A Difference-differential Analogue of the Burgers Equation and Some Models of Economic Development

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CEMI RAS

1998

ЦЕНТРАЛЬНЫЙ ЭКОНОМИКО-МАТЕМАТИЧЕСКИЙ ИНСТИТУТ
CENTRAL ECONOMICS AND MATHEMATICS INSTITUTE

РОССИЙСКАЯ
АКАДЕМИЯ НАУК

RUSSIAN
ACADEMY OF SCIENCES

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Working Paper # WP/98/051

МОСКВА
1998

Работа посвящена исследованию ряда дифференциально-разностных уравнений, среди которых центральную роль играет следующее:

$$dF_n/dt = \varphi(F_n)(F_{n-1} - F_n), \qquad (*)$$

где $\{F_n(t), n = 0, 1, 2, ...\}$ при любом t является функцией распределения, а φ - положительная функция на отрезке [0, 1]. Уравнение (*) возникло как описание экономического развития отрасли, учитывающее процессы создания и распространения новых технологий. Статья содержит обзор ранее полученных результатов, в том числе, касающихся многомерного обобщения и применения этого уравнения к теории экономического роста.

Если φ убывает, то решения проблемы Коши для уравнения (*) сходятся к семейству волновых решений. Для возрастающего φ асимптотическое поведение имеет характер диффузии. Сформулирована общая гипотеза для немонотонного случая и доказан аналог теоремы Вайнбергера (1990). Приведены аргументы показывающие, что уравнение (*) можно считать аналогом уравнения Бюргерса.

The paper is devoted to investigation of a number of difference-differential equations, among them the following one plays the central role:

$$dF_n/dt = \varphi(F_n)(F_{n-1} - F_n) \qquad (*)$$

where, for every t, $\{F_n(t), n = 0, 1, 2, ...\}$ is a probability distribution function, and φ is a positive function on [0, 1]. The equation (*) arose as a description of industrial economic development taking into account processes of creation and propagation of new technologies. The paper contains a survey of the earlier received results including a multy-dimensional generalisation and an application to the economic growth theory.

If φ is decreasing then solutions of the Cauchy problem for (*) approach to a family of wave-trains. We show that diffusion-wise asymptotic behaviour takes place if φ is increasing. For the nonmonotonic case a general hypothesis about asymptotic behaviour is formulated and an analogue of a Weinberger's (1990) theorem is proved. It is argued that the equation can be considered as an analogue of Burgers equation.

Key words: Difference-differential equations, Burgers equations, non-linear diffusion, long-time asymptotic of Cauchy problem, evolution of industries, economic growth, innovation and imitation processes.

1. Introduction

In this paper having partially survey character we present a number of results, obtained during last 10 years, concerning some nonlinear difference-differential equations, their solutions reveal stable wave-vise behavior. In a sense the main of the equations can be considered as an analogue of the famous Burgers equation. We come to them thinking about an old economic problem: how to describe endogenous economic growth or technical progress.

This application merits to be discussed before we concentrate on mathematical essence of the theory under consideration.

Our main initial point is the idea developed by Josef Schumpeter (1939), who divided the mechanism of technological changes into two components: creation of new technologies by a firm (innovation process) and adoption of technologies created by other firms (imitation process).

There are many papers about the transition processes from one technology to another. Most of them use the simple equation such as $dF_1/dt = -\beta(1-F_1)F_1$; $F_1(-\infty) = 1$, where F_1 is the share of firms (or capacities) which use an old technology; the speed of the transition is proportional to F_1 and the proportionality coefficient increases with expansion of the share of the firms which have adopted the new technology. So the imitation is the main force in the development of the system. This equation has an explicit solution—famous logistic curve which was used in many hundreds empirical works (Schumpeter have mentioned it and Zvi Griliches was the first who used it for some case studies of technological change (Griliches(1957)). In many cases it is in good accordance with empirical data (Davies (1979)).

Almost all of these works consider only transition process between two technologies. But in many big industries we have permanent interactions between innovation and imitation processes: some firms create new technologies and others imitate them.

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Even in industries producing a homogeneous good, technologies of different effectiveness often coexist. Therefore, considering an industry with many firms, one can describe its development in terms of efficiency distribution, that is the distribution of firms on efficiency. Efficiency may be defined as profit or added value per unit of capacity. It is assumed that each firm wants to increase its level of efficiency. We will consider this level as a discreet variable n which may take any integer values. Empirical observations show that the efficiency distributions in a "mature" industry are similar at different time moments (Sato (1975)).

So we have some empirical observations to explain.

Iwai (1984a, b) undertook the first attempt to show that both of these empirical facts, that is, the "logistic" character of diffusion curves and stability of the form of the efficiency distribution in some industries, are consequences of a "dynamic equilibrium" between innovation and imitation processes. The Iwai model is based on two main assumptions. He supposes that the probability of transition to an efficiency level is the same for all less efficient firm. Therefore the rate of change of the integral distribution function at every point is defined by its value at that point. Moreover, the exponential velocity of the emergence of new, the most effective technologies is postulated directly and thus the displacement velocity of the efficiency distribution is established a priori. Both assumptions seem to be artificial.

Developing the Iwai approach, we proposed an alternative model (Polterovich, Henkin, 1988a, 1989) based on the assumption that transitions are realized in succession from each level to the higher neighboring one.

In the next Section we formulate four hypothesis that lead to a very simple difference-differential equation

$$\frac{dF_n}{dt} = (\alpha + \beta(1 - F_n))(F_{n-1} - F_n), \qquad (1.1)$$

where n is integer and for every t $\{F_n^{(t)}\}_{n=-\infty}^{\infty}$ is a distribution function. The equation looks like a discretization of the shock-wave equation. But it turns out that the Cauchy problem for equation (1.1) admits explicit solutions that approach to a family of wave-trains uniformly by n. The linearizing substitution and the behavior are very similar to the Burgers equation ones. There is no straightforward connection between (1.1) and Burgers equation but in the Burgers equation literature [Hopf (1950), Il'jin, Olejnic (1960), Weinberger (1990)] we found a number of ideas that turned out to be useful for exploration of (1.1). It is the reason why we called (1.1) as difference-differential analogue of Burgers equation.

In the Section 3 we consider a natural generalization of (1.1)

$$\frac{dF_n}{dt} = \varphi(F_n)(F_{n-1} - F_n) \qquad (1.2)$$

for non-increasing φ. This equation seems to be not linearizable. But qualitative behavior of its solutions is substantially the same as for (1.1).

Section 4 is devoted to several applications and variations of the basic model (1.2) including an evolutionary model of economic growth.

Section 5 contains the following two-dimensional generalization of (1.1)

$$\frac{dF_{mn}}{dt} = \varphi_1(F_m^{(1)})(F_{(m-1)n} - F_{mn}) + \varphi_2(F_n^{(2)})(F_{m(n-1)} - F_{mn}), \qquad (1.3)$$

where

$$F_m^{(1)} = \sup_n F_{mn}, \qquad (1.4)$$

$$F_n^{(2)} = \sup_m F_{mn}, \qquad (1.5)$$

where $\{F_{mn}\}_{m,n=-\infty}^{\infty}$ is a two-dimensional distribution. If φ_i, $i = 1, 2$ are non-increasing then solutions of the Cauchy problem for (1.3)–(1.5) approach to a family of wave solutions. Each of them is a product of one-dimensional wave-trains. We do not know any continuous analogues of the system (1.3).

If φ is an increasing function then equation (1.2) has no wave solutions. In this case we observe a diffusion behavior which is similar to heat expansion: here the distribution density tends uniformly to zero. Its asymptotics is described in Section 6 in detail.

In Section 7 we do not assume that in (1.2) φ is monotonic and formulate necessary and sufficient conditions for wave trains to exist. Stability problem for general case of nonmonotonic φ is considered in Section 8. We have no general results but considering some particular cases we come to hypothesis that general asymptotic behavior is a combination of wave train behavior and diffusion movement. In the concluding Section 9 we compare (1.2) with Burgers equation.

Sections 2–4 contain a short survey of main already published results. Sections 6–8 contain mainly new results. We give them with complete proofs. As a step to demonstrate the hypothesis we prove an analogue of the theorem received by Weinberger (1990) for Burgers equation.

2. The simplest model

Let F_n be the share of firms which have efficiency level n or less. So $\mathcal{F} = \{F_n\}$ is a distribution function.

Our model describes the evolution of the distribution curve F_n in time. We introduce four hypotheses.

1. The firm can not jump over levels: if a firm has the level n then it may transit to the level $n + 1$ only.

2. The speed of the transition is the sum of two components: an innovation component and an imitation component.

3. The speed of the transition from the level n to the next level per unit of time as a result of the imitation is proportional to the share of more efficient firms.

4. The speed of the transition as a result of the innovation is constant.

One can doubt in each of these assumptions. But it is the simplest system of assumptions which saves a reasonable differences between innovation and imitation: innovation processes are spontaneous whereas the imitation propensity depends on the position of the firm among other firms.

So we have the following infinite system of difference-differential equations.

$$\frac{dF_n}{dt} = \alpha(F_{n-1} - F_n) + \beta(1 - F_n)(F_{n-1} - F_n)$$
$$= (\alpha + \beta(1 - F_n))(F_{n-1} - F_n) \quad (2.1)$$

under initial conditions

$$F_n(0) = 0, \quad \text{for } n \leq 0; \qquad 0 \leq F_n \leq 1, \quad \text{for all } n; \qquad (2.2)$$

$$\sum_{n=1}^{\infty} (1 - F_n) < \infty;$$

where the constant $\alpha > 0$ characterizes the velocity of the properly innovatory process, and the value $\beta(1 - F_n)$, $\beta > 0$, defines the share of firms going over from the level n to the next higher level $n + 1$ per unit of time due to imitation.

In accordance to the hypothesis 2 the speed of transition from level n to the level $n+1$ is the sum of innovation and imitation components:

$$\varphi(F_n) = \alpha + \beta(1 - F_n). \quad (2.3)$$

This equation can be linearized by substitution [Levi, Ragnisco, Bruschi (1983)]

$$F_n = 1/\beta(\mu - z_{n-1}/z_n), \qquad 1 \leq n < \infty, \quad (2.4)$$
$$z_0 = e^{\mu t}, \qquad \mu = \alpha + \beta.$$

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Since $F_n(t) \equiv 0$ under $n \leq 0$ we can consider positive n only. We get: $dz_n/dt = z_{n-1}$, $1 \leq n \leq \infty$, $z_0 = e^{\mu t}$. Hence $dz_n/dt = e^{\mu t}$.

The explicit formula of the solution F_n is rather complicated and it is not necessarily to write it down. But it is remarkable that the equation (2.1) has the following family of solutions

$$F_n^*(t, A) = (1 + A(\alpha/\mu)^n e^{\beta t})^{-1}, \qquad (2.5)$$

where A is an arbitrary parameter of a shift. The function (2.3) is a wave solution which runs to the right (that is to the direction of increase of efficiency) with a constant speed

$$c = \beta/\ln(\mu/\alpha), \qquad \mu = \alpha + \beta. \qquad (2.6)$$

Theorem 2.1 (Polterovich, Henkin, 1988a). *Let $F = \{F_n\}$ be a solution of the problem (2.1), (2.2).*

1. *Then one can find d such that*

$$\sup_n |F_n(t) - F_n^*(t, d)| \to \infty, \qquad t \to \infty.$$

2. *If the initial conditions are finite, that is $F_n(0) = 1$ under $n \geq N$, N—some positive integer, then*

$$|F_n(t) - F_n^*(t, d)| \leq \lambda e^{-\gamma t}, \qquad 0 \leq n \leq \infty, t \geq T_0,$$

where $\gamma = \gamma(\alpha, \beta)$, λ, T_0 depend on α, β, N, and on the value of the first integral (see (3.4) below). So every solution approaches to some wave-solution uniformly by n.

If n is fixed then the wave solution (2.5) is the well known logistic diffusion curve; so we have a result in accordance with many empirical works. Under fixed t, the wave solution is the logistic probability distribution. Theorem 2.1 says that asymptotic behavior

of the solution doesn't depend on initial conditions. After some time any solution has the shape and the speed of a wave solution.

The linearizing substitution (2.4) is similar to the well-known Florin–Cole–Hopf substitution for the Burgers equation, and the theorem 2.1 is quite similar to the corresponding Hopf theorem about Burgers equation (Hopf [1950]). Due to these facts we consider (2.1) as a difference-differential analogue of the Burgers equation. This analogy is discussed in the last section.

An important generalization of the equation (2.1) arises if one considers nonlinear speed of transition $\varphi(F_n)$ instead of linear one (2.3). The most complete results can be obtained for the case of non-increasing φ.

3. The case of nonlinear non-increasing φ

We assume that at every moment a fraction $\varphi(F_n)$ firms goes over from level n to level $n + 1$; leaps through several levels are not admitted. Obviously $F_n - F_{n-1}$ is the proportion of all firms that are at the level n, and in a unit of time decreasing of the share is equal to $\varphi(F_n)(F_n - F_{n-1})$. Hence we obtain the following system of difference-differential equations describing the evolution of the efficiency distribution in the industry

$$\frac{dF_n}{dt} = \varphi(F_n)(F_{n-1} - F_n), \qquad -\infty < n < \infty \qquad (3.1)$$

under initial conditions

$$a \leq F_n(0) \leq b, \qquad \sum_{-\infty}^{0} (F_n(0) - a) < \infty, \qquad \sum_{0}^{\infty} (b - F_n(0)) < \infty, \qquad (3.2)$$

where a, b—constants, $a < b$, $\varphi: [a, b] \to R^1$.

We accept the following assumption.

A1. φ is positive, bounded in $[a, b]$, and $1/\varphi$ is integrable.

Let us introduce a function $\Phi(z)$

$$(b-a)\Phi(z) = \int_z^b dy/\varphi(y), \qquad z \in [a,b]. \tag{3.3}$$

For a sequence of functions

$$\mathcal{F} = \{F_n\}_{n=-\infty}^{\infty}$$

define

$$B_{\mathcal{F}}(t) = \sum_{n=1}^{\infty} \Phi(F_n(t)) - \sum_{n=-\infty}^{0} (\Phi(a) - \Phi(F_n(t)) - t). \tag{3.4}$$

Theorem 3.1. *Under A1 there exists a unique solution*

$$\mathcal{F} = \{F_n(t)\}_{-\infty}^{\infty}$$

of the problem (3.1), (3.2). For all $t \geq 0$ one has

$$F_n(t) \to a, \qquad \text{as } n \to -\infty,$$
$$F_n(t) \to b, \qquad \text{as } n \to +\infty,$$
$$B_{\mathcal{F}}(t) \equiv B_{\mathcal{F}}(0);$$

$F_n(t) \geq F_{n-1}(t)$ *for all n, if $F_n(0) \geq F_{n-1}(0)$ for all n.*

In accordance to theorem 3.1 the problem (3.1), (3.2) has a nontrivial first integral $B_{\mathcal{F}}(t)$.

It is not surprisingly but important that the equation saves the property to be a distribution.

