spot price. No agent cares specifically about the identity or name of other traders, but they do care about the composition of traders (or in our set up with homotheticity, the ratio of pre-trade endowments). So the new market mechanism needs to track who people are and what commitments they have made, in a certain well defined sense. Importantly, the formation of exchanges is determined by the market, so the government or a planner is not directing traffic. Specifically, to support the new constrained efficient allocation, there must be an ex ante market for ex post spot markets, with prices (fees and receipts) paid ex ante at the time of contracting for participation in these ex post exchanges, depending on each agents type (his/her pre trade endowment inclusive of savings to support securities). Agents are in effecting buying and selling their rights to trade in clubs, the set of agents with whom they will execute their promises and unwind positions (but we do not use the word club in a pejorative sense, as each club has a continuum of price taking agents).

In order to visualize more clearly the mechanics of our proposed market structure, we try to place it in a contemporary setting. We imagine that there are two commodities, money (good 1) and treasury obligations (good 2, the collateral good). Even though money and treasuries can not be consumed directly, as can the commodities of our model, each participant (financial institutions, e.g., banks, insurance companies, hedge funds) derives an indirect utility from holding them today and also from holding them tomorrow (due to reasons that we do not model here). But the utility is less from treasuries when they used as collateral backing promises to pay. In the initial date these financial players borrow and lend in securities markets and buy insurance obligations, with loans and insurance contracts dominated in money. The market fundamental in future spot markets then is determined by the relative ratio of money to treasuries, equivalently the interest rate at that time. In addition, some market participants buy for cash one among a range of market exchange certificates designating the future spot price of treasures. Other participants are paid to hold one of a range market exchange certificates. (In some instances certificates are issued at random, after the price is paid). This arrangement is essentially offers insurance against the uncertain spot price of treasuries, but the market in the certificates in effect restricts the set of traders with whom there is a trade today and unwinding of positions tomorrow, in such a way that the contracted insured price is the market clearing spot price. Broker-dealers will clear all the markets for securities and markets of certificates. Of course these institutional arrangements will require an excellent registration, to keep track of which exchange market traders are allowed to use (and hence the securities which are held). It is important to ensure that agents cannot participate across markets where they do not have the right to buy and sell and unwind trades, to forestall the obvious arbitrage when multiple exchanges emerge in equilibrium.

Asset-backed securities are allowed in our set up and do not cause a problem. Neither are they essential here in that various combinations of securities and markets are equivalent. Essentially asset back security trades mimic spot market trade, and are an essentially part of the set up if and only if spot market exchange is for some reason more limited. As a result, all arguments stated in terms of spot markets can be restated using asset back securities. In particular, we can solve the externality problem by creating segregated exchanges where agents can trade ex ante collateralized and asset backed securities indexed by the market fundamental.

Our methods extend to other set ups in which spot market exchange is desirable or cannot be limited a priori. First, this model can be readily extended to incorporate the contract-specific collateralization without pyramiding and tranching as in Geanakoplos (2003), among others. In this case, spot trades will be necessary and cannot be substituted by ex-ante contracting. Second, this model can also be extended to general preferences and dynamic environments. This extended version will be used to study equilibrium cascades. This is again closely related to Geanakoplos (2003). Third, we can use our approach to study retrading or anonymous trading in spot markets under moral hazard environments. This is related to Acemoglu and Simsek (2008).

References

Acemoglu, D. and A. Simsek (2008): "Moral Hazard and Efficiency in General Equilibrium with Anonymous Trading," *Working Paper*.

Acemoglu, D. and F. Zilibotti (1997): "Was Prometheus Unbound by Chance? Risk, Diversification, and Growth," *Journal of political economy*, 105, 709–751.

Aghion, P., A. Banerjee, and T. Piketty (1999): "Dualism and Macroeconomic Volatility," *Quarterly Journal of Economics*, 114, 1359–1397.

Allen, F. and D. Gale (2004): "Financial Intermediaries and Markets," *Econometrica*, 1023–1061.

Alvarez, F. and U. Jermann (2000): "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica*, 68, 775–797.

Bernanke, B. and M. Gertler (1989): "Agency Costs, Net Worth, and Business Fluctuations," *The American Economic Review*, 14–31.

Bernanke, B., M. Gertler, and S. Gilchrist (1996): "The Financial Accelerator and the Flight to Quality," *The Review of Economics and Statistics*, 1–15.

Bewley, T. (1986): "Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers," *Contributions to Mathematical Economics in Honor of Gérard Debreu*, 79.

Bisin, A., J. Geanakoplos, P. Gottardi, E. Minelli, and H. Polemarchakis (2001): "Markets and Contracts," *Journal of Mathematical Economics.*

Bisin, A. and P. Gottardi (2006): "Efficient Competitive Equilibria with Adverse Selection," *Journal of Political Economy*, 114, 485–516.

Caballero, R. and A. Krishnamurthy (2001): "International and Domestic Collateral Constraints in a Model of Emerging Market Crises," *Journal of Monetary Economics*, 48, 513–548.

——— (2004): "Smoothing Sudden Stops," *Journal of Economic Theory*, 119, 104–127.

Cooley, T., R. Marimon, and V. Quadrini (2004): "Aggregate Consequences of Limited Contract Enforceability," *Journal of Political Economy*, 112, 817–847.

Debreu, G. (1954): "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences of the United States of America*, 40, 588.

Farhi, E., M. Golosov, and A. Tsyvinski (2009): "A Theory of Liquidity and Regulation of Financial Intermediation," *Review of Economic Studies*, 76, 973–992.

Geanakoplos, J. (1997): "Promises Promises," in *Santa Fe Institute Studies in the Sciences of Complexity*, Addison-Wesley Publishing Co, vol. 27, 285–320.

——— (2003): "Liquidity, Default, and Crashes," *Advances in Economics and Econometrics: Theory and Applications: Eighth World Congress*.

Geanakoplos, J. and H. Polemarchakis (1986): "Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete," *Uncertainty, Information and Communication: Essays in Honor of KJ Arrow*, 3, 65–96.

Geanakoplos, J. and W. Zame (2002): "Collateral and the Enforcement of Intertemporal Contracts," Tech. rep., UCLA Working Paper.

Golosov, M. and A. Tsyvinski (2007): "Optimal Taxation With Endogenous Insurance Markets," *The Quarterly Journal of Economics*, 122, 487–534.

Greenwald, B. and J. Stiglitz (1986): "Externalities in Economies with Imperfect Information and Incomplete Markets," *Quarterly Journal of Economics*, 101, 229–264.

Hansen, G. (1985): "Indivisible Labor and the Business Cycle," *Journal of Monetary Economics*, 16, 309–327.

Jacklin, C. (1987): "Demand Deposits, Trading Restrictions, and Risk Sharing," *Contractual Arrangements for Intertemporal Trade*, 26–47.

Kehoe, T. and D. Levine (1993): "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60, 865–888.

Kilenthong, W. T. (2008): "Collateral Premia and Risk Sharing under Limited Commitment," *Working Paper*.

Kiyotaki, N. and J. Moore (1997): "Credit Cycles," *Journal of Political Economy*, 105, 211–248.

KOCHERLAKOTA, N. R. (1996): "Implications of Efficient Risk Sharing without Commitment," *Review of Economic Studies*, 63, 595–609.

KRISHNAMURTHY, A. (2003): "Collateral Constraints and the Amplification Mechanism," *Journal of Economic Theory*, 111, 277–292.

LAMONT, O. (1995): "Corporate-Debt Overhang and Macroeconomic Expectations," *The American Economic Review*, 1106–1117.

LORENZONI, G. (2008): "Inefficient Credit Booms," *Review of Economic Studies*, 75, 809–833.

LUSTIG, H. (2007): "The Market Price of Aggregate Risk and the Wealth Distribution," *UCLA Working Paper*.

MAKOWSKI, L. (1980): "Perfect Competition, the Profit Criterion, and the Organization of Economic Activity," *Journal of Economic Theory*, 22, 222–242.

NEGISHI, T. (1960): "Welfare Economics and Existence of an Equilibrium for a Competitive Economy," *Metroeconomica*, 12, 92–97.

PESENDORFER, W. (1995): "Financial Innovation in a General Equilibrium Model," *Journal of Economic Theory*, 65, 79–116.

PRESCOTT, E. C. AND R. M. TOWNSEND (1984a): "General Competitive Analysis in an Economy with Private Information," *International Economic Review*, 25, 1–20.

——— (1984b): "Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard," *Econometrica*, 52, 21–45.