A special solution of (3.1) is called a wave train if

$$F_n(t) = F(x), \qquad x = n - ct - d,$$

where speed c of the wave depends on asymptotic values $\lim_{x\to-\infty} F(x) = a$, $\lim_{x\to+\infty} F(x) = b$, and d is an arbitrary shift parameter.

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For wave trains the equation (3.1) can be written in the following form

$$c\frac{dF}{dx} = \varphi(F)(F(x) - F(x-1)). \tag{3.5}$$

Our second assumption plays the decisive role in this section.

A2. φ is nonincreasing, $\varphi(a) > \varphi(b)$, φ satisfies the Lipshitz condition.

In the economic interpretation described above the monotonicity of φ means that the "intensity" of the imitation process increases as the share $1 - F_n$ of more advanced firms increases (see (2.3) as an example).

Theorem 3.2. *Let A1, A2 be valid. Then a wave train $\widetilde{F}(x)$ exists iff*

$$c = 1/\Phi(a), \qquad \Phi(a) = \int_a^b dy/\varphi(y) \ \Big/ \ (b-a).$$

Every wave train has the form $\widetilde{F}(x-d)$, where d is a constant. There exist positive numbers $\lambda_0, \lambda_1, \lambda_2, \overline{x} > 0$ such that

$$e^{\lambda_0 x} \geq \widetilde{F}(x) - a \geq e^{\lambda_1 x}, \qquad \text{for all } x \leq -\overline{x}, \tag{3.6}$$
$$e^{-\lambda_2 x} \geq b - \widetilde{F}(x), \qquad \text{for all } x \geq \overline{x}. \tag{3.7}$$

For applications it is useful to have conditions ensuring that the distribution density is unimodal.

Theorem 3.3. *Let A1, A2 be fulfilled, φ be twice differentiable and $1/\varphi$ be convex. Then the wave train density dF/dx has a unique local maximum point.*

Note that $1/\varphi$ is convex for positive and concave φ.

Theorem 3.4. *Let A1, A2, and $\widetilde{F}$ be a wave train. Then for every solution $\mathcal{F} = \{F_n\}$ of the problem (3.1), (3.2) one can find a constant d such that*

$$\sup_n |F_n(t) - \widetilde{F}(n - ct - d)| \to 0$$

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as $t \to \infty$.

The constant d is the solution of the equation

$$B_{\mathcal{F}}(0) = \sum_{n=-\infty}^{0} (\Phi(\widetilde{F}(n-d)) - \Phi(0)) + \sum_{n=1}^{\infty} \Phi(\widetilde{F}(n-d)).$$

Theorem 3.4 shows that as a result of the interaction of imitation and innovation processes the form of the efficiency distribution stabilizes with time; this curve moves with almost constant velocity along the abscissa axis; in asymptotics neither the form nor the velocity depends on initial conditions. Thus the model explains the stability of the form of observed distribution curves.

This kind of behavior is typical for some classes of physical and biological systems being studied in non-linear wave theory (see, for example, Whitham (1974)).

Theorems 3.1–3.4 were proved in Henkin, Polterovich (1991). The proof of the central Theorem 4 uses substantially some ideas contained in Iljin, Olejnic (1960) where generalized Burgers equation was studied. The existence of wave solutions and the stability result hold under much weaker assumptions (see Section 7).

4. Applications and variations of the model

The model considered above is very stylized. In more concrete situations one needs to deal with its modifications and searches for a way to apply the results of previous sections. Below we describe three simple applications.

A. Evolutionary model of economic growth

We describe a simple model taking into account qualitative improvement of production capacities in the processes of imitation and innovation as well as quantitative growth.

Let consider some industry and let M_n be the volume of production capacities which bring the profit λ_n per unit of capacities per unit of time. The quantity λ_n is considered

as a measure of efficiency of the level n. The function of capacity distribution on efficiency levels is defined by the formula

$$F_n = \left(\sum_{0}^{n} M_k \right) \left(\sum_{0}^{\infty} M_k \right)^{-1}, \qquad n = 0, 1, \ldots \tag{4.1}$$

It is supposed that the profit $\lambda_n M_n$ of the level n is divided onto two streams of investment. The share $\varphi_0(F_n)$ of this profit creates new capacities of the next level $n+1$, and the quantity $(1 - \varphi_0(F_n))\lambda_n M_n$ is spent on the expansion of the capacities of the level n. For simplicity, consumption of the produced good is not taken into account. Thus we receive the following equation of economic growth

$$dM_n/dt = (1 - \varphi_0(F_n))\lambda_n M_n + \varphi_0(F_{n-1})\lambda_{n-1} M_{n-1} \tag{4.2}$$

under boundary and initial conditions

$$M_0(t) \equiv 0, \quad M_n(0) \geq 0, \quad \sum_{1}^{N} M_n(0) > 0, \quad M_n(0) = 0 \text{ if } n > N. \tag{4.3}$$

If $\lambda_n = \text{const} = \lambda$ then (4.2) can be rewritten as (3.1) under $\varphi = \lambda\varphi_0$. If numbers λ_n are different then equation (4.2) turns out to be a perturbation of (3.1). Using Theorem 3.4 one can prove the following statement.

Theorem 4.1 (Polterovich, Henkin, 1989). *Let φ_0 satisfy A1, A2 and λ_n be a positive increasing sequence convergent to some λ fast enough such that*

$$\sum_{1}^{\infty} k(\lambda - \lambda_k) < \infty.$$

Then for every solution $\{F_n\}$ of the problem (4.1), (4.2), (4.3) *there exists a constant d such that*

$$\sup_n |F_n(t) - F^*(n - ct - d)| \to 0, \qquad t \to \infty,$$

where F^* is the solution of equation (3.5) *under $\varphi = \lambda\varphi_0$ and*

$$c^{-1} = \int_0^1 dy/\varphi(y).$$

It is naturally to assume that efficiency is bounded and converges to a limit. The most restrictive assumption of Theorem 4.1 requires convergency to be fast enough. The result shows that stable wave train behavior can arise in a process of economic growth.

B. Belenky' model

An interesting modification of the basic model (3.1), (3.2) was proposed by Belenky (1990a). He assumes that the speed of transition Ψ from efficiency level n to level $n+1$ depends on a proportion of more advanced firms among all firms that are not worse than the firms of level n. This assumption entails the following equation

$$\frac{d\theta_n}{dt} = \psi(\theta_n/\theta_{n-1})(\theta_{n-1} - \theta_n), \tag{4.4}$$

where $\theta_n = 1 - F_n$. Let us denote

$$r_n = \frac{1 - F_n}{1 - F_{n-1}}. \tag{4.5}$$

Then (4.4) might be rewritten as follows

$$\frac{d \ln \theta_n}{dt} = \Psi(r_n), \tag{4.6}$$

where $\Psi(r) = \psi(r)\left(\frac{1}{r} - 1\right)$. From (4.6) one has

$$\frac{d \ln r_n}{dt} = \Psi(r_n) - \Psi(r_{n-1}). \tag{4.7}$$

Let Φ be the following function

$$\Phi(Y_n) = \ln \Psi^{(-1)}(Y_n), \tag{4.8}$$

where $Y_n = \Psi(r_n)$, and let $-\varphi(Y) = 1/\Phi'(Y)$. Then we obtain from (4.7), (4.8)

$$\frac{dY_n}{dt} = \varphi(Y_n)(Y_{n-1} - Y_n).$$

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that is the equation (3.1).

If Ψ is positive, strictly decreasing and continuously differentiable then φ is positive, continuous and fulfills conditions of a generalized theorem 3.1 (see Theorem 8.1 of Section 8). Assume that the initial equations (3.2) are valid with $a = 0$, $b = 1$, and there exists

$$\lim_{n \to \infty} r_n(0) = \lim_{n \to \infty} (1 - F_n(0))/(1 - F_{n-1}(0)) = \bar{r}, \qquad 0 < \bar{r} < 1.$$

Then the original problem (4.4), (3.2) is equivalent to (3.1), (3.2) with overfall $[0, \Psi(\bar{r})]$, and stability Theorem 8.1 is applicable. Thus in this case overfall and speed of the wave train depend on parameter $\bar{r}$ of initial conditions.

C. A model with depreciation

In the models considered above capacities can improve but not diminish their efficiency levels. In reality capacities can depreciate, and it means their movement to the left along the efficiency axe. For this case one gets the following generalization of (3.1)

$$\frac{dF_n}{dt} = -\varphi(F_n)(F_n - F_{n-1}) + \mu(F_{n+1} - F_n), \tag{4.9}$$

where μ is depreciation velocity. Equation (4.9) was introduced in Polterovich, Henkin (1988a) but was not studied theoretically. In Gelman and oth. (1993) a discrete version of (4.9) was used as a model to describe the development of ferrous metallurgy in USSR during 1976–1988. A group of 35 enterprises was considered that produced more than 90% of total output of the industry. Profit per unit of material expenditure was used as an efficiency indicator. The function φ was supposed to be linear: $\varphi(F) = \alpha + \beta(1 - F)$. Three coefficients α, β, μ were calculated to reach the best coincidence of the real evolution and the output of the model. The result turns out to be quite well for subperiods 1976–1982 and 1982–1988 when prices were constant. Note that the industry

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declined during the period considered, therefore the distribution function moved to the left due to depreciation.

D. A difference-differential analogue of Kolmogoroff–Petrovsky–Piskunoff equation

It was assumed above that capacities can transite from an efficiency level to the next one only. If jumps on two or several levels are permitted we get a number of generalizations of (3.1). For example if capacities can transite from levels $n-1$ and $n-2$ to level n one has the following equation

$$\frac{dF_n}{dt} = (\varphi_1(F_n) + \varphi_2(F_n))(F_{n-1} - F_n) + \varphi_2(F_{n-1})(F_{n-2} - F_{n-1}), \quad (4.10)$$

where $\varphi_1(F_n)$, $\varphi_2(F_n)$ are speeds of transition from level n to the level $n+1$ and the level $n+2$, correspondingly.

The most simple model arises if it is possible to transite from a level to any other level with larger efficiency, and the probabilities of all transitions due to imitation are equal (see Iwai (1984b)). If the innovation speed is constant we obtain the following equation

$$\frac{dF_n}{dt} = -\alpha(F_n - F_{n-1}) - \beta F_n(1 - F_n). \quad (4.11)$$

It looks like an analogue of an equation studied by Kolmogoroff, Petrovsky, Piskunoff (1938) and Fisher (1938). Equations (4.10) and (4.11) are not studied yet.

5. Evolutionary equation for efficiency distributions under several efficiency indicators

The assumption that in the process of development firms orient themselves at the unique efficiency indicator seems to be rather restrictive. A firm can prefer to pay particular attention to profit maximization, to increasing its market share, etc. If efficiency is characterized by a set of indicators, then we should consider innovation and imitation processes aimed at improving each of them.

For simplicity we consider a situation with two efficiency parameters m and n.

We accept the following postulates.

(I). The transition from the state (m, n) can take place into one of two neighboring higher levels: $(m+1, n)$ and $(m, n+1)$.

(II). The proportion of firms per unit of time moving from the state (m, n) to the state $(m+1, n)$ is proportional to the share of firms in the state (m, n) and the proportion coefficient is positive and non-decreasing in the share of firms which are more advanced according to the first indicator. A similar hypothesis is admitted for the transition from (m, n) to $(m, n+1)$.

By assumption (I) transitions under which one of the efficiency indicators decreases are not admitted. Moreover, it is postulated that it is possible to represent a process of development as a union of elementary acts of improving one of the indicators under constancy of the other.

The assumptions are restrictive enough. Real economic actions often increase some parameters while decreasing others.

Let m, n be the levels of two efficiency parameters

$$m, n = 0, 1, \ldots,$$

and f_{mn} be the proportion of firms at level (m, n); $F_{mn} = \sum_{k=1}^{m} \sum_{r=1}^{n} f_{kr}$ be a distribution function.

$$F_m^{(1)} = \sum_{k=1}^{m} \sum_{r=1}^{\infty} f_{kr}; \qquad F_n^{(2)} = \sum_{k=1}^{\infty} \sum_{r=1}^{n} f_{kr}.$$

In accordance to axioms (I), (II) we have the following two-dimensional equation:

$$\frac{df_{mn}}{dt} = -\varphi_1(F_m^{(1)})f_{mn} - \varphi_2(F_n^{(2)})f_{mn} + \varphi_1(F_{m-1}^{(1)})f_{(m-1)n} + \varphi_2(F_{n-1}^{(2)})f_{m(n-1)},$$

$$m, n = 0, 1, \ldots$$

It can be rewritten by the following way:

$$\frac{dF_{mn}}{dt} = \varphi_1(F_m^{(1)})(F_{(m-1)n} - F_{mn}) + \varphi_2(F_n^{(2)})(F_{m(n-1)} - F_{mn}), \qquad (5.1)$$

under $F_{0n}(t) \equiv 0$, $F_{m0}(t) \equiv 0$.

$$F_{mn}(0) = \sum_{k=1}^{m} \sum_{r=1}^{n} f_{kr}(0), \qquad f_{kr}(0) \geq 0, \qquad F_{mn}(0) = 1, \qquad (5.2)$$

$m \geq m_0$, $n \geq n_0$ for some given m_0, n_0.

We would like to note that the relation (5.2) is not local since functions $F_m^{(i)}$ are not defined by values F_{kl} in a vicinity of (m, n). The equation could be an analogue of an integro-differential equation.

It turns out that the two-dimensional motion (5.1) approaches to Cartesian product of two one-dimensional motions studied above. Equation (5.1) was introduced in Polterovich, Henkin (1989). A proof of the following theorem is contained in Henkin, Polterovich (1991).

Theorem 5.1. *Let φ_1 satisfy A1, A2, and $\widetilde{F}^{(i)}$, $i = 1, 2$, are one-dimensional wave trains, and c_i are their speeds. Then*

1. *$\widetilde{F}_{mn}(t, d) = \prod_{i=1}^{2} \widetilde{F}^{(i)}(n_i - c_i t - d_i)$ is a "wave solution" of the two-dimensional problem (5.1), (5.2) under every $d = (d_1, d_2)$.*

2. *For every solution $\{F_{mn}(t)\}$ of the problem (5.1), (5.2) there exists d such that*

$$\sup_{m,n} |F_{mn}(t) - \widetilde{F}_{mn}(t, d)| \to 0, \qquad t \to \infty.$$

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6. The case of nondecreasing φ: diffusion

It seems to be natural to suppose that the imitation speed decreases when the relative efficiency of a firm grows. But as regards innovation, opinions vary. If the innovation speed increases quickly with F_n our assumption A2 may turn out to be not realistic: the function φ could be not decreasing.

If so, the asymptotic behavior of solutions can be very different from studied above. The next sections are devoted to exploration of this more general case. Most results presented below are new. We still have no complete description of asymptotic behavior for general case of φ, but have a hope that the results are important steps in understanding of this problem.

As a first step let us consider the case of nondecreasing φ.

We begin with two simple statements about the diffusion case.