PRESCOTT, E. S. AND R. M. TOWNSEND (2005): "Firms as Clubs in Walrasian Markets with Private Information: Technical Appendix," *mimeo Federal Reserve Bank of Richmond*.

——— (2006): "Firms as Clubs in Walrasian Markets with Private Information," *Journal of Political Economy*, 114, 644–671.

RAMPINI, A. (2004): "Entrepreneurial Activity, Risk, and the Business Cycle," *Journal of Monetary Economics*, 51, 555–573.

RANCIERE, R., A. TORNELL, AND F. WESTERMANN (2008): "Systemic Crises and Growth," *Quarterly Journal of Economics*, 123, 359–406.

ROGERSON, R. (1988): "Indivisible Labor, Lotteries and Equilibrium," *Journal of Monetary Economics*, 21, 3–16.

A. More Results and Proofs

A.1. Pooling Collateral versus Tranching

This section shows that the markets economize on collateral; that is, there is no gain from pooling collateral across agents type h. Let the collateral constraints with pooling be:

$$p(z_s)R_sK \geq -\sum_h \alpha^h p(z_s)\min\left\{0,\psi_s^h\right\} - \sum_h \alpha^h \min\left\{0,\hat{\psi}_s^h\right\} \tag{58}$$

where the average collateral $K = \sum_h \alpha^h k^h$. We then show that the group collateral constraint is equivalent to individuals collateral constraints (4).

Lemma 9. *For any allocation $\left(k^h,\psi_s^h,\tau_s^h,\hat{\tau}_s^h\right)$ satisfying the collateral constraints (58), then there exists there exists an equivalent allocation $\left(k'^h,\psi_s'^h,\tau_s^h,\hat{\tau}_s^h\right)$ with*

$$k'^1 = \frac{\sum_h \alpha^h k_s^h}{\alpha^1}, \text{ and } k'^h = 0 \text{ for } h \neq 1 \tag{59}$$

$$\psi_s'^h = \left(R_s k_s^h + \psi_s^h\right) - R_s k'^h \tag{60}$$

where $k_s^h = \frac{-p(z_s)\min\left(0,\psi_s^h\right)-\min\left(0,\hat{\psi}_s^h\right)}{p(z_s)R_s}, \forall s.$

Proof. This result can be proved in two steps: (i) show that the collateral constraints (58) hold if and only if there exists k_s^h such that (4) hold, (ii) then show that any allocation with state-contingent collateral, k_s^h, can be replicated by an allocation with fixed collateral allocation k^h.

Step I: $\Longrightarrow$ Suppose that collateral constraints (58) hold. Now consider an alternative allocation with

$$k_s^h = \frac{-p(z_s)\min\left(0,\psi_s^h\right)-\min\left(0,\hat{\psi}_s^h\right)}{p(z_s)R_s}, \forall s \tag{61}$$

This clearly implies no default. We then only need to show that the average collateral needed $\sum_h \alpha^h k_s^h$ is no larger than K. Summing the above equation over h with weight α^h gives, for each s,

$$\sum_h \alpha^h k_s^h = \sum_h \alpha^h \frac{-p(z_s)\min\left(0,\psi_s^h\right)-\min\left(0,\hat{\psi}_s^h\right)}{p(z_s)R_s} \tag{62}$$

$$\leq K \tag{63}$$

where the last inequality follows from the group collateral constraints (58).

$\Longleftarrow$ This can be done by summing over the individuals collateral constraints with weight α^h.

Step II: Let $\left(k_s^h,\psi_s^h,\tau_s^h,\hat{\tau}_s^h\right)$ be an attainable allocation with contingent collateral; that is, it satisfies the collateral constraint for each h and s:

$$R_s k_s^h \geq -\min\left(0,\psi_s^h\right) \tag{64}$$

and the average collateral is the same in every state; $K = \sum_h \alpha^h k_s^h$ for all s. In addition, the consumption allocation of agent h in state s is given by

$$c_{1s}^h = e_{1s}^h + \hat{\tau}_s^h \tag{65}$$
$$c_{2s}^h = e_{2s}^h + \left(R_s k_s^h + \psi_s^h\right) + \tau_s^h \tag{66}$$

where the spot trade satisfies:

$$\hat{\tau}_s^h + p(z_s)\tau_s^h = 0 \tag{67}$$

Now consider a candidate allocation $\left(k'^h, \psi_s'^h, \tau_s'^h, \hat{\tau}_s'^h\right)$ with

$$k'^1 = \frac{K}{\alpha^1}, \text{ and } k'^h = 0 \text{ for } h \neq 1 \tag{68}$$
$$\psi_s'^h = \left(R_s k_s^h + \psi_s^h\right) - R_s k'^h \tag{69}$$
$$\hat{\tau}_s'^h = \hat{\tau}_s^h, \text{ and } \tau_s'^h = \tau_s^h \tag{70}$$