If $\beta = 0$ the equation (2.1) is linear

$$\frac{dF_n}{dt} = \alpha(F_{n-1} - F_n). \qquad (6.1)$$

Let us consider initial conditions $F_n(0)$ that are positive for $n \geq 0$ and equal to zero for $n < 0$. Then the equation (6.1) has the following solution

$$F_n(t) = e^{-\alpha t} \sum_{k=0}^{n-1} (\alpha t)^k F_{n-k}(0)/k! \qquad (6.2)$$

Proposition 6.1. *Let $F_{n-1}(0) \leq F_n(0)$; $F_n(0) \to 1$, $n \to \infty$. Then $f_n(t) \stackrel{def}{=} F_n(t) - F_{n-1}(t) = O(1/\sqrt{t})$, where, by definition, $O(x)$ is a function such that $|O(x)| \leq const \cdot x$.*

Proof. From (6.2) we have

$$e^{\alpha t} f_n = \frac{(\alpha t)^{n-1} F_1(0)}{(n-1)!} + \sum_{k=0}^{n-2} \frac{(\alpha t)^k}{k!} f_{n-1}(0) \leq const \cdot \max_k \frac{(\alpha t)^k}{k!}.$$

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It follows from Stirling formulae that

$$\max_k \frac{(\alpha t)^k}{k!} \le \text{const} \cdot e^{\alpha t}/\sqrt{t}.$$

The proposition is proved.

If φ has a positive derivative then the speed of diffusion is larger.

In the rest of this section we assume to be valid the following conditions.

(i) Function φ and its derivatives φ', φ'' are bounded,

(ii) φ' positive on $[0,1]$,

(iii) $F_n(t)$ is a solution of equation (3.1), (3.2) with $a = 0$, $b = 1$, $F_n(0) = 0$ for $n \le 0$, $F_{n-1}(0) \le F_n(0)$ for $n \ge 1$.

Theorem 6.1. *Let* $\varphi'(y) \ge \xi > 0$ $\forall y \in [0,1]$. *Then*

$$f_n(t) = F_n(t) - F_{n-1}(t) \le 1/(\xi t + 1) \quad \text{for all } n, t.$$

Proof of Theorem 6.1. The equation (3.1) may be rewritten in the terms of the density

$$df_n/dt = -\varphi(F_n)f_n + \varphi(F_{n-1})f_{n-1}.$$

Let $v_n(t) = (\xi t + 1)f_n(t)$. Then

$$dv_n/dt = -\varphi(F_n)v_n + \varphi(F_{n-1})v_{n-1} + \xi(\xi t + 1)^{-1}v_n. \qquad (6.3)$$

We must show that $v_n \le 1$. Suppose the contrary. Then for some constant $\varkappa > 1$

$$t_0 = \inf\{t | v_n(t) = \varkappa \text{ for some } n\} < \infty.$$

Obviously $t_0 > 0$. Since $v_n(t_0) \to 0$ as $n \to \mp\infty$ there is a number k such that $v_k(t_0) = \varkappa \ge v_{k-1}(t_0)$, $dv_k(t_0)/dt \ge 0$. But using (6.3) we come to contradiction

$$-\xi(\xi t + 1)^{-1}\varkappa \le -(\varphi(F_{k-1} + f_k) - \varphi(F_{k-1}))\varkappa \le -\xi(\xi t_0 + 1)^{-1}\varkappa^2.$$

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The theorem is proved.

Let us introduce the following function

$$F_n^*(t) = \begin{cases} 0, & n \le \varphi(0)t, \\ \varphi^{-1}(n/t), & \varphi(0)t < n < \varphi(1)t, \\ 1, & n \ge \varphi(1)t. \end{cases} \qquad (6.4)$$

The next theorem is the main result of this section.

Theorem 6.2. *There exists a constant* K *such that*

$$\sup_n |F_n(t) - F_n^*(t)| \le K/\sqrt{t}, \qquad t \ge 1. \qquad (6.5)$$

It means that with time all solutions of equation (3.1), (3.2) approach to $F_n^*(t)$ independently on initial conditions.

The proof of Theorem 6.2 is not very short but its basic idea is simple and is contained in the following statement.

Lemma 6.1. *Let* $h(y)$ *be a continuously differentiable function on* $\varphi([0,1])$, *and*

$$H(y,t) = \varphi^{-1}(y) + \frac{1}{t}h(y). \qquad (6.6)$$

Then the function $H_n(t) = H(n/t, t)$ *satisfies the following inequality*

$$\left| \frac{dH_n(t)}{dt} - \varphi(H_n(t))(H_{n-1}(t) - H_n(t)) \right| \le K/t^2, \qquad (6.7)$$

where the constant K *depends on* φ *and* h.

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Proof of Lemma 6.1. Let $g(y) = \varphi^{-1}(y)$. One has from (6.6) under $y = n/t$

$$\frac{dH}{dt} = -\frac{y}{t}g'(y) - \frac{y}{t^2}h'(y) - \frac{1}{t^2}h(y), \tag{6.8}$$

$$\varphi(H(y,t)) = \varphi(g) + \frac{\varphi'(g)}{t}h(y) + O(1/t^2), \tag{6.9}$$

$$\begin{aligned}
H\left(\frac{n-1}{t}, t\right) - H\left(\frac{n}{t}, t\right) &= H(y - 1/t, t) - H(y, t) \\
&= -\frac{1}{t}H'_y(y, t) + O(1/t^2) \\
&= -\frac{1}{t}[g'(y) + h'(y)/t] + O(1/t^2) \\
&= -\frac{1}{t}g'(y) + O(1/t^2). \tag{6.10}
\end{aligned}$$

Having in mind that $H_n(t) = H(n/t, t)$ we get from (6.8), (6.9), (6.10)

$$\frac{dH_n}{dt} - \varphi(H_n)(H_{n-1} - H_n) = \frac{1}{t}[-yg'(y) + \varphi(g(y))g'(y)] + O(1/t^2) = O(1/t^2),$$

since $\varphi(g(y)) = y$. Lemma 6.1 is proved.

To prove theorem 6.2 we need several other lemmas.

Lemma 6.2. Let φ (or $\widehat{\varphi}$) be a nondecreasing function, and $\varphi(F) \leq \widehat{\varphi}(F)$ for all $F \in [0,1]$. Suppose that $\{G_n\}$, $\{\widehat{G}_n\}$ fulfill the following inequalities

$$\frac{dG_n}{dt} \geq \varphi(G_n)(G_{n-1} - G_n), \tag{6.11}$$

$$\frac{d\widehat{G}_n}{dt} \leq \widehat{\varphi}(\widehat{G}_n)(\widehat{G}_{n-1} - \widehat{G}_n), \tag{6.12}$$

$$\widehat{G}_n(t) \geq \widehat{G}_{n-1}(t) \tag{6.13}$$

for all $\gamma_2 t > n \geq \gamma_1 t$, where $\infty \geq \gamma_2 > \gamma_1 \geq 0$. Let us assume that the following conditions hold

$$V_n(t) = G_n(t) - \widehat{G}_n(t) \geq 0 \quad \text{if} \quad -1 \leq n - \gamma_1 t \leq 0$$

$$\text{or } n - \gamma_2 t = 0, \quad t \geq T_0 > 0, \tag{6.14}$$

$$V_n(T_0) \geq 0 \quad \text{if} \quad \gamma_2 T_0 \geq n \geq \gamma_1 T_0, \tag{6.15}$$

where by definition $\gamma_2 t = \infty$ if $\gamma_2 = \infty$ and $V_\infty(t) = \underline{\lim}_{n\to\infty} V_n(t)$. Then

$$V_n(t) \geq 0 \quad \text{for all } n, t \text{ such that } \gamma_2 t \geq n \geq \gamma_1 t, \quad t \geq T.$$

Proof. From (6.11), (6.12) one has

$$\frac{dV_n}{dt} \geq \varphi(G_n)(V_{n-1} - V_n) + (\varphi(G_n) - \widehat{\varphi}(\widehat{G}_n))(\widehat{G}_{n-1} - \widehat{G}_n). \tag{6.16}$$

Let us fix $T > T_0$, and denote

$$g_n = \min_t V_n(t) \quad \text{under} \quad t \in E_n = [n/\gamma_2, n/\gamma_1] \cap [T_0, T].$$

If E_n is empty we take $g_n = +\infty$.

We prove that $g_n \geq 0$ for all n. By contradiction, suppose that the set $A = \{n : g_n < 0\}$ is not empty. Let $r = \min_{n\in A} n$, and $V_r(\tau) = g_r < 0$. Due to monotonicity of φ we have

$$\varphi(G_r(\tau)) \leq \varphi(\widehat{G}_r(\tau)) \leq \widehat{\varphi}(\widehat{G}_r(\tau)). \tag{6.17}$$

(If $\widehat{\varphi}$ is nondecreasing then $\varphi(G_r) \leq \widehat{\varphi}(G_r) \leq \widehat{\varphi}(\widehat{G}_r)$.) Obviously, the choice of r, τ and (6.14) entail $V_{r-1}(\tau) \geq 0$, therefore $V_{r-1}(\tau) > V_r(\tau)$. Hence $\frac{dV_r(\tau)}{dt} > 0$ due to (6.16), (6.17), (6.13). Due to (6.15), $\tau > T_0$, and (6.14) entails $\tau > r/\gamma_2$. Since τ is a minimal point of $V_r(t)$ on the segment E_r we obtain $\frac{dV_r(\tau)}{dt} \leq 0$. It is a contradiction. Lemma 6.2 is proved.

Remark 1. Let $\Gamma = \{(n,t)|\gamma_2 t \geq n \geq \gamma_1 t\}$ and $G_n(t), \widehat{G}_n(t) \in [\theta_1, \theta_2] \subset [0,1]$. Then the statement of Lemma 6.2 is valid if inequality $\varphi(F) \leq \widehat{\varphi}(F)$ and the monotonicity of φ (or $\widehat{\varphi}$) are fulfilled for $F \in [\theta_1, \theta_2]$.

Remark 2. It is substantial in the proof of Lemma 6.2 that φ (or $\widehat{\varphi}$) is a nondecreasing function but its smoothness is not used.

Denote

$$\Psi(z) = \frac{1}{2\beta}(z - \sqrt{b + z^2}), \tag{6.18}$$

where β, b are some positive numbers, and

$$\check{F}_n(t) = \frac{1}{\sqrt{t}}\Psi\left(\frac{n - c_1 t}{\sqrt{t}}\right) + 1. \tag{6.19}$$

$$\widehat{F}_n(t) = -\frac{1}{\sqrt{t}}\Psi\left(\frac{c_0 t - n}{\sqrt{t}}\right). \tag{6.20}$$

Lemma 6.3. *Assume that φ fulfills* (i) *from Section 6, and* $0 < \beta \leq \min_{F \in [0,1]} \varphi'(F)$, $c_1 = \varphi(1)$, $c_0 = \varphi(0)$. *Then*

$$\frac{d\check{F}_n}{dt} \leq \varphi(\check{F}_n)(\check{F}_{n-1} - \check{F}_n) \quad \text{for all } n, t, \tag{6.21}$$

$$\frac{d\widehat{F}_n}{dt} \geq \varphi(\widehat{F}_n)(\widehat{F}_{n-1} - \widehat{F}_n) \quad \text{for all } n, t. \tag{6.22}$$

Proof. It is simple to check that

$$\frac{d\Psi}{dz} = \frac{\Psi}{2\beta\Psi - z}, \quad \frac{d\Psi}{dz} > 0, \quad \frac{d^2\Psi}{dz^2} < 0. \tag{6.23}$$

For $z = \frac{n - c_1 t}{\sqrt{t}}$ we have

$$\frac{d\check{F}_n}{dt} = -\frac{1}{2t^{3/2}}\Psi + \frac{1}{\sqrt{t}}\frac{d\Psi}{dz}\left(-\frac{1}{2}t^{-3/2}(n - c_1 t) - c_1 t^{-1/2}\right)$$

$$= -\frac{1}{2t^{3/2}}\Psi - \frac{1}{\sqrt{t}}\frac{d\Psi}{dz}\left(\frac{z}{2t} + \frac{c}{\sqrt{t}}\right). \tag{6.24}$$

Obviously, $\check{F}_{n-1} \leq \check{F}_n$, $\varphi(F) \leq \varphi(1) - \beta(1 - F)$, and Ψ is a concave function. Therefore

$$\varphi(\check{F}_n)(\check{F}_{n-1} - \check{F}_n) \geq (\varphi(1) - \beta(1 - \check{F}_n))(\check{F}_{n-1} - \check{F}_n)$$

$$= (c_1 + \beta\Psi(z)/\sqrt{t})\frac{1}{\sqrt{t}}(\Psi(z - 1/\sqrt{t}) - \Psi(z))$$

$$\geq -(c_1 + \beta\Psi/\sqrt{t})(1/t)\, d\Psi/dz. \tag{6.25}$$

In view of (6.23) the right hand sides of (6.24) and (6.25) are equal. It gives (6.21). The inequality (6.22) can be proved by similar considerations.

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Lemma 6.4. *Let the assumptions* (i), (ii) *of Section 6 be fulfilled. Then for every a_0, a_1 there exist constants γ_0, γ_1 such that*

$$F_n(t) \leq \gamma_0/\sqrt{t} \quad \text{if } n \leq c_0 t + a_0, \ t \geq 0, \tag{6.26}$$

$$F_n(t) \geq 1 - \gamma_1/\sqrt{t} \quad \text{if } n \geq c_1 t - a_1, \ t \geq 0, \tag{6.27}$$

where $c_0 = \varphi(0)$, $c_1 = \varphi(1)$.

Proof. Let $\check{F}_n$ be defined by (6.19). It is straightforward to check that $z\Psi(z) + \frac{b}{4\beta} \to 0$ under $z \to \infty$, $\Psi(z) - z/\beta \to 0$ under $z \to -\infty$, $\Psi(z) < 0$ for all z. Let $\beta \leq \min_{F \in [0,1]} \varphi'(F)$. In view of (3.2) and (iii) for every $\varepsilon > 0$ $F_n(0) > 1 - \frac{\varepsilon}{n}$ if n is large enough. At other hand $\check{F}_n(1) < 1 - \frac{b}{5\beta(n-c_1)}$ for large n. Therefore one can find b such that

$$\check{F}_n(1) < F_n(0) \quad \text{for all } n, \text{ and } \quad \check{F}_n(1) < 0 \quad \text{under } n \leq 0. \tag{6.28}$$

Let us show that b can be chosen to fulfill the inequality

$$\check{F}_n(t+1) \leq F_n(t) \quad \text{for all } n \leq 0, \ t \geq 0. \tag{6.29}$$

One can check that

$$\limsup_{t \to \infty}\, _{n \leq 0} \check{F}_n(t) \leq \lim_{t \to \infty} \check{F}_0(t) = 1 - c_1/\beta.$$

Since $\varphi(1) - \varphi(0) \geq \beta = \min_{F \in [0,1]} \varphi'(F)$ we obtain: $\check{F}_n(t) \leq \text{const} < 0$ for t large enough and $n \leq 0$. Therefore (6.29) follows. Let $\widehat{G}_n(t) = \check{F}_n(t+1)$, $G_n(t) = F_n(t)$. Then due to (6.28), (6.29) and Lemmas 6.2, 6.3 we have $\check{F}_n(t+1) \leq F_n(t)$ for all n and all $t \geq 0$.

Hence

$$F_n(t) \geq \frac{1}{2\beta\sqrt{t+1}}(z - \sqrt{b + z^2}) + 1, \quad z = \frac{n - c_1(t+1)}{\sqrt{t+1}}.$$

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If $n \geq c_1 t - a_1$ then

$$F_n(t) \geq 1 - \frac{b}{2\beta[-a_1 - c_1 + \sqrt{b(t+1) + (a_1 + c_1)^2}]} \geq 1 - \frac{\gamma_1}{\sqrt{t}}$$

for some constant γ_1. The inequality (6.27) is proved. The inequality (6.26) can be proved by a similar way.

Let us introduce functions

$$\widehat{F}_n(t) = F_n^*(t) + \gamma/\sqrt{t}, \qquad (6.30)$$

$$\check{F}_n(t) = F_n^*(t) - \gamma/\sqrt{t}, \qquad (6.30a)$$

where F_n^* is defined in (6.4). Let $\{F_n(t)\}$ be a solution of (3.1), (3.2), $c_0 = \varphi(0)$, $c_1 = \varphi(1)$.

Lemma 6.5. *There exist $\gamma > 0$ and $T > 0$ such that the following inequalities are valid*

$$\frac{d\widehat{F}_n(t)}{dt} \geq \varphi(\widehat{F}_n)(\widehat{F}_{n-1} - \widehat{F}_n), \quad c_0 t \leq n \leq c_1 t;\ t \geq T; \qquad (6.31)$$

$$\widehat{F}_n(t) \geq F_n(t), \quad -1 \leq n - c_0 t \leq 0,\ n - c_1 t \geq 0,\ t \geq T; \qquad (6.32)$$

$$\widehat{F}_n(T) \geq F_n(T), \quad c_0 T \leq n \leq c_1 T. \qquad (6.33)$$

$$\frac{d\check{F}_n(t)}{dt} \leq \varphi(\check{F}_n)(\check{F}_{n-1} - \check{F}_n), \quad c_0 t \leq n \leq c_1 t;\ t \geq T; \qquad (6.31a)$$

$$\check{F}_n(t) \leq F_n(t), \quad -1 \leq n - c_0 t \leq 0,\ n - c_1 t \geq 0,\ t \geq T; \qquad (6.32a)$$

$$\check{F}_n(T) \leq F_n(T), \quad c_0 T \leq n \leq c_1 T. \qquad (6.33a)$$

Proof. To fulfill (6.31) let us note that

$$\varphi(F_n^*)(F_{n-1}^* - F_n^*) = \varphi(\widehat{F}_n - \gamma/\sqrt{t})(\widehat{F}_{n-1} - \widehat{F}_n)$$

$$= \varphi(\widehat{F}_n)(\widehat{F}_{n-1} - \widehat{F}_n) - \varphi'(A_n)(\widehat{F}_{n-1} - \widehat{F}_n)\gamma/\sqrt{t}, \qquad (6.34)$$

where $A_n = A_n(t)$ fulfills the inequalities

$$\varphi^{-1}(n/t) - \gamma/\sqrt{t} \leq A_n(t) \leq \varphi^{-1}(n/t), \qquad c_0 t \leq n \leq c_1 t.$$

Besides

$$\widehat{F}_{n-1} - \widehat{F}_n = -1/t\varphi'(B_n), \qquad (6.35)$$

where $\varphi^{-1}(\frac{n-1}{t}) \leq B_n(t) \leq \varphi^{-1}(\frac{n}{t})$, $c_0 t \leq n \leq c_1 t$.

Hence $A_n - B_n \to 0$ under $t \to \infty$ uniformly in n. Therefore and due to (i) one can find T such that

$$\varphi'(A_n(t))/\varphi'(B_n(t)) \geq 3/4 \quad \text{for all } t \geq T \text{ and } c_0 t \leq n \leq c_1 t.$$

Combining this inequality with (6.34), (6.35) we get

$$\varphi(F_n^*)(F_{n-1}^* - F_n^*) \geq \varphi(\widehat{F}_n)(\widehat{F}_{n-1} - \widehat{F}_n) + \frac{3\gamma}{4t^{3/2}}. \qquad (6.36)$$

Let us check now (6.31). In view of Lemma 6.1

$$\frac{d\widehat{F}_n}{dt} = \frac{dF_n^*}{dt} - \frac{1}{2}\frac{\gamma}{t^{3/2}} = \varphi(F_n^*)(F_{n-1}^* - F_n^*) + O(1/t^2) - \frac{1}{2}\frac{\gamma}{t^{3/2}}.$$

Due to (6.36)

$$\frac{d\widehat{F}_n}{dt} \geq \varphi(\widehat{F}_n)(\widehat{F}_{n-1} - \widehat{F}_n) + \frac{1}{4}\frac{\gamma}{t^{3/2}} + O(1/t^2),$$

and (6.31) is valid if $\gamma = \sqrt{T}$, $t \geq T$, T is large enough. Due to the choice of γ

$$\widehat{F}_n(t) = F_n^*(t) + \sqrt{\frac{T}{t}} \geq 1 \quad \text{under } t = T,$$

hence (6.33) is fulfilled. The first condition (6.32) is also valid due to Lemma 6.4 if $\gamma = \sqrt{T} \geq \gamma_0$, where γ_0 is a constant from (6.26). The validity of the second condition (6.32) is obvious since $F_n^*(t) = 1$ for $n = c_1 t$.

We omit the proof of (6.31a)-(6.33a) since it is completely similar.

Proof of Theorem 6.2. For $n \leq c_0 t$ and $n \geq c_1 t$ the statement of Theorem follows from Lemma 6.4. From (6.30)–(6.33), (6.30a)–(6.33a) and from Lemmas 6.5 and 6.2 one has

$$|F_n(t) - F_n^*(t)| \leq \gamma/\sqrt{t}, \qquad t \geq T, \quad c_0 t \leq n \leq c_1 t.$$

Hence (6.5) is valid as well. It completes the proof.

7. Wave-trains for nonmonotonic φ

We say that $\widetilde{F}(x)$ is a wave train if $\widetilde{F}(x)$ is nondecreasing, fulfills (3.5) and $\widetilde{F}(-\infty) = a$, $\widetilde{F}(+\infty) = b$.

It follows from the definition that $\widetilde{F}(x) < b$.

The nonmonotonic case was considered in Henkin, Polterovich (1991, 1994), and in Belenky (1990). Denote

$$\Gamma(z) = (1/(z - a)) \int_a^z dx/\varphi(x), \qquad (7.1)$$

and let Φ be defined by (3.3).

Below we will use A1 from Section 3 jointly with some of the following assumptions.

A3. φ is larger than a positive constant.

A4$_a$ (A4$_b$). φ is continuous at a (at b).

A5$_a$ (A5$_b$). $\varphi'(a) \neq 0$ ($\varphi'(b) \neq 0$).

A6$_a$ (A6$_b$). φ has continuous second derivative in a neighborhood of the point a (point b).

The results by Belenky (1990) entail the following necessary and sufficient conditions of wave-train existence.

Theorem 7.1. *Let φ satisfy A1, A3. Then a wave-train $\widetilde{F}(x)$ exists if and only if $c = 1/\Phi(a) = 1/\Gamma(b)$ and $\Gamma(z) < \Gamma(b)$ for all $z \in (a, b)$. Any other solution of (3.5)*

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takes a form $\widetilde{F}(x - d)$ for some constant d. If moreover A4$_a$, A4$_b$ are fulfilled then

$$\lim_{x \to -\infty} 1/x \ln(\widetilde{F}(x) - a) = \lambda_1, \qquad \lim_{x \to +\infty} -1/x \ln(b - \widetilde{F}(x)) = \lambda_2, \qquad (7.2)$$

where λ_1, λ_2 are roots of the equations

$$\lambda_1 = \varphi(a)\Phi(a)(1 - e^{-\lambda_1}), \qquad (7.3)$$

$$\lambda_2 = \varphi(b)\Phi(a)(e^{\lambda_2} - 1). \qquad (7.4)$$

Remark 7.1. If a wave train exists then

$$\varphi(a) \geq c \geq \varphi(b).$$

To see it we note that due to $\Gamma(z) < \Gamma(b)$ the following inequalities hold

$$\frac{1}{z - a} \int_a^z \frac{dx}{\varphi(x)} < \Gamma(b) = 1/c < \frac{1}{b - z} \int_z^b \frac{dx}{\varphi(x)}.$$

Under $z \to a$ and $z \to b$ they entail the inequalities above.

Remark 7.2. Let A1 be fulfilled. If $\Gamma(z_0) = \Gamma(b)$ for some $z_0 \in (a, b)$ and $\Gamma(z) < \Gamma(b)$ for all $z \in (a, b)$, $z \neq z_0$, then there are no wave trains on $[a, b]$ but for segments $[a, z_0]$, $[z_0.b]$ there are wave trains moving with equal speeds $c_1 = c_2 = 1/\Gamma(b)$.

Indeed, let

$$\Gamma_y(z) = \frac{1}{z - y} \int_y^z dx/\varphi(x).$$

It is simple to check that $\Gamma_{z_0}(b) = \Gamma(b) > \Gamma_{z_0}(z)$ for $z \in (z_0, b)$. Hence the statement follows from Theorem 7.1.

Obviously (see (7.3), (7.4))

$$\lambda_1 > 0 \qquad \text{iff} \qquad \varphi(a)\Phi(a) > 1, \qquad (7.5)$$

$$\lambda_2 > 0 \qquad \text{iff} \qquad \varphi(b)\Phi(a) < 1. \qquad (7.6)$$

These conditions jointly with $\Gamma(z) < \Gamma(b)$, $z \in (a, b)$ were used in Henkin, Polterovich (1991) to prove wave-train existence for Lipshitz-continuous φ. If (7.5), (7.6) are valid

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then, due to (7.2), the wave train $F(x)$ approaches to 0 under $x \to -\infty$ and to 1 under $x \to +\infty$ not slowly than some exponential functions. But if $\lambda_1 = 0$ or $\lambda_2 = 0$ then (7.2) gives rather crude estimates of wave train asymptotics. To study the stability of wave trains for nonmonotonic φ we need more precise asymptotic estimates.

Remark 7.3. Let $a \in [a_1, a_2]$ and let φ fulfill A1, A3, A4$_a$, A4$_b$ for all $[a, b]$, $a \in [a_1, a_2]$. Suppose $\varphi(a)\Phi(a) > 1$. Then

$$1/x \ln(\widetilde{F}(x) - a) \to \lambda_1(a), \qquad x \to -\infty.$$

Uniformly with respect to a, where λ_1 is the root of the equation

$$\lambda_1 = \varphi(a)\Phi(a)(1 - e^{-\lambda_1}).$$

One can check that this statement is entailed by the proof of the existence of the wave-train given in Henkin, Polterovich (1991).

Theorem 7.2. *Let φ fulfill A1, A3 and $\widetilde{F}$ be a wave train. 1) Assume that A5$_b$, A6$_b$ are valid and*

$$\varphi(b)\Phi(a) = 1.$$

Then

$$\lim_{x \to +\infty}(1 - \widetilde{F}(x))x = \varphi(1)/\varphi'(1). \tag{7.7}$$

2) *Assume that A5$_a$, A6$_a$ are valid, and*

$$\varphi(a)\Phi(a) = 1.$$

Then

$$\lim_{x \to -\infty}\widetilde{F}(x)x = -\varphi(0)/\varphi'(0). \tag{7.8}$$

To prove Theorem 7.2 we need the following Lemma.

Lemma 7.1. *Suppose that A1, A3 are valid and there exists a wave train with limits a, b. If A4$_b$, A6$_b$, and $\varphi(b)\Phi(a) = 1$ then $\varphi'(b) \geq 0$. If A4$_a$, A6$_a$, and $\varphi(a)\Phi(a) = 1$ then $\varphi'(a) \geq 0$.*

Proof. Let $\varphi(b)\Phi(a) = 1$. It follows from Theorem 7.1 that $\Gamma(z)$ reaches its maximum on $[a, b]$ at the point b. Since

$$(b - a)\Gamma'(b) = -\Phi(a) + 1/\varphi(b) = 0$$

one should have

$$0 \geq (b - a)^2\Gamma''(b) = 2\Phi(a) - 2/\varphi(b) - \varphi'(b)/\varphi^2(b)(b - a) = -\frac{\varphi'(b)}{\varphi^2(b)}(b - a),$$

and the first statement follows.

Let $\varphi(a)\Phi(a) = 1$ and define $\Gamma(a) = 1/\varphi(a)$. Then $\Gamma(z)$ is continuous on $[a, b]$ and has continuous derivative such that $\Gamma'(a) = \lim_{z \to a}\Gamma'(z)$, where

$$\Gamma'(z) = -\frac{1}{(z - a)^2}\int_a^z\left(\frac{1}{\varphi(x)} - \frac{1}{\varphi(z)}\right) dx.$$

Hence

$$\Gamma'(a) = -\frac{\varphi'(a)}{\varphi^2(a)}.$$

In view of the condition of the second statement and by definitions we have

$$\Gamma(a) = 1/\varphi(a) = \Phi(a) = \Gamma(b) > \Gamma(z), \qquad z \in (a, b).$$

It means that a is the second maximum of $\Gamma(z)$, hence $\Gamma'(a) \leq 0$. Therefore $\varphi'(a) \geq 0$, and Lemma 7.1 is proved.

Due to Theorem 7.1 there exists a wave train $\widetilde{F}(x)$, and $\widetilde{F}(x) \to 1$ under $x \to \infty$. Since $\widetilde{F}(x + d)$ is also a wave train for arbitrary d one can think that the values of $\widetilde{F}(x)$ belong to a small neighborhood of 1 for all $x \geq 0$ so that second derivative $\varphi''(\widetilde{F}(x))$ exists and $\varphi'(\widetilde{F}(x)) \neq 0$ for all $x \geq 0$. In view of Lemma 7.1, $\varphi'(\widetilde{F}(x)) > 0$ for all $x \geq 0$.

Let us introduce a function

$$F_A^0(x) = \varphi^{-1}(\varphi(b)(1 - l/(x + A)), \tag{7.9}$$

where l is an arbitrary positive number, and A is a constant.

Lemma 7.2. *Let φ fulfill A1, A3, A5$_b$, A6$_b$. There exists $A_1 = A_1(l, a, b)$ such that*

$$\widetilde{F}(x) \geq F_A^0(x) \quad \text{for all } x \geq 0, l > b$$

and for all $A \leq A_1$.

If moreover $\varphi(b)\Phi(a) = 1$ then there exists $A_2 = A_2(l, a, b)$ such that

$$\widetilde{F}(x) \leq F_A^0(x) \quad \text{for all } x \geq 0, l < b$$

and for all $A \geq A_2$.

The functions $A_1(l, a, b)$, $A_2(l, a, b)$ are continuos.

Proof. We assume $b = 1$.