Using (68), we can write the securities as

$$\psi_s'^1 = \left(R_s k_s^1 + \psi_s^1\right) - k'^1 \tag{71}$$
$$\psi'^h = R_s k_s^h + \psi_s^h \text{ for } h \neq 1 \tag{72}$$

Using the collateral constraint (64) we can show that for each $h \neq 1$:

$$\psi_s'^h = R_s k_s^h + \psi_s^h \geq R_s k_s^h + \min\left\{0, \psi_s^h\right\} \geq 0 \tag{73}$$

where the last inequality follows from the collateral constraint (64). This, $\psi_s'^h \geq 0$, implies that the collateral constraint for any $h \neq 1$ holds (since he does not issue securities at all).

We hence only need to show that the collateral constraint also holds for $h = 1$. We can rewrite (71) as

$$k'^1 = \left(R_s k_s^1 + \psi_s^1\right) - \psi_s'^1 \geq -\psi_s'^1 \tag{74}$$

where the last inequality follows from the collateral constraint (64) for $h = 1$. This shows that the collateral constraint also holds for $h = 1$.

Given that $\sum_h \alpha^h k_s'^h = K = \sum_h \alpha^h k_s^h$, the market fundamentals are the same for every state. With the same market fundamental, z_s, the spot trade is satisfied, using (70).

Now we will show that the consumption allocations ar also the same.

$$c_{1s}'^h = e_{1s}^h + \hat{\tau}_s'^h = e_{1s}^h + \hat{\tau}_s^h = c_{1s}^h \tag{75}$$

where the second equality follows from (70), and the last one follows from (65). Similarly,

$$\begin{aligned} c_{2s}'^h &= e_{2s}^h + \left(R_s k_s'^h + \psi_s'^h\right) + \tau_s'^h = e_{2s}^h + \left(R_s k_s'^h + \left(R_s k_s^h + \psi_s^h - R_s k'^h\right)\right) + \tau_s^h \\ &= e_{2s}^h + \left(R_s k_s^h + \psi_s^h\right) + \tau_s^h = c_{2s}^h \end{aligned} \tag{76}$$

where the second equality follows from (69) and (70), and the last one follows from (66). *Q.E.D.*

Proof of Lemma 4. The first statement can be proved as follows. First, it is clear that conditions (48) imply (10)-(11). We now only need to show that (45), (46), and (47) imply (4). Summing up all collateral requirement conditions, (45), (46), and (47), and using the fact that $\max(0,x) + \min(0,x) = x$ give, for an agent h in state s,

$$p(z_s)R_s k^h + \left[\hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h\right] + p(z_s)\left[\psi_s^h + \sigma_s^h + \nu_s^h\right] \quad \geq \quad 0$$

which is the collateral constraint for an agent h in state s where $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$ and $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$.

The second statement is proved as follows. Consider an allocation $\left(k^h, \hat{\theta}_s^h, \theta_s^h\right)_h$ that satisfies (4) and (10)-(11). We will now choose a corresponding allocation $\left(k^h, \hat{\psi}_s^h, \psi_s^h, \hat{\sigma}_s^h, \sigma_s^h, \hat{\nu}_s^h, \nu_s^h\right)_h$ that satisfies $\hat{\theta}_s^h = \hat{\psi}_s^h + \hat{\sigma}_s^h + \hat{\nu}_s^h$, $\theta_s^h = \psi_s^h + \sigma_s^h + \nu_s^h$, the collateral requirement conditions (45), (46), (47), and the market-clearing conditions (48). Consider the following candidate allocation:

$$\begin{aligned} \hat{\psi}_s^h &= \hat{\theta}_s^h + p(z_s)\theta_s^h && (77) \\ \psi_s^h &= \hat{\nu}_s^h = \nu_s^h = 0 && (78) \\ \hat{\sigma}_s^h &= \hat{\theta}_s^h - \hat{\psi}_s^h = -p(z_s)\theta_s^h && (79) \\ \sigma_s^h &= \theta_s^h && (80) \end{aligned}$$

(78) implies that agents hold no $\psi_s^h, \hat{\nu}_s^h, \nu_s^h$; they will borrow or lend through directly collateralized contract paying in good-1 $\hat{\psi}_s^h$ only.