Denote $V(x) = F_A^0(x) - \widetilde{F}(x)$. One has from (3.5)

$$\frac{dV}{dx} = \frac{\varphi(\widetilde{F}(x))}{c}(V(x) - V(x-1))$$
$$+ \frac{dF_A^0}{dx} - \frac{\varphi(\widetilde{F}(x))}{c}(F_A^0(x) - F_A^0(x-1)), \tag{7.10}$$

$$\frac{dV}{dx} = \frac{\varphi(\widetilde{F}(x))}{c}(V(x) - V(x-1)) + R(x), \tag{7.11}$$

where

$$R(x) = \frac{dF_A^0}{dx} - \frac{\varphi(\widetilde{F}(x))}{\varphi(1)}(F_A^0(x) - F_A^0(x-1)). \tag{7.12}$$

Let us prove that if A is negative enough then

$$\widetilde{F}(x) \geq F_A^0(x), \qquad x \geq 0. \tag{7.13}$$

Firstly, one can chose A_0 such that for $A \leq A_0$

$$\widetilde{F}(x) \geq F_A^0(x), \qquad x \in [0, 1]. \tag{7.14}$$

Let us consider $V^* = \sup_{x \geq 0} V(x)$, and prove that $V^* \leq 0$. By contradiction, suppose $V^* > 0$. Since $V(\infty) = 0$ and $V(x) \leq 0$, $x \in [0, 1]$ there exists $x^* > 1$ such that $V^* = V(x^*)$. Then $\frac{dV(x^*)}{dx} = 0$, and in view of (7.11)

$$V(x^* - 1) = V(x^*) + \frac{c}{\varphi(\widetilde{F}(x^*))}R(x^*). \tag{7.15}$$

We will come to contradiction by demonstration that $R(x^*) > 0$ if A is negative enough.

Since $F_A^0(x^*) > \widetilde{F}(x^*)$ and since F_A^0, φ are increasing functions

$$R(x^*) \geq R_0(x^*) = \frac{dF_A^0}{dx} - \frac{\varphi(F_A^0(x^*))}{c}(F_A^0(x^*) - F_A^0(x^* - 1)). \tag{7.16}$$

Let $y = x + A$. One can check that

$$dF_A^0/dx = \varphi(1)l/\varphi'(F_A^0(x))y^2, \tag{7.17}$$

$$\varphi(F_A^0(x)) = \varphi(1)(1 - l/y), \tag{7.18}$$

$$F_A^0(x) - F_A^0(x-1) = \frac{dF_A^0(x)}{dx} - \frac{1}{2}\frac{d^2F_A^0(\overline{x})}{dx^2}, \tag{7.19}$$

where $x - 1 \leq \overline{x} \leq x$.

$$\frac{d^2F^0(x)}{dx^2} = -\frac{2\varphi(1)l}{\varphi'(F_A^0(x))y^3} - \frac{(\varphi(1)l)^2\varphi''(F_A^0(x))}{(\varphi'(F_A^0(x))^3y^4}. \tag{7.20}$$

It follows from (7.16)–(7.19) that

$$R_0(x^*) = \left[\frac{l}{y^*}\frac{dF_A^0(x^*)}{dx} + (1 - l/y^*)\frac{1}{2}\frac{d^2F_A^0(\overline{x})}{dx^2}\right]\left(\frac{\varphi(1)}{c}\right)$$
$$+ \left(1 - \frac{\varphi(1)}{c}\right)\frac{dF_A^0}{dx}. \tag{7.21}$$

Since $\varphi'(\widetilde{F}(x)) > 0$ one can find $A_1 \leq A_0$ such that $\varphi'(F_A^0(x^*)) \geq \delta > 0$. Since φ, φ', φ'' are bounded one has from (7.9), (7.17), (7.20) under $A \to -\infty$ and $y^* = x^* + A$

$$y^* \sim \overline{y}, \quad \frac{dF_A^0(x^*)}{dx} \sim \frac{\varphi(1)l}{\varphi'(1)y^{*2}}, \quad \frac{d^2F_A^0(\overline{x})}{dx^2} \sim -\frac{2\varphi(1)l}{\varphi'(1)y^{*3}}.$$

Due to (7.21) one has

$$R_0(x^*) \sim \frac{1}{y^{*3}} \frac{\varphi(1)l(l-1)}{\varphi'(1)} \frac{\varphi(1)}{c} + \left(1 - \frac{\varphi(1)}{c}\right) \frac{\varphi(1)l}{\varphi'(1)y^{*2}}. \qquad (7.22)$$

Since $l > 1$ and $\varphi(1) \leq c$ (see Theorem 7.1 and Remark 7.2) the value $R_0(x^*)$ is positive under large negative A. Thus (7.13) is proved which is the first statement of Lemma 7.2.[1]

To prove the second statement one could consider $V^{**} = \inf_{x\geq 0} V(x)$ and take into account that $\varphi(1) = c$. The considerations are very similar to the arguments given above and we omit them. Lemma 7.2 is proved.

Proof of Theorem 7.2. The first statement is a strict consequence of Lemma 7.2. We omit the proof of the second statement of Theorem 7.2 since it can be proved by the similar arguments if we take

$$F_A^0(x) = \varphi^{-1}\left(\varphi(a)\left(1 + \frac{l}{A-x}\right)\right)$$

instead of (7.9).

Remark 7.4. Let $a \in [a_1, a_2]$, $a_2 < b$. One can check that the same constants A_1, A_2 of Lemma 7.2 can be chosen for all segment $[a_1, a_2]$.

8 Stability problem

Theorem 3.4 about stability of wave trains can be generalized on nonmonotonic case.

Theorem 8.1. *Let φ satisfy A1, A3, A4, and $\Gamma(z) < \Gamma(b)$, $z \in (a, b)$.*

Suppose that

$$\varphi(a)\Phi(a) > 1, \qquad \varphi(b)\Phi(a) < 1. \qquad (8.1)$$

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[1]If $\varphi(1)\Phi(a) < 1$ then the first statement of Lemma 7.2 is valid for arbitrary $l > 0$. It follows from (7.22).

Then for every solution $\mathcal{F} = \{F_n\}$ of the problem (3.1), (3.2) *there exists a constant d such that*

$$\sup |F_n(t) - \widetilde{F}(n - ct - d)| \to 0 \qquad as\ t \to \infty.$$

Here $\widetilde{F}$ is a wave train which exists and is unique up to translation due to Theorem 7.1.

Theorem 8.1 was proved as Theorem 4 in Henkin, Polterovich (1991) under an additional assumption of Lipshitz-continuity of φ. But this assumption was used only to guarantee existence of a wave train. The formulation above takes into account that the wave train existence Theorem 7.1 by Belenky does not demand Lipshitz condition.

The next statement shows that the conditions (8.1) can not be omitted.

Theorem 8.2. *Let φ fulfill A1, A3, A5$_b$, A6$_b$ and a Lipshitz condition, and let $\Gamma(z) < \Gamma(b)$, $z \in (a, b)$. Suppose that*

$$\varphi(a)\Phi(a) > 1, \qquad \varphi(b)\Phi(a) = 1. \qquad (8.2)$$

If $\widetilde{F}$ is a wave train and $F_n(t)$ is a solution of (3.1), (3.2) *such that $F_n(0) = 0$ under $n \leq 0$, then for every d the following relation holds*

$$\varlimsup_{t\to\infty} \sup_n |F_n(t) - \widetilde{F}(n - ct - d)| > 0. \qquad (8.3)$$

To prove Theorem 8.2 we need the following statement about comparison of solutions. It has an independent interest.

Lemma 8.1. *Let φ be a Lipshitz-continuous function on $[0,1]$, $x(t)$ be a continuous function on $[0,\infty)$, $x(t) \geq 1$. Suppose that $\{G_n\}_0^\infty$, $\{\widehat{G}_n\}_0^\infty$ fulfill the inequalities*

$$\frac{dG_n}{dt} \geq \varphi(G_n)(G_{n-1} - G_n), \qquad n \geq 1,\ t \geq 0; \qquad (8.4)$$

$$\frac{d\widehat{G}_n}{dt} \leq \varphi(\widehat{G}_n)(\widehat{G}_{n-1} - \widehat{G}_n), \qquad n \geq 1,\ t \geq 0; \qquad (8.5)$$

where $G_n, \widehat{G}_n : R_+^1 \to [0,1]$. If

$$G_n(0) > \widehat{G}_n(0), \qquad x(0) \geq n \geq 1, \tag{8.6}$$

$$G_0(t) \geq \widehat{G}_0(t), \qquad t \geq 0, \tag{8.7}$$

$$G_{[x(t)]}(t) > \widehat{G}_{[x(t)]}(t), \qquad t \geq 0, \tag{8.8}$$

then $G_n(t) > \widehat{G}_n(t)$ for all $n \geq 1$, $n \leq x(t)$.

Here $[x]$ means integer part of x.

Proof. Denote $V_n = G_n - \widehat{G}_n$. From (8.4), (8.5) we have the following inequality

$$\frac{dV_n}{dt} \geq \varphi(G_n)(V_{n-1} - V_n) + (\varphi(G_n) - \varphi(\widehat{G}_n))(\widehat{G}_{n-1} - \widehat{G}_n). \tag{8.9}$$

Let us consider

$$t^* = \sup\{t : V_n(t) > 0 \quad \forall n \geq 1, \ n \leq x(t)\}.$$

Due to (8.6), $t^* > 0$. We prove that $t^* = \infty$. By contradiction, let $t^* < \infty$. Then due to continuity of $x(t)$ and $V(t)$ one can find $r \geq 1$, $r \leq x(t^*)$ such that $V_r(t^*) = 0$. Due to (8.8), $r < x(t^*)$. Let $0 \leq t^1 < t^*$ and $r < x(t)$ for all $t \in [t^1, t^*]$. In view of (8.7) $V_{r-1}(t) \geq 0$ under $t \in [t^1, t^*]$. Therefore and due to Lipshitz condition one has from (8.9) taking into account that $\widehat{G}_n(t) \in [0,1]$

$$\frac{dV_r}{dt} \geq V_r(-\varphi(G_r) - L) \geq hV_r, \qquad t \in [t^1, t^*], \tag{8.10}$$

where L is a Lipshitz constant for φ, and $h \leq -\varphi(G_r) - L$ for $t \in [t^1, t^*]$. It follows from (8.10) that

$$V_r(t) \geq V_r(t^1)e^{h(t-t^1)} > 0 \quad \text{for } t \in [0, t^*],$$

a contradiction. Thus $t^* = \infty$, and Lemma 8.1 is proved.

Proof of Theorem 8.2. Without losses of generality we take $a = 0$, $b = 1$. Let us introduce notations

$$F(t, x) = F_n(t) \quad \text{under } n - 1 < x \leq n.$$

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$$V(t, M, F) = \int_0^{ct+M} \Phi(F(t,x)) \, dx - \int_{-\infty}^0 [\Phi(0) - \Phi(F(t,x))] \, dx - t, \tag{8.11}$$

where $c = 1/\Phi(a)$, M is a constant;

$$W(t, M) = \int_{ct}^{ct+M} [\Phi(\widetilde{F}(x - ct - d)) - \Phi(F(x, t))] \, dx,$$

where d is a constant. The second integral in (8.11) converges due to Theorem 3.1. The same is true for $V(t, M, \widetilde{F})$ in view of Theorem 7.1 because of (8.2) and (7.5).

We have

$$W(t, M) = V(t, M, \widetilde{F}) - V(t, M, F) - V(t, 0, \widetilde{F}) + V(t, 0, F). \tag{8.12}$$

In view of Theorem 3.1 the problem (3.1), (3.2) has a first integral $B_{\mathcal{F}}(t)$ (see (3.4)), therefore

$$V(t, M, F) \leq B_{\mathcal{F}}(t) = B_{\mathcal{F}}(0). \tag{8.13}$$

Let us take $x = y + ct$ in (8.11). We have

$$V(t, M, F) = \int_0^M \Phi(F(t, y + ct)) \, dy - \int_{-\infty}^0 [\Phi(0) - \Phi(F(t, y + ct))] \, dy. \tag{8.14}$$

Since $\widetilde{F}(x - ct - d) = \widetilde{F}(y - d)$ we get

$$V(t, M, \widetilde{F}) = \int_0^M \Phi(\widetilde{F}(y - d)) \, dy$$
$$\qquad - \int_{-\infty}^0 [\Phi(0) - \Phi(\widetilde{F}(y - d))] \, dy$$
$$= V(0, M, \widetilde{F}). \tag{8.15}$$

Let us fix $x^+ > 0$. Due to Theorem 7.2 there exists $\delta > 0$ such that

$$1 - \widetilde{F}(x) \geq \frac{\delta}{x} \quad \text{for all } x \geq x^+. \tag{8.16}$$

Besides

$$\Phi(\widetilde{F}(x)) = \int_{\widetilde{F}(x)}^1 dy / \varphi(y) \geq \gamma(1 - \widetilde{F}(x)), \tag{8.17}$$

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where $\gamma = \min_{y \in [0,1]}(1/\varphi(y))$.

It follows from (8.15)–(8.17) that

$$V(t, M, \widetilde{F}) \geq \int_{x^+}^{M} \frac{\gamma\delta}{x}\, dx - V(0, 0, \widetilde{F}) = \gamma\delta \ln \frac{M}{x^+} - V(0, 0, \widetilde{F}). \qquad (8.18)$$

Due to (8.12), (8.13), (8.15) one has

$$W(t, M) \geq \gamma\delta \ln \frac{M}{x^+} - B_{\mathcal{F}}(0) + V(t, 0, F) - 2V(0, 0, \widetilde{F}). \qquad (8.19)$$

Let us suppose that (8.3) is not correct, and there exist $d, T(\varepsilon)$ such that

$$\sup_{n} |F_n(t) - \widetilde{F}(n - ct - d)| \leq \varepsilon \quad \text{for } t \geq T(\varepsilon). \qquad (8.20)$$

Then $W(t, M) \leq M\varepsilon \max_{y \in [0,1]} 1/\varphi(y)$. It contradicts (8.19) if

$$V(t, 0, F) \geq \gamma, \qquad t \geq 0 \qquad (8.21)$$

for some constant γ. Thus the proof will be finished if we show that (8.20) entails (8.21).

Let us apply Lemma 8.1 under $\widehat{G}_n = F_n$, $x(t) = ct + 1$ and $G_n(t) = \widetilde{F}(n - ct + D)$, where constant D will be chosen below. Since $F_0(t) \equiv 0$ the conditions (8.7) is valid. Due to (3.1) $\frac{dF_1(t)}{dt} < 0$ if $F_1(t) > 0$, hence $F_1(t) < 1$ for all $t > 0$. Therefore without losses of generality we can assume that $F_1(0) < 1$. Since $\widetilde{F}(1 + D) \to 1$ under $D \to \infty$, (8.6) holds for all D large enough.

Since $\widetilde{F}$ is nondecreasing and $\widetilde{F}(x) < 1$ the inequality (8.20) entails

$$F_{[x(t)]}(t) \leq \widetilde{F}(1 - d) + \varepsilon < 1 \qquad (8.22)$$

for all ε small enough. Let D be so large that $\widetilde{F}(-1 + D) > \widetilde{F}(1 - d) + \varepsilon$. Since $\widetilde{F}([x(t)] - ct + D) \geq \widetilde{F}(-1 + D)$ the inequality (8.22) entails (8.8). Thus all assumptions of Lemma 8.1 are fulfilled, hence

$$\widetilde{F}_D = \widetilde{F}(n - ct + D) > F_n(t) \quad \text{for all } n \geq 1, n \leq ct.$$

It follows from (8.11) and (3.3) that $V(t, 0, F)$ is a nonincreasing function of F, therefore

$$V(t, 0, F) \geq V(t, 0, \widetilde{F}_D) = V(0, 0, \widetilde{F}_D),$$

and (8.21) is valid for $\gamma = V(0, 0, \widetilde{F}_D) = -\int_{-\infty}^{0}[\Phi(0) - \Phi(\widetilde{F}(y + D))]\, dy$ (see (8.15)). Theorem 8.2 is proved.

One could think that the condition $\varphi(b)\Phi(a) = 1$ of Theorem 8.2 holds only for exceptional cases. But it is not so. Similar conditions arise for a broad class of nonmonotonic φ if one tries to investigate asymptotic behavior of the solutions of (3.1), (3.2).

This behavior can be more complicated than in the case of monotonic φ. General results are unknown, but taking into account Theorems 7.1, 8.1, 6.2 one could suppose that the following picture takes place.

Let us consider a function

$$\Psi(z) = \int_0^z \frac{dx}{\varphi(x)} = \Phi(0) - \Phi(z), \qquad z \in [0, 1] \qquad (8.23)$$

and let $\Psi_0(z)$ be the upper boundary of the convex hull of the set

$$\begin{aligned} W &= \{(z, v) : v \leq \Psi(z), \quad 0 \leq z \leq 1\}; \\ \text{Conv } W &= \{(z, v) : v \leq \Psi_0(z), \quad 0 \leq z \leq 1\}. \end{aligned} \qquad (8.24)$$

The set

$$E = \{z : \Psi(z) < \Psi_0(z), \quad 0 \leq z \leq 1\} \qquad (8.25)$$

is open in $[0, 1]$ and can be represented as a union of intervals of the form $[0, b)$, (a_i, b_i), $(a, 1]$, or $[0, 1]$. Let Σ be the set of these intervals and $l(\sigma)$ be the length of interval $\sigma \in \Sigma$.

We assume that conditions A1, A3 are fulfilled.

Proposition 8.1. *For every $\sigma \in \Sigma$ there exists a wave train with overfall $l(\sigma)$. Besides*

$$\varphi(0) \geq c = \varphi(b) \quad \text{if } \sigma = [0, b),\ 0 < b < 1;$$

$$\varphi(a) = c = \varphi(b) \quad \text{if } \sigma = (a_i, b_i),\ 0 < a < b < 1;$$

$$\varphi(a) = c \geq \varphi(1) \quad \text{if } \sigma = (a, 1),\ 0 < a < 1;$$

$$\varphi(0) \geq c \geq \varphi(1) \quad \text{if } \sigma = [0, 1],$$

If $z \notin E = \bigcup_{\sigma \in \Sigma} \sigma$ then $\varphi'(z) \geq 0$.

Here c is a speed of the wave train (see (3.5)), c depends on an interval.

Proof. Let $(a, b) \in \Sigma$. By definition

$$\Psi(z) = \int_0^z \frac{dx}{\varphi(x)} < \Psi_0(z) = \frac{b-z}{b-a}\Psi(a) + \frac{z-a}{b-a}\Psi(b) \qquad (8.26)$$

for all $z \in (a, b)$ (see Fig. 1). It follows from (8.26) that

$$\Gamma(a, z) = \frac{1}{z-a}\int_a^z \frac{dx}{\varphi(x)} < \Gamma(a, b) \quad \text{for all } z \in (a, b).$$

Therefore Theorem 7.1 entails the existence of a wave-train with overfall $b - a$. Furthermore in view of (8.25) the function

$$u(z) = \Psi(z) - \frac{b-z}{b-a}\Psi(a) - \frac{z-a}{b-a}\Psi(b)$$

has its local maxima at $z = a$ and at $z = b$. One has

$$u'(z) = \frac{1}{\varphi(z)} - \Gamma(a, b) = \frac{1}{\varphi(z)} - \frac{1}{c};$$

and $u'(a) \leq 0$, $u'(b) \geq 0$ with equalities for $a \in (0, 1)$, $b \in (0, 1)$. It proves the second part of the proposition. The third part immediately follows from the formulae

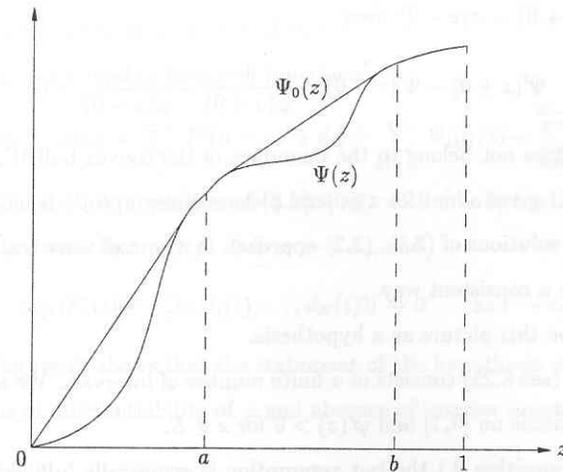$$\Psi''(z) = -\varphi'(z)/\varphi^2(z).$$

Fig. 1.

Indeed if $\varphi'(z) < 0$ then $\Psi(z)$ is convex in a vicinity of z and therefore z should belong to an interval from Σ. The Proposition 8.1 is proved,

Remark 8.1. It may be checked that an interval σ with endpoints a, b belongs to Σ if and only if the following inequalities hold

$$\Gamma(a, z) < \Gamma(a, b) < \Gamma(y, b) \quad \text{for all } z > a,\ y < b,$$

$$\Gamma(a, z) \geq \Gamma(a, b), \quad z \leq a,$$

$$\Gamma(y, b) \leq \Gamma(a, b), \quad y \geq b.$$

Remark 8.2. Assume that φ is a piecewise continuous differentiable function with finite number of discontinuity points which are jumps of φ'.

Let $z \notin \bigcup_{\sigma \in \Sigma} \sigma$. Then $\varphi'(z) \geq 0$ if z is a continuity point and $\varphi(z+0) \geq \varphi(z-0)$ if z is a jump point.

Indeed if $\varphi(z+0) < \varphi(z-0)$ then

$$\Psi'(z+0) - \Psi'(z-0) = \frac{1}{\varphi(z+0)} - \frac{1}{\varphi(z-0)} > 0.$$

It means that z does not belong to the boundary of the convex hull W.

Proposition 8.1 gives a hint for a general picture of asymptotic behavior. It seems to be plausible that solutions of (3.1), (3.2) approach to a sum of wave trains and diffusion curves moving by a consistent way.

Let us describe this picture as a hypothesis.

Let the set E (see 8.25) consists of a finite number of intervals. We assume that φ is positive, differentiable on $[0,1]$ and $\varphi'(z) > 0$ for $z \notin E$.

In view of Proposition 8.1 the last assumption is generically fulfilled.

The set $E' = [0,1] \setminus E$ is a union of finite set of segments. We assume that E' does not contain interior isolated points. Let $b_0 = 0$, $b_N = 1$ and b_i, $i = 1, \ldots, N-1$ be endpoints of our intervals inside $[0,1]$. Obviously if $\sigma_i = (b_i, b_{i+1}) \subset E$ then $\sigma_{i+1} = [b_{i+1}, b_{i+2}] \subset E'$. For every $\sigma_i = [b_i, b_{i+1}] \subset E'$ we define a function

$$\Psi_i(n/t) = \begin{cases} b_i & \text{for } n < \varphi(b_i)t, \\ \varphi^{-1}(n/t) & \text{for } \varphi(b_i)t \le n \le \varphi(b_{i+1})t, \\ b_{i+1} & \text{for } n > \varphi(b_{i+1})t. \end{cases}$$

Due to Proposition 8.1, for every $\sigma_i \subset E$ there exists a corresponding wave train F^i with a speed c_i. If $N > 1$ and $i \ge 1$ then $c_i = \varphi(b_i)$,

$$c_i^{-1} = \frac{1}{b_{i+1} - b_i} \int_{b_i}^{b_{i+1}} \frac{dF}{\varphi(F)}.$$

Our hypothesis is that asymptotic behavior of solutions of (3.1), (3.2) looks like a wave-train for the sets

$$\{(n,t) \,|\, F_n(t) \in \sigma_i\}$$

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if $\sigma_i \in E$ and like a diffusion or $\Psi_i(n,t)$ if $\sigma_i \in E'$.

Let us define the following family of functions

$$\tilde{F}_n(t, d_1, \ldots, d_N) = \sum_{\sigma_i \subset E} F^i(n - c_i t + d_i) + \sum_{\sigma_i \subset E'} \Psi_i(n/t) - \sum_{i=1}^{N-1} b_i. \qquad (8.27)$$

Hypothesis. If F_n is a solution of (3.1), (3.2) with $a = 0$, $b = 1$, then there exist $d_i(t)$, $i = 1, \ldots, N$ such that $d_i(t)/t \to 0$ as $t \to \infty$, and

$$\sup_n |F_n(t) - \tilde{F}_n(t, d_1(t), \ldots, d_N(t))| \to 0 \qquad \text{as } t \to \infty.$$

The following result shows that the statement of the hypothesis can be valid even if the assumptions of differentiability of φ and absence of interior isolated points in E' do not hold.

Let φ have the special form (see Fig. 2)

$$\varphi(F) = \begin{cases} \varphi_1(F), & F \le \varkappa, \\ \varphi_2(F), & F > \varkappa, \, 0 < \varkappa < 1, \end{cases}$$

where φ_1, φ_2 satisfy A1, A2 in $[0, \varkappa]$ and in $[\varkappa, 1]$ respectively.

Let us denote

$$c_1^{-1} = \frac{1}{\varkappa} \int_0^\varkappa \frac{dF}{\varphi_1(F)}, \qquad c_2^{-1} = \frac{1}{1 - \varkappa} \int_\varkappa^1 \frac{dF}{\varphi_2(F)}.$$

The values c_1, c_2 are velocities of wave trains F^1, F^2 for φ_1, φ_2 and with overfalls $\varkappa$, $1 - \varkappa$. In this case E consists of two intervals $(0, \varkappa)$ and $(\varkappa, 1)$ and $E' = \{0; \varkappa; 1\}$, $\varkappa$ is the interior isolated point. The second term in (8.27) disappears, and every solution of (3.1), (3.2) converges to a function $\tilde{F}(t, d_1, d_2) = F^1(c - c_1 t - d_1) + F^2(n - c_2 t - d_2) - \varkappa$ for some constant d_1, d_2 uniformly in n (see Henkin, Polterovich (1994)). Specifics of this example is the absence of diffusion segments.

The following statement is some step to prove the Hypothesis formulated above. It shows that the Hypothesis is valid outside of an arbitrary linearly increasing neighborhood of wave trains.
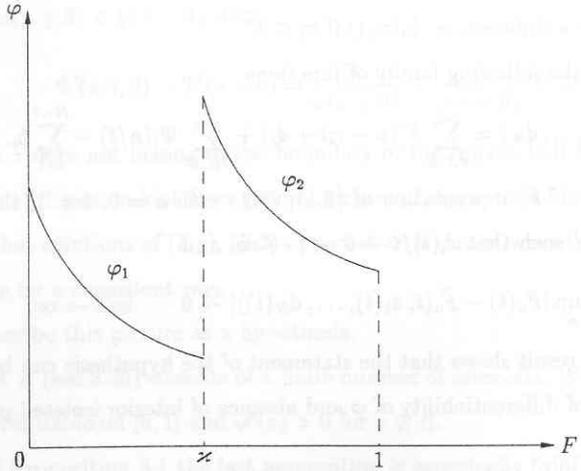
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Fig. 2. Piecewise decreasing φ

Let $\beta > 0$ and c_i, $i = 1, \ldots, N$ be speeds of wave trains F_i from (8.27). We introduce the following subset of the set $\mathbb{R}$ of real numbers

$$Z_\beta(t) = \mathbb{R} \setminus \cup_i [c_i t - \beta\sqrt{t}, c_i t + \beta\sqrt{t}].$$

Theorem 8.3. *Let φ be positive, φ', φ'' be bounded, $\varphi'(z) > 0$ for $z \notin E$. Let us assume also that*

$$F_n(0) = 0 \quad \forall n \le 0, \quad F_n(0) \ge F_{n-1}(0) \quad \forall n > 0.$$

Then for every β large enough

$$\sup_{n \in Z_\beta(t)} |F_n(t) - \widetilde{F}_n(t, d_1(t), \ldots, d_N(t))| \to 0 \quad as \quad t \to \infty$$

for arbitrary $d_i(t)$ such that $d_i(t)/t^{1/2} \to 0$.

This Theorem is an analogue of the result received by Weinberger (1990) for Burgers equation.

We describe a plan of the proof of Theorem 8.3 and then give a complete proof for a particular case.

Let us assume that $\sigma_0 \in E$, then $\sigma_i \in E$ for even i and $\sigma_i \in E'$ if i is odd. We expect that wave-wise behavior prevails for $\eta_i = \{(n,t) | F_n(t) \in \sigma_i\}$ if $\sigma_i \in E$ and diffusion-wise behavior takes place if $\sigma_i \in E'$. Theorem 8.3 asserts that if $(n,t) \in \eta_i$, $\sigma_i \in E$, then n is located in a segment $[c_i t - \beta\sqrt{t}, c_i t + \beta\sqrt{t}]$ for β large enough (see Fig. 3).
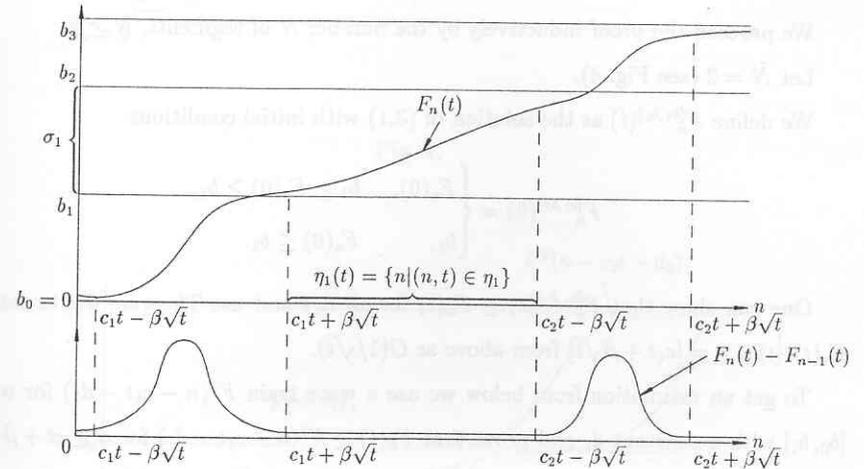

Fig. 3.