It is straightforward to show that resource constraints (48) hold. Since the resource constraints are satisfied and the collateral allocations k^h are the same, the market fundamentals are the same. We now would like to show that collateral requirement conditions (45), (46), (47) also hold. First, we will show that (46) and (47) hold. There are two cases to consider; (i) $\theta_s^h > 0$, (ii) $\theta_s^h < 0$. Case I: Suppose that $\theta_s^h > 0$. Using (80), this implies that $\sigma_s^h > 0$, which in turn leads to $\min\left(0, \sigma_s^h\right) = 0$. On the other hand, it is true that

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) = \max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h \geq 0$$

where the first equality follows from (78). Since $\min\left(0, \sigma_s^h\right) = 0$, we have

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) = \max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h \geq -p(z_s)\min\left(0, \sigma_s^h\right)$$

which is (47). On the other hand, (79) implies that $\hat{\sigma}_s^h < 0$ when $\theta_s^h > 0$. As a result, $\min\left(0, \hat{\sigma}_s^h\right) = \hat{\sigma}_s^h$. Using (78), (79), (80), we then can show that

$$\begin{aligned} p(z_s)\max\left(0, \psi_s^h\right) + p(z_s)\max\left(0, \sigma_s^h\right) + p(z_s)\nu_s^h + \min\left(0, \hat{\sigma}_s^h\right) \\ = 0 + p(z_s)\sigma_s^h + 0 + \hat{\sigma}_s^h = p(z_s)\theta_s^h - p(z_s)\theta_s^h = 0 \end{aligned}$$

where the second equality follows from (79) and (80). This shows that (46) holds.

Case II: Suppose that $\theta_s^h < 0$. (79) and (80) imply that $\max\left(0, \hat{\sigma}_s^h\right) = \hat{\sigma}_s^h = -p(z_s)\theta_s^h$ and $\min\left(0, \sigma_s^h\right) = \sigma_s^h = \theta_s^h$, respectively. We then can write

$$\max\left(0, \hat{\psi}_s^h\right) + \max\left(0, \hat{\sigma}_s^h\right) + \hat{\nu}_s^h = \max\left(0, \hat{\psi}_s^h\right) - p(z_s)\theta_s^h \geq -p(z_s)\theta_s^h = -p(z_s)\min\left(0, \sigma_s^h\right)$$

which is exactly (47). Note that the first equality follows from (78), the second inequality follows from the fact that $\max\left(0, \hat{\psi}_s^h\right) \geq 0$. Similarly, using , we can show that $\max\left(0, \sigma_s^h\right) = \min\left(0, \hat{\sigma}_s^h\right) = 0$. This implies that

$$p(z_s)\max\left(0, \psi_s^h\right) + p(z_s)\max\left(0, \sigma_s^h\right) + p(z_s)\nu_s^h + \min\left(0, \hat{\sigma}_s^h\right) = 0 + 0 + 0 + 0 = 0$$

which is exactly (46).

Similarly, we can now show that (45) also holds. There are two cases to be considered as well.

Case I: suppose that $\hat{\theta}_s^h + p(z_s)\theta_s^h < 0$. (77) implies that $\hat{\psi}_s^h < 0$, which in turn implies that $\min\left(0, \hat{\psi}_s^h\right) = \hat{\psi}_s^h = \hat{\theta}_s^h + p(z_s)\theta_s^h$. Using (78), we now can show that

$$p(z_s)R_s k^h + \min\left(0, \hat{\psi}_s^h\right) + p(z_s)\min\left(0, \psi_s^h\right) = p(z_s)R_s k^h + \hat{\theta}_s^h + p(z_s)\theta_s^h + 0 \geq 0$$

where the last inequality follows (4). This implies that (45) holds.

Case II: we can use a similar argument to show that (45) holds when $\hat{\theta}_s^h + p(z_s)\theta_s^h = \hat{\psi}_s^h > 0$. In summary, we have show that all collateral requirement conditions hold. *Q.E.D.*