The proof includes two types of statement. We show that

$$F_{n(t)}(t) = b_i + O(1/\sqrt{t}) \qquad (*)$$

if $c_i t + \beta\sqrt{t} - 1 \leq n(t) \leq c_i t + \beta\sqrt{t}$, $i = 1, 3, \ldots$, $c_i t - \beta\sqrt{t} - 1 \leq n(t) \leq c_i t - \beta\sqrt{t}$, $i = 0, 2, 4, \ldots$, where $b_0 = 0$, $c_0 = c_1 = \varphi(b)$, $c_i = \varphi(b_i)$. Besides we demonstrate that for segments $\sigma_i = [b_i, b_{i+1}] \in E'$ the relations $(*)$ entail

$$F_n(t) - \Psi_i(n/t) = O(1/\sqrt{t}), \quad (n, t) \in \eta_i. \qquad (**)$$

Particularly the density $F_n(t) - F_{n-1}(t)$ vanishes in η_i, $\sigma_i \in E'$.

We proceed the proof inductively by the number N of segments, $N \geq 2$.

Let $N = 2$ (see Fig. 4).

We define $F_n^{[b_1, b_2]}(t)$ as the solution of (3.1) with initial conditions

$$F_n^{[b_1, b_2]}(0) = \begin{cases} F_n(0), & b_2 \geq F_n(0) \geq b_1, \\ b_1, & F_n(0) \leq b_1. \end{cases}$$

One can show that $F_n^{[b_1, b_2]}(t) \geq F_n(t)$ for all n, t and use Theorem 6.2 to estimate $F_n(t) - b_1$, $n = [c_1 t + \beta\sqrt{t}]$ from above as $O(1/\sqrt{t})$.

To get an estimation from below we use a wave train $F^1(n - c_1 t - d_1)$ for overfall $[b_0, b_1]$ with a constant d_1 and prove that $F_n(t) \geq F^1(n - c_1 t - d_1)$ for $n \geq ct + \beta\sqrt{t}$. It entails the relation $F_n(t) - b_1 \geq O(1/\sqrt{t})$, $n = [c_1 t + \beta\sqrt{t}]$ due to Theorem 7.2. So we get $(*)$ for $i = 1$, $N = 2$.

To receive $(*)$ for $i = 2$ we consider a solution $\widehat{F}_n(t)$ of (3.1) for $\widehat{\varphi}(F) = \varphi(F) + \alpha(1 - F)$, $\alpha > 0$ with initial conditions

$$\widehat{F}_n(0) = \begin{cases} F_n(0) & \text{if } b_1 < F_n(0) \leq b_2, \\ b_1 & \text{otherwise.} \end{cases}$$
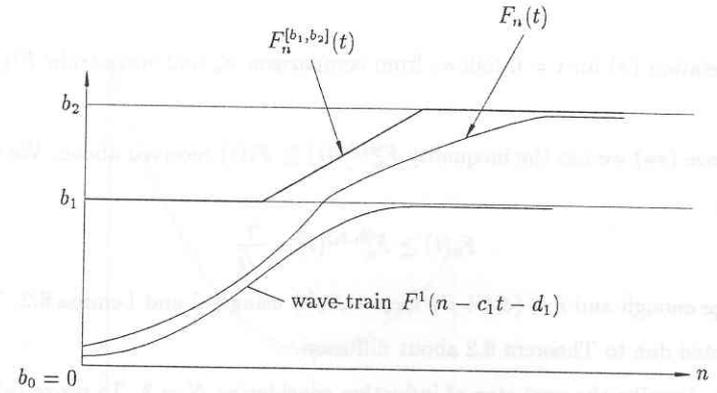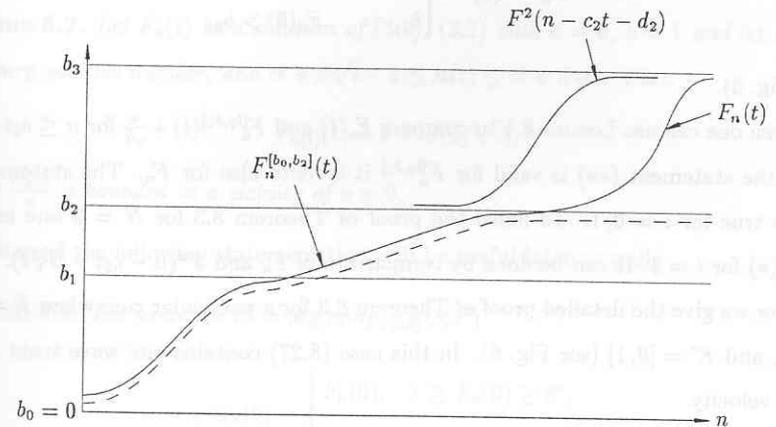
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Fig. 4.

Fig. 5.

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The relation (*) for $i = 2$ follows from comparison of F_n and $\widehat{F}_n$ and from Theorem 6.2.

The relation (*) for $i = 0$ follows from comparison F_n and wave-train $F^1(n - c_1 t + \beta\sqrt{t})$.

To prove (**) we use the inequality $F_n^{[b_1,b_2]}(t) \geq F_n(t)$ received above. We can prove also that

$$F_n(t) \geq F_n^{[b_1,b_2]}(t) - \frac{\gamma}{\sqrt{t}}$$

for γ large enough and $n \in (c_1 t + \beta\sqrt{t}, c_2 t - \beta\sqrt{t})$ using (*) and Lemma 6.2. The proof is completed due to Theorem 6.2 about diffusion.

Let us describe the next step of induction considering $N = 3$. To prove (*) for $i = 2$ we compare F_n with $F_n^{[b_0,b_2]}$ and $F^2(n - c_2 t + d_2)$ where F^2 is the wave-train for overfall $[b_2, b_3]$ and $F_n^{[b_0,b_2]}$ is the solution of (3.1) under initial conditions

$$F_n^{[b_0,b_2]}(0) = \begin{cases} F_n(0), & F_n(0) \leq b_2, \\ b_2, & F_n(0) > b_2 \end{cases}$$

(see Fig. 5).

Then one can use Lemma 8.1 to compare $F_n(t)$ and $F_n^{[b_0,b_2]}(t) + \frac{\gamma}{\sqrt{t}}$ for $n \leq c_2 t - \beta\sqrt{t}$. Since the statement (**) is valid for $F_n^{[b_0,b_2]}$ it is valid also for F_n. The statement (*) is also true for $i = 0, 1$. To finish the proof of Theorem 8.3 for $N = 3$ one needs to show (*) for $i = 3$. It can be done by comparison of F_n and $F^2(n - c_2 t - \beta\sqrt{t})$.

Now we give the detailed proof of Theorem 8.3 for a particular case when $E = [0, \theta]$, $\theta < 1$, and $E' = [\theta, 1]$ (see Fig. 6). In this case (8.27) contains one wave train. Let c_0 be its velocity.

Let us remind that θ is defined by the following equality

$$1/\varphi(\theta) = (1/\theta) \int_0^\theta dy/\varphi(y). \tag{8.28}$$
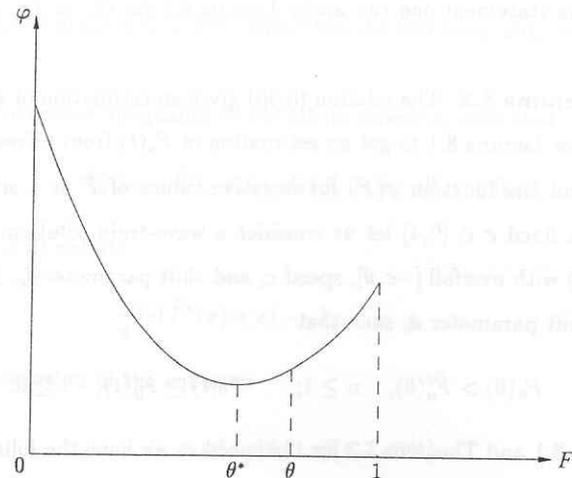

Fig. 6.

Lemma 8.2. *Let $F_n(t)$ be a solution of* (3.1), (3.2) *with $a = 0$, $b = 1$ and let β be an arbitrary positive number, and $ct + \beta\sqrt{t} - 1 \leq n(t) \leq ct + \beta\sqrt{t}$. Then*

$$F_{n(t)}(t) = \theta + O(1/\sqrt{t}),$$

where $\frac{O(x)}{x}$ is bounded in a vicinity of $x = 0$.

We need the following statement that will be useful later as well.

Lemma 8.3. *Let us define $\theta^* = \arg\min_{F \in [0,\theta]} \varphi(F)$*

$$G_n(0) = \begin{cases} F_n(0), & 1 \geq F_n(0) \geq \theta^*, \\ \theta^*, & 0 \leq F_n(0) \leq \theta^*, \end{cases} \tag{8.29}$$

and let $G_n(t)$ be the solution of equation (3.1) *under initial data $G_n(0)$. Then*

$$G_n(t) \geq F_n(t) \quad \text{for all} \ \ n, t. \tag{8.30}$$

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To prove this statement one can apply Lemma 6.2 for $\widehat{G}_n = F_n$, $\gamma_2 = \infty$, $\gamma_1 = 0$, $T_0 = 0$.

Proof of Lemma 8.2. The relation (8.30) gives an estimation of $F_n(t)$ from above. Let us apply now Lemma 8.1 to get an estimation of $F_n(t)$ from below.

Let us extend the function $\varphi(F)$ for negative values of F as a smooth decreasing function. For a fixed $\varepsilon \in [0, 1]$ let us consider a wave-train solution of (3.1) $\widehat{F}_n^\varepsilon(t) = \widehat{F}^\varepsilon(n - c_\varepsilon t - d_\varepsilon)$ with overfall $[-\varepsilon, \theta]$, speed c_ε and shift parameter d_ε. For any $\varepsilon > 0$ we can choose a shift parameter d_ε such that

$$F_n(0) > \widehat{F}_n^\varepsilon(0), \quad n \geq 1; \qquad F_0(t) \geq \widehat{F}_0^\varepsilon(t), \quad t \geq 0. \tag{8.31}$$

By Proposition 8.1 and Theorem 3.2 for the speed c_ε we have the following equality

$$\frac{1}{c_\varepsilon} = \frac{1}{\theta + \varepsilon} \int_{-\varepsilon}^{\theta} \frac{dy}{\varphi(y)} = \frac{1}{\theta + \varepsilon} \int_{-\varepsilon}^{0} \frac{dy}{\varphi(y)} + \frac{\theta}{\theta + \varepsilon} \frac{1}{c_0},$$

where c_0 is the speed of the wave-train $\widehat{F}^0(n - c_0 t - d_0)$ with the overfall $[0, \theta]$.

For small ε we have the equality

$$c_\varepsilon = c_0(1 + O(\varepsilon)). \tag{8.32}$$

Let us show that, to satisfy (8.31), we can take a shift parameter d_ε in the form

$$d_\varepsilon = l \ln \frac{1}{\varepsilon}, \tag{8.33}$$

where l is a positive constant large enough.

It follows from continuous dependence of the values of $\widehat{F}_n^\varepsilon(t)$ on parameter $\varepsilon \in [0, 1]$ and from the properties

$$F_n(0) \to 1, \qquad n \to +\infty,$$
$$\widehat{F}^\varepsilon(n) \to \theta < 1, \qquad n \to +\infty,$$
$$\widehat{F}^\varepsilon(-d) \to -\varepsilon, \qquad d \to +\infty$$

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that there exists a constant $l_0 = O(1)$ such that the first inequality in (8.31) is fulfilled for any $d_\varepsilon \geq l_0$.

To satisfy the second inequality (8.31) let us choose d_ε such that

$$\widehat{F}_0^\varepsilon(t) = \widehat{F}^\varepsilon(-c_\varepsilon t - d_\varepsilon) \leq 0 \quad \text{for all } t \geq 0. \tag{8.34}$$

Estimation (7.2) entails

$$\frac{1}{x} \ln(\widehat{F}^\varepsilon(x) + \varepsilon) \to \lambda_1, \quad x \to -\infty, \tag{8.35}$$

where λ_1 is the solution of the equation

$$\lambda_1 = \varphi(-\varepsilon)\Phi(-\varepsilon)(1 - e^{-\lambda_1}).$$

Inequalities (8.34) and (8.31) are satisfied due to (8.35) if we take $d_\varepsilon = l \ln \frac{1}{\varepsilon}$, where l is large enough. Due to Remark 7.3 number l can be chosen independently on ε.

Let us choose a continuous function $x(t)$, $t \in [0, \infty)$, such that

$$x(t) \geq 1 \quad \text{and} \quad F_{[x(t)]}(t) > \theta. \tag{8.36}$$

The inequalities (8.31), (8.36) together with inequality $\widehat{F}_n^\varepsilon(t) < \theta$ for any $n \geq 0$, $t > 0$, give us conditions for applying Lemma 8.1. From this Lemma and (8.33) we obtain the following inequality

$$F_n(t) \geq \widehat{F}_n^\varepsilon\left(n - c_\varepsilon t - l \ln \frac{1}{\varepsilon}\right) \tag{8.37}$$

for all $n \geq 0$, $t > 0$ and $0 < \varepsilon \leq 1$.

Let us choose, further, $\varepsilon = \varepsilon(t)$ and $n = n(t)$ in the form

$$\varepsilon(t) = \exp(-t^{1/3}) \quad \text{and} \quad n(t) = [ct + \beta\sqrt{t}] + 1. \tag{8.38}$$

Using these $\varepsilon = \varepsilon(t)$ and $n = n(t)$ we get from (8.37) and (8.32)

$$F_{n(t)}(t) \geq \widehat{F}^{\varepsilon(t)}(\beta\sqrt{t} + ct - c_\varepsilon t - lt^{1/3}), \quad c_\varepsilon = c(1 + O(\exp(-t^{1/3}))), \quad t \geq 0. \tag{8.39}$$

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Due to Lemma 7.2 and Remark 7.4 we obtain the following: uniformly for $\varepsilon \in [0, 1]$, for every $l > 1$ there exists $A > 0$ large enough such that

$$\widehat{F}^\varepsilon(x) \geq \varphi^{-1}\left(\varphi(\theta)\left(1 - \frac{l}{x + A}\right)\right), \quad x \geq 0. \tag{8.40}$$

From (8.39) and (8.40) it follows the existence of $T = T(A, l)$ such that

$$F_{n(t)}(t) \geq \theta - \frac{\varphi(\theta)}{\varphi'(\theta)}\frac{(l+1)}{\beta\sqrt{t}} \quad \text{for } t \geq T. \tag{8.41}$$

Besides, using (8.30) and Theorem 6.2 we obtain

$$F_{n(t)}(t) \leq G_{n(t)}(t) \leq \varphi^{(-1)}\left(\frac{n(t)}{t}\right) + O(1/\sqrt{t}) \leq$$

$$\varphi^{(-1)}(\varphi(\theta) + O(1/\sqrt{t})) + O(1/\sqrt{t}) \leq \theta + O(1/\sqrt{t}). \tag{8.42}$$

Lemma 8.2 follows from (8.41) and (8.42).

Lemma 8.4. Let $F_n(t)$ be a solution of (3.1), (3.2) with $a = 0$, $b = 1$ and let $\widehat{F}(n - ct)$ be a wave train of (3.1) with overfall $[0, \theta]$. Then there exist $T > 0$ and $B > 0$ such that

$$F_n(t) \leq \widehat{F}(n - ct + \beta\sqrt{t}) \quad \text{for } n \leq ct - B\sqrt{t}, \quad t \geq T$$

for all $\beta > 2B$.

Proof. For proving Lemma 8.4 it is sufficient to check the conditions of Lemma 8.1 for $\widehat{G}_n = F_n$, $G_n = \widehat{F}(n - ct + \beta\sqrt{t})$, $x(t) = ct - B\sqrt{t} + 1$ and apply this lemma.

Firstly, we have the inequality

$$\widehat{G}_0(t) = F_0(t) = 0 < G_0(t) = \widehat{F}(-ct + \beta\sqrt{t})$$

for every β and d.

So, we can choose B large enough to have also inequalities

$$\widehat{G}_n(T) = F_n(T) < G_n(T) = \widehat{F}(n - cT + \beta\sqrt{T})$$

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for $n = 1, \ldots, [x(T)]$, $\beta \geq 2B$.

Secondly, we have the inequality

$$\frac{dG_n}{dt} = \left(-c + \frac{\beta}{2\sqrt{t}}\right) \cdot \frac{d\widehat{F}}{dx}(n - ct + \beta\sqrt{t})$$

$$= \varphi(G_n)(G_{n-1} - G_n) + \frac{\beta}{2\sqrt{t}}\frac{d\widehat{F}}{dx}(n - ct + \beta\sqrt{t})$$

$$\geq \varphi(G_n)(G_{n-1} - G_n).$$

Let us show, at last, that if B is large enough, then

$$\widehat{G}_{[x(t)]}(t) < G_{[x(t)]}(t) \quad \text{for } t \geq T, \beta \geq 2B. \tag{8.43}$$

Let $\widetilde{F}_n(t)$ be the solution of (3.1) under $\varphi = \widetilde{\varphi}$ and initial data $\widetilde{F}_n(0)$ defined by (8.29). Then (8.30) is valid.

Theorem 6.2 and (8.30) entail the following inequalities for $t \geq T$ and some $\gamma > 0$

$$F_{[ct - B\sqrt{t} + 1]}(t) \leq \widetilde{F}_{[ct - B\sqrt{t} + 1]}(t)$$

$$\leq \varphi^{(-1)}\left(\frac{ct - B\sqrt{t} + 1}{t}\right) + \frac{\gamma}{\sqrt{t}}$$

$$= \varphi^{(-1)}\left(\varphi(\theta) - \frac{B}{\sqrt{t}} + \frac{1}{t}\right) + \frac{\gamma}{\sqrt{t}}$$

$$\leq \theta - \frac{(1 - \delta_1(T))}{\varphi'(\theta)}\frac{B}{\sqrt{t}} + \frac{\gamma}{\sqrt{t}}$$

$$= \theta - \left(\frac{(1 - \delta_1(T))B}{\varphi'(\theta)} - \gamma\right)\frac{1}{\sqrt{t}}, \tag{8.44}$$

where $\delta_1(T) > 0$ and $\delta_1(T) \to 0$ as $T \to \infty$.

Due to Lemma 7.2 and Remark 7.4 there exists $T = T(l)$, $l > 1$ such that

$$G_{[x(t)]} \geq \widehat{F}_{[ct + (\beta - B)\sqrt{t} + d]} \geq \theta - \frac{\varphi(\theta)(l + 1)}{\varphi'(\theta)(\beta - B)\sqrt{t}}, \quad t \geq T. \tag{8.45}$$

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Let $\beta \geq 2B$. In view of (8.44), (8.45), to satisfy (8.43) it is sufficient to have the inequality:

$$\theta - \left(\frac{(1 - \delta_1(T))B}{\varphi'(\theta)} - \gamma\right)\frac{1}{\sqrt{t}} \leq \theta - \frac{\varphi(\theta)}{\varphi'(\theta)}\frac{(l+1)}{B\sqrt{t}}.$$

The last inequality is fulfilled for every B large enough. Hence, all conditions of Lemma 8.1 are valid, and Lemma 8.4 is proved.

Proof of Theorem 8.3 (for the case $E = [0, \theta]$, $\theta < 1$). For this case the function (8.27) takes the following form

$$\widetilde{F}_n(t, d) = \widehat{F}(n - ct + d) + \Psi(n/t) - \theta \tag{8.46}$$

where $\widehat{F}$ is a wave train of (3.1) with overfall $[0, \theta]$, $c = \varphi(\theta)$,

$$\Psi(n/t) = \begin{cases} \theta & \text{for } n \leq ct, \\ \varphi^{-1}(n/t) & \text{for } ct \leq n \leq \varphi(1)t, \\ 1 & \text{for } n \geq \varphi(1)t. \end{cases}$$

Let $d(t)$ be such that $d(t)/\sqrt{t} \to 0$, $t \to \infty$. Due to Lemma 8.4 one can find B and T such that

$$F_n(t) \leq \widehat{F}(n - ct + 2B\sqrt{t} + d(t)) \quad \text{for } n \leq ct - B\sqrt{t}, t \geq T. \tag{8.47}$$

To prove Theorem 8.3 we will show that

$$\sup_{n \leq ct - 3B\sqrt{t}} |F_n(t) - \widetilde{F}_n(t, d(t))| \to 0, \tag{8.48}$$

and

$$\sup_{n \geq ct + 3B\sqrt{t}} |F_n(t) - \widetilde{F}_n(t, d(t))| \to 0 \tag{8.49}$$

as $t \to \infty$.

If $n \leq ct - 3B\sqrt{t}$ then $\widetilde{F}_n(t, d) = \widehat{F}(n - ct + d)$ and $n - ct + 2B\sqrt{t} + d(t) \leq -B\sqrt{t} + d(t) \to -\infty$ as $t \to \infty$. Since $\widehat{F}(x) \to 0$ as $x \to -\infty$ the relation (8.48) follows from (8.47).

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If $n \geq ct + 3B\sqrt{t}$ then $n - ct + d(t) \geq 3B\sqrt{t} + d(t) \to +\infty$ as $t \to \infty$. Since $\widehat{F}(x) \to \theta$ as $x \to +\infty$ we have

$$\sup_{n \geq ct + 3B\sqrt{t}} |\widetilde{F}_n(t, d(t)) - \Psi(n/t)| \to 0 \quad \text{as } t \to \infty. \tag{8.50}$$

The proof of Theorem 8.3 will be completed if we show that

$$\sup_{n \geq ct + 3B\sqrt{t}} |F_n(t) - \Psi(n/t)| \to 0 \quad \text{as } t \to \infty. \tag{8.51}$$

Due to Lemma 8.3 the inequality (8.30) holds. Let us define

$$\Psi_g(n/t) = \begin{cases} \theta^* & \text{for } n \leq g(\theta^*)t, \\ g^{-1}(n/t) & \text{for } g(\theta^*)t \leq n \leq g(1)t, \\ 1 & \text{for } n \geq g(1)t, \end{cases}$$

where g is defined by (8.29). It follows from (8.30) and Theorem 6.2 that

$$F_n(t) \leq G_n(t) \leq \Psi_g(n/t) + \frac{\gamma}{\sqrt{t}}.$$

Since $\theta > \theta^*$ and $g(F) = \varphi(F)$ for $F \geq \theta^*$ one gets $\Psi_g(n/t) = \Psi(n/t)$ for $n \geq ct$. Therefore

$$F_n(t) \leq \Psi(n/t) + \frac{\gamma}{\sqrt{t}}, \quad n \geq ct. \tag{8.52}$$

Let us introduce functions

$$\check{F}_n(t) = \Psi(n/t) - \frac{\gamma}{\sqrt{t}}.$$

Due to Lemma 6.5 there exist $\gamma > 0$, $T > 0$ such that

$$\frac{d\check{F}_n}{dt} \leq \varphi(\check{F}_n)(\check{F}_{n-1} - \check{F}_n) \quad \text{for } \varphi(\theta)t \leq n \leq \varphi(1)t, t \geq T. \tag{8.53}$$

If $\gamma = \gamma(T)$ is large enough then the following inequalities obviously hold

$$\check{F}_n(T) \leq F_n(T) \quad \text{for } \varphi(0)T \leq n \leq \varphi(1)T. \tag{8.54}$$

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If $\varphi(\theta)t \le n \le \varphi(\theta)t + \beta\sqrt{t}$ then $\check{F}_n(t) \le \varphi^{-1}(\varphi(\theta) + \beta\sqrt{t}) - \gamma/\sqrt{t} = \theta + O(1/\sqrt{t}) - \gamma/\sqrt{t}$. Since $F_n(t) = \theta + O(1/\sqrt{t})$ for $\varphi(\theta)t + \beta\sqrt{t} - 1 \le n \le \varphi(\theta)t + \beta\sqrt{t}$ (see Lemma 8.2) one can choose γ large enough so that

$$\check{F}_n(t) \le F_n(t), \quad -1 < n - \varphi(\theta)t - \beta\sqrt{t} \le 0, \ t \ge T. \tag{8.55}$$

Let us prove now that

$$\check{F}_n(t) \le F_n(t), \quad n \ge \varphi(1)t, \ t \ge T. \tag{8.56}$$

By definition of $\check{F}_n$ the inequality (8.56) is equivalent to the following relation

$$F_n(t) \ge 1 - \gamma/\sqrt{t}, \quad n \ge \varphi(1)t, \ t \ge T. \tag{8.57}$$

Let $\widehat{\varphi} = \varphi(F) + \alpha(1 - F)$, $F \in [\theta - \varepsilon, 1]$, where $\alpha > 0$, $\varepsilon > 0$, ε is small enough. Consider a solution of the equation

$$\frac{d\widehat{F}_n}{dt} = \widehat{\varphi}(\widehat{F}_n)(\widehat{F}_{n-1} - \widehat{F}_n)$$

with initial conditions

$$\widehat{F}_n(0) = \begin{cases} F_n(0) & \text{if } \theta - \varepsilon \le F_n(0) \le 1, \\ \theta - \varepsilon, & \text{otherwise.} \end{cases}$$

Due to Lemma 8.2 and since $\varphi(\theta) < \widehat{\varphi}(\theta - \varepsilon)$ one has

$$F_n(t) \ge \theta + O(1/\sqrt{t}), \quad n \ge \widehat{\varphi}(\theta - \varepsilon)t - 1.$$

Using Theorem 6.2 we have

$$\widehat{F}_n(t) = \theta - \varepsilon + O(1/\sqrt{t}), \quad n \le \widehat{\varphi}(\theta - \varepsilon)t.$$

Therefore there exists $T > 0$ such that $F_n(t) \ge \widehat{F}_n(t)$, $-1 \le n - \widehat{\varphi}(\theta - \varepsilon)t \le 0$, $t \ge T$.

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Using Lemma 6.2 we get $F_n(t) \ge \widehat{F}_n(t)$ for all $n \in [(\theta - \varepsilon)t, \infty)$. Due to Theorem 6.2

$$\widehat{F}_n(t) \ge 1 - \frac{\gamma}{\sqrt{t}}, \quad n \ge \widehat{\varphi}(1)t = \varphi(1)t,$$

therefore (8.57) is proved.

The inequalities (8.53)–(8.57) permit to use Lemma 6.2 again to compare F_n and $\check{F}_n$ on $[\varphi(\theta)t + \beta\sqrt{t}, \varphi(1)t]$, $\beta = 3B$. We get

$$F_n(t) \ge \Psi(n/t) - \frac{\gamma}{\sqrt{t}} \tag{8.58}$$

for $t \ge T$, $n \ge \varphi(\theta)t + \beta\sqrt{t}$.

From Lemma 8.3 and Theorem 6.2 it follows also that

$$F_n(t) \le G_n(t) \le \Psi(n/t) + \frac{\gamma}{\sqrt{t}} \tag{8.59}$$

for $t \ge T$, $n \ge \varphi(\theta)t + \beta\sqrt{t}$.

Now the inequality (8.51) and the statement of Theorem 8.3 for the case $E = [0, \theta]$, $\theta < 1$ follows from (8.58) and (8.59).

9. Comparison with Burgers equation

At first sight equation (3.1) looks like a discretization (in space variable) of the following shock wave equation

$$\frac{\partial F(x, t)}{\partial t} = \Psi(F)\frac{\partial F(x, t)}{\partial x} \tag{9.1}$$

that was studied in many works (see for example Lax (1973), Whitham (1984)). But solutions of (3.1) do not reveal shock wave behavior. The dynamics of (3.1) for monotonic case reminds the wave train or diffusion behavior studied by Hopf (1950) and Iljin and Olejnic (1960) for Burgers equation

$$\frac{\partial F}{\partial t} = -\varepsilon\frac{\partial^2 F}{\partial x^2} + \Psi(F)\frac{\partial F}{\partial x}, \tag{9.2}$$

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where Ψ is a monotonic function, ε is a constant. The analogy holds not only for results but for schemes of some proofs. The first term in (9.2) is called as dissipative term with viscosity ε. One could remark that the difference $F_n - F_{n-1}$ (see (3.1)) "contains" a dissipative term in a sense due to Tailor expansion, but the same is true for terms with third and forth derivatives, and it is not clear what is the nature of the analogue between (3.1) and (9.2).

Let us compare these two equations in greater detail (see Henkin, Polterovich (1991)). Consider a variant of equation (3.1) with an arbitrary "step of discretization" δ

$$\frac{\partial F(x,t)}{\partial t} = -\varphi(F)\frac{F(x,t) - F(x-\delta,t)}{\delta}, \tag{9.3}$$

and compare (9.2) with (9.3) for $\varphi(F) = \alpha + \beta(1 - F) = -\Psi(F)$. Wave trains of both equations are logistic curves that have the form

$$F^*(x,t) = 1/(1 + e^{-p(x-ct)}),$$

where $p = p' = (1/\delta)\ln(\mu/\alpha)$, $c = c' = \beta\ln(\mu/\alpha)$ for (9.3), $\mu = \alpha + \beta$, and $p = p'' = \beta/2\alpha$, $c = c'' = \alpha + \beta/2$ in the case (9.2). It is simple to check that if $\varepsilon = \alpha\delta/2$, $\alpha > 0$, and $\beta \sim \delta \to 0$, then the differences $p' - p''$, $c' - c''$ have the order δ^2 as well as the difference of the right-hand sides of both the equations. Thus under these artificial conditions, one can consider equation (9.3) as an approximation of (9.2). But if β/α is not small, then the distinction between (9.3) and (9.2) is substantial. The formulas of p and c show that solutions of (9.3) do not converge to feasible solutions of (9.1) as $\varepsilon \to 0$.

Nevertheless all main facts of the theory of the Burgers equation have their counterparts for the equation studied above.

We believe that this analogy will be expanded for the nonmonotonic case as well and similar general results will be received for both problems.

